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On weighted Ramsey numbers

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Abstract

The weighted Ramsey number, $\mathrm{wR}(n,k)$, is the minimum q such that there is an assignment of nonnegative real numbers (weights) to the edges of K_n with the total sum of the weights equal to $\binom{n}{2}$ and there is a Red/Blue coloring of edges of the same K_n, such that in any complete k-vertex subgraph H, of K_n, the sum of the weights on Red edges in H is at most q and the sum of the weights on Blue edges in H is at most q. This concept was introduced recently by Fