AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **38** (2007), Pages 255–266

Some new results on the integer-magic spectra of tessellation graphs

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Abstract

Let A be an abelian group with non-identity elements A^*. A graph is A-magic if there exists an edge-labeling using elements of A^*, which induces a constant vertex labeling of the graph. In Low and Lee (*Australas. J. Combin.* 34 (2006), 195–210), the $\mathbb{Z}_k$-magic property for certain classes of triominoes and polyominoes was studied. In this paper, we resolve an open question involving the $\mathbb{Z}_k$-magic property for pyramid graphs. Furthermore, for certain classes of honeycomb graphs, we determine for which values of $k \geq 2$ the graphs are $\mathbb{Z}_k$-magic.

1 Introduction

Let $G = (V, E)$ be a connected simple graph. For any non-trivial abelian group A (written additively), let $A^* = A - \{0\}$. A function $f : E(G) \to A^*$ is called a *labeling* of G. Any such labeling induces a map $f^+ : V(G) \to A$, defined by $f^+(v) = \Sigma f(u, v)$, where the sum is over all $(u, v) \in E(G)$. If there exists a labeling f whose induced map on $V(G)$ is a constant map, we say that f is an *A-magic labeling* of G and that G is an *A-magic* graph. The corresponding constant is called an *A-magic value*. The *integer-magic spectrum* of a graph G is the set $\mathrm{IM}(G) = \{k : G \text{ is } \mathbb{Z}_k\text{-magic and } k \geq 2\}$. By convention, $\mathbb{Z}$-magic graphs are considered to be $\mathbb{Z}_1$-magic.

$\mathbb{Z}$-magic graphs were considered by Stanley [21, 22], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [1, 2, 3] and others [7, 9, 15, 16, 19] have studied A-magic graphs and $\mathbb{Z}_k$-magic graphs were investigated in [4, 6, 8, 10, 11, 12, 13, 14, 20].

Within the mathematical literature, various definitions of magic graphs have been introduced. The original concept of an A-magic graph is due to J. Sedlácek [17, 18], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Previously, Kotzig and Rosa [5] had introduced yet another definition of a magic graph. Over the years, there has been great research interest in graph labeling problems. The interested reader is directed to Wallis' [23] recent monograph on magic graphs.

2 Tessellation graphs

In [14], Low and Lee introduced the concept of a tessellation graph. For convenience, we include the relevant definitions in this section.

A *tessellation* is a tiling of the plane, using polygons. If a tessellation consists of congruent polygons, it is a *regular tessellation*. Thus, there are only three regular tessellations, utilizing equilateral triangles, squares, or regular hexagons. A *tessellation graph* is a finite subgraph of a regular tessellation, consisting of a grid of congruent polygons where each polygon shares at least one common edge with another.

Definition 1 *A region Ω in the plane is n-connected if the complement of Ω has exactly n components.*

Definition 2 *For $n \geq 2$, an n-tessellation graph is a graph which tessellates an n-connected region in the plane.*

For example, a 1-tessellation graph tessellates a simply-connected, bounded region in the plane. The reader should note the following remarks.

Remarks. Let G be an n-tessellation graph, $n \geq 2$.

1. In G, the boundaries of a hole and the outer boundary of G have no vertices in common.

2. If G is an n-tessellation graph with $n \geq 3$, then the boundaries of any two holes have no vertices in common.

Low and Lee [14] determined the entire integer-magic spectra of n-tessellation graphs constructed from squares. In addition, the integer-magic spectra for certain classes of n-tessellation graphs, constructed from equilateral triangles, were analyzed. In this paper, we continue to study the integer-magic spectra of triominoes and begin the study of honeycomb graphs.

3 Triominoes

Consider a tessellation of the plane, using congruent equilateral triangles. Two triangles are *connected* if they share a common edge. Let T be a connected collection of triangles. Then, T is a connected planar graph, consisting of a grid of C_3's with each C_3 sharing at least one common edge with another. A connected collection of triangles is called a *triomino*. T is called an n-*triomino* if it is an n-tessellation graph.

In [14], the integer-magic spectra for various classes of 1-triominoes were analyzed, one of which was the class of *pyramid graphs* (see Figure 1).

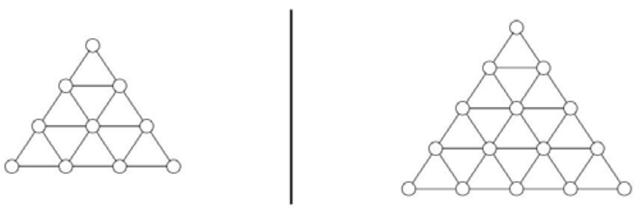

Figure 1: Pyramids of heights 3 and 4, respectively.

In particular, Low and Lee [14] used an intricate argument to establish the following result.

Theorem A *Let P be a pyramid of height n, $n \geq 3$. If $n \equiv 0$ or $3 \pmod 4$, then* $\mathrm{IM}(P) = \mathbb{N} - \{1\}$. *Otherwise,* $\{2, 4, 5, 6, ...\} \subseteq \mathrm{IM}(P)$.

It remained unresolved as to whether 3 was contained in the integer-magic spectrum of a pyramid of height n, when $n \equiv 1$ or $2 \pmod 4$. We now give a complete answer for these remaining cases and thus, establish the integer-magic spectra of pyramids.

Theorem 2 *Let P be a pyramid of height n, where $n \equiv 1 \pmod 4$. Then,* $\mathrm{IM}(P) = \mathbb{N} - \{1\}$.

Proof. It is clear that if P is a pyramid of height 1, then $\mathrm{IM}(P) = \mathbb{N} - \{1\}$. First, we give a $\mathbb{Z}_3$-magic labeling for a pyramid of height 5.
Now, every pyramid P of height n where $n \equiv 1 \pmod 4$ and $n \geq 5$, has n edges at the bottom of the pyramid. In particular, n is odd. Thus, the $\mathbb{Z}_3$-magic labeling illustrated in Figure 2 can be extended to give a $\mathbb{Z}_3$-magic labeling for any pyramid P of height n, where $n \equiv 1 \pmod 4$ and $n \geq 5$. That is, all diagonal edges of P are labeled 2, all horizontal edges in the first $n-1$ layers of P are labeled 1, and the horizontal edges on the bottom of the last layer of P are labeled $2, 1, 2, 1, \ldots, 1, 2$, respectively. This, along with Theorem A, establishes the result. □

In [6], it was shown that if P is a pyramid of height 2, then $\mathrm{IM}(P) = \mathbb{N} - \{1, 3\}$. The next theorem finishes the analysis of the integer-magic spectra of P.

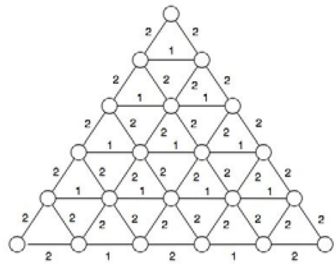

Figure 2: $\mathbb{Z}_3$-magic labeling of a pyramid of height 5.

Theorem 3 *Let P be a pyramid of height n, where $n \equiv 2 \pmod 4$ and $n \geq 6$. Then,* $\mathrm{IM}(P) = \mathbb{N} - \{1\}$.

Proof. Figure 3 illustrates a $\mathbb{Z}_3$-magic labeling for a pyramid of height 6. This particular labeling has $\mathbb{Z}_3$-magic value 1. We now construct $\mathbb{Z}_3$-magic labelings for pyramids of height n, where $n \equiv 2 \pmod 4$ and $n \geq 10$. This is accomplished by successively adding the labeled block illustrated in Figure 4 to the bottom of the pyramid in Figure 3. At each stage, we identify the vertices at the bottom of the pyramid with the vertices at the top of the block. Hence, we obtain a $\mathbb{Z}_3$-magic labeling of P, with $\mathbb{Z}_3$-magic value 1. This, along with Theorem A, establishes the result.

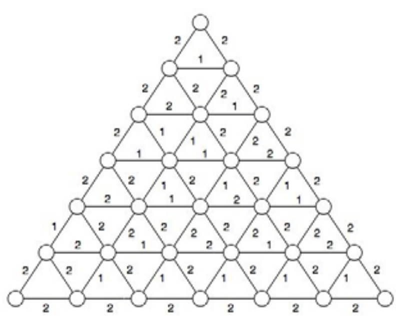

Figure 3: $\mathbb{Z}_3$-magic labeling of a pyramid of height 6.

□

4 Honeycomb graphs

Consider a tessellation of the plane, using congruent hexagons. Two hexagons are *connected* if they share a common edge. Let H be a connected collection of hexagons.

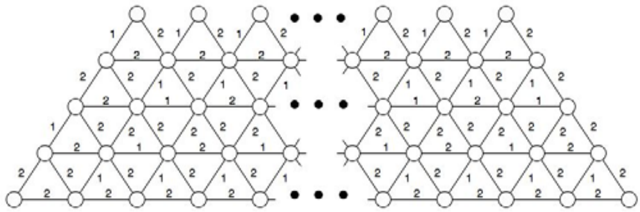

Figure 4: Labeled block.

Then, H is a connected planar graph, consisting of a grid of C_6's with each C_6 sharing at least one common edge with another. A connected collection of hexagons is called a *honeycomb* graph. H is called an *n-honeycomb* if it is an n-tessellation graph made up of hexagons.

Observations.

1. Every regular graph is $\mathbb{Z}_k$-magic, for $k \geq 2$. Thus, if $H = C_6$, then $\mathrm{IM}(H) = \mathbb{N} - \{1\}$.

2. A graph is $\mathbb{Z}_2$-magic if and only if all of its vertices are of the same parity. Thus, if H is a 1-honeycomb graph made up of two or more hexagons, then H is not $\mathbb{Z}_2$-magic.

Theorem 4 *Let H be a 1-honeycomb graph which can be constructed by successively adjoining hexagons, via the identification of at most two pairs of edges (and their corresponding vertices) at each step. Then,* $\{4, 5, 6, 7, ...\} \subseteq \mathrm{IM}(H)$.

Proof. Let $k \in \mathbb{Z}$ and $k \geq 4$. To obtain a $\mathbb{Z}_k$-magic labeling of H, we use the two labelings of a hexagon illustrated in Figure 5.

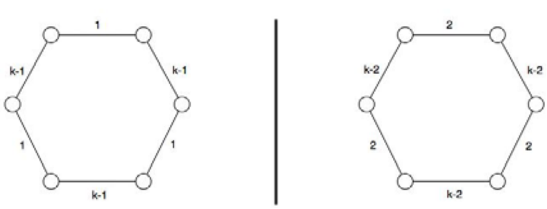

Figure 5: $\mathbb{Z}_k$-magic labelings of a hexagon, with $\mathbb{Z}_k$-magic value 0.

We proceed by induction on the number of hexagons used to form H. If $H = C_6$, we obtain a $\mathbb{Z}_k$-magic labeling with $\mathbb{Z}_k$-magic value 0, by using either of the labelings found in Figure 5. Now, assume that every 1-honeycomb graph H' made up of n hexagons (and constructed as described in the hypothesis of the theorem) has a $\mathbb{Z}_k$-magic labeling with $\mathbb{Z}_k$-magic value 0. Let H be a 1-honeycomb graph (consisting of

$n + 1$ hexagons) which is constructed by adjoining a hexagon (via identification of at most two pairs of edges) to H'.

CASE 1. H is constructed by identifying one edge of a hexagon to an edge of H'. By the induction hypothesis, H' has a $\mathbb{Z}_k$-magic labeling with $\mathbb{Z}_k$-magic value 0. Since all of the exterior edges of H' are 1, $k-1$, 2, or $k-2$, we use the appropriate labeling of a hexagon from Figure 5 so that the identified edge has a non-zero value. This yields a $\mathbb{Z}_k$-magic labeling of H, with $\mathbb{Z}_k$-magic value 0.

CASE 2. H is constructed by identifying two edges of a hexagon to two edges of H'. In this case, there are only ten possible situations which can occur. Figure 6 below gives $\mathbb{Z}_k$-magic labelings ($k \geq 4$) with $\mathbb{Z}_k$-magic value 0, for H.

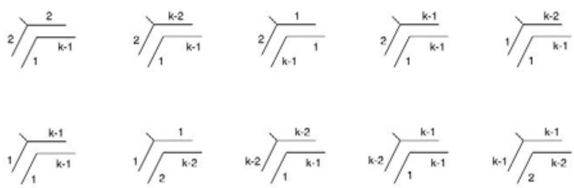

Figure 6: [CASE 2.] $\mathbb{Z}_k$-magic labelings of H ($k \geq 4$), with $\mathbb{Z}_k$-magic value 0.

□

An example illustrating Theorem 4 is given in Figure 7.

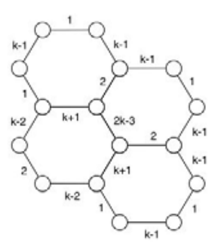

Figure 7: A $\mathbb{Z}_k$-magic labeling of a honeycomb graph, with $\mathbb{Z}_k$-magic value 0, for $k \geq 4$.

We now analyze the integer-magic spectra for the class of *hexagonal pyramids* (see Figure 8). Each layer of a hexagonal pyramid is called a *hexagonal snake.* A hexagonal snake consisting of n hexagons is of *length* n.

Lemma 1 *Let S be a hexagonal snake. Then, for $k \geq 4$, there exists a $\mathbb{Z}_k$-magic labeling of S with $\mathbb{Z}_k$-magic value* 0*, where the exterior edges of S are labeled* 1, $k-1$, 2*, or* $k-2$.

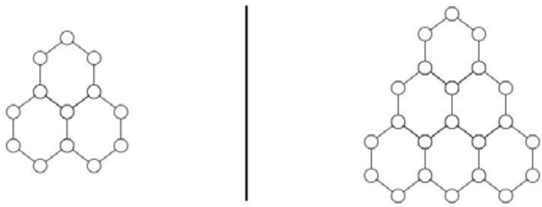

Figure 8: Hexagonal pyramids of heights 2 and 3, respectively.

Proof. To construct the required $\mathbb{Z}_k$-magic labeling of S with $\mathbb{Z}_k$-magic value 0, we will use the two labelings of C_6 as described in Figure 5. Each hexagon of S will be labeled using one of these labelings and a labeling of S will be obtained by identifying pairs of edges. For a snake S of length n, we use the notation $L_n = (x_1, x_2, \ldots, x_{2n})$ to denote the labeling in Figure 9.

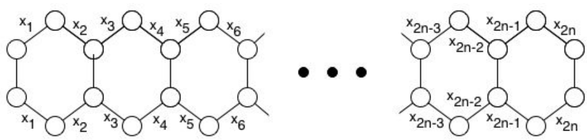

Figure 9:

For $k \geq 4$, a $\mathbb{Z}_k$-magic labeling (with $\mathbb{Z}_k$-magic value 0) of S can be obtained using the recursive construction described in Table 1. Let n be the length of S. Then:

n	L_n
1	$(2, k-2)$
2	$(1, k-1, k-1, 1)$
3	$(k-1, 1, 2, k-2, 1, k-1$
4	$(2, k-2, 1, k-1, k-1, 1, k-2, 2)$
5	$(1, k-1, k-1, 1, 2, k-2, 1, k-1, k-1, 1)$
6	$(k-1, 1, 2, k-2, 1, k-1, k-1, 1, k-2, 2, 1, k-1)$
7	$(2, k-2, 1, k-1, k-1, 1, 2, k-2, 1, k-1, k-1, 1, k-2, 2)$
8	$(1, k-1, k-1, 1, 2, k-2, 1, k-1, k-1, 1, k-2, 2, 1, k-1, k-1, 1)$
$\vdots$	$\vdots$

Table 1: Recursive construction of $\mathbb{Z}_k$-magic labelings of S, for $k \geq 4$.

Formally, the construction is described in the following way: Let $L_1 = (2, k-2)$, $L_2 = (1, k-1, k-1, 1)$ and $[L_r]_s$ denote the sth entry of L_r. Then for $n \geq 3$,

$L_n = (x_1, x_2, \ldots, x_{2n})$ is defined by

$$x_i = [L_{n-2}]_{i-2}, \quad \text{for } 3 \leq i \leq 2n-2,$$

$$x_2 = x_{2n-1} = \begin{cases} 1, & \text{if } [L_{n-1}]_1 = 1 \\ k-1, & \text{if } [L_{n-1}]_1 = 2 \\ k-2, & \text{if } [L_{n-1}]_1 = k-1, \end{cases}$$

and $x_1 = x_{2n} = k - x_2$. For example, Figure 10 gives a $\mathbb{Z}_k$-magic labeling (with $\mathbb{Z}_k$-magic value 0) for a hexagonal snake of length 5, where $k \geq 4$.

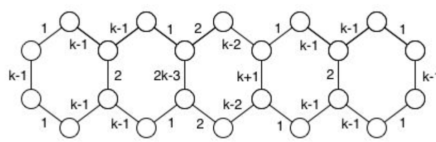

Figure 10: $L_5 = (1, k-1, k-1, 1, 2, k-2, 1, k-1, k-1, 1)$.

Since the labelings of C_6 (in Figure 5) have $\mathbb{Z}_k$-magic value 0, the L_n labelings of S described in Table 1 have $\mathbb{Z}_k$-magic value 0. Furthermore for $k \geq 4$, the only time that an interior edge is labeled 0 is if L_n is of one of the following forms:

- $L_n = (\ldots, 2, k-2, 2, k-2, \ldots)$
- $L_n = (\ldots, k-2, 2, k-2, 2, \ldots)$
- $L_n = (\ldots, 1, k-1, 1, k-1, \ldots)$
- $L_n = (\ldots, k-1, 1, k-1, 1, \ldots)$.

In our recursive construction of L_n, this can never occur. Thus for a hexagonal snake S of length n and $k \geq 4$, L_n is a $\mathbb{Z}_k$-magic labeling with $\mathbb{Z}_k$-magic value 0. □

Theorem 5 *Let G be a hexagonal pyramid of height $n \geq 2$. Then,* $\{4, 5, 6, 7, ...\} \subseteq$ IM(G).

Proof. Let $k \geq 4$ and suppose that G is a hexagonal pyramid of height $n \geq 2$. Using Lemma 1, we first consider the hexagonal snakes of length $1, 2, \ldots, n$, along with their respective $L_1, L_2, \ldots, L_n$ labels. The labeled hexagonal snakes are then layered to form G, by the appropriate identification of vertices and edges. Note that when two edges are identified with each other, the resulting edge is labeled with the sum of the two original labels. Since each L_n has Z_k-magic value 0, the resulting labeling of G has $\mathbb{Z}_k$-magic value 0. Furthermore, every pair of edges which are identified with each other forms an edge which is labeled with a non-zero number (mod k). This is easily seen since for $n \geq 2$ and $k \geq 4$,

$$[L_n]_i + [L_{n-1}]_{i-1} \neq 0, \ (\text{mod } k),$$

where $2 \leq i \leq 2n-1$. □

For example, Figure 11 gives a $\mathbb{Z}_k$-magic labeling (with $\mathbb{Z}_k$-magic value 0) for a hexagonal pyramid of height 4, where $k \geq 4$.

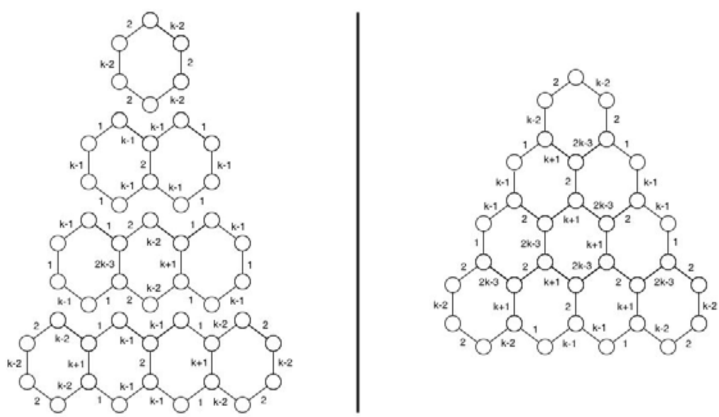

Figure 11: A $\mathbb{Z}_k$-magic labeling of a hexagonal pyramid of height 4, where $k \geq 4$.

Using Theorem 5, we easily obtain $\mathbb{Z}_k$-magic labelings ($k \geq 4$) for the class of *hexagonal diamonds* (see Figure 12).

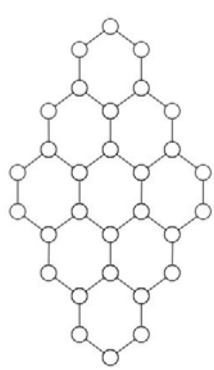

Figure 12: A hexagonal diamond of length 5.

Theorem 6 *Let D be a hexagonal diamond of length $2n+1$, where $n \geq 1$. Then, $\{4, 5, 6, 7, ...\} \subseteq \mathrm{IM}(D)$.*

Proof. Let $k \geq 4$ and D be a hexagonal diamond of length $2n+1$, where $n \geq 1$. Using Theorem 5, obtain $\mathbb{Z}_k$-magic labelings (with $\mathbb{Z}_k$-magic value 0) for pyramids P of height $n+1$ and P' of height n, respectively. Form the labeled hexagonal diamond D by identifying the appropriate (bottom) vertices and edges of P with the corresponding (bottom) vertices and edges of P'. Each identified pair of edges forms a new single edge which is labeled with the sum of the original labels. This gives a $\mathbb{Z}_k$-magic labeling of D, for $k \geq 4$. □

Figure 13 illustrates a $\mathbb{Z}_k$-magic labeling ($k \geq 4$) of a hexagonal diamond, using Theorem 6.

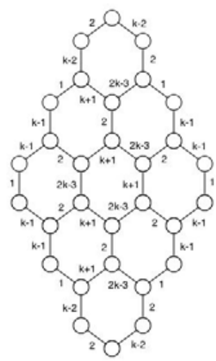

Figure 13: A $\mathbb{Z}_k$-magic labeling of a hexagonal diamond of length 5, where $k \geq 4$.

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(Received 30 May 2006)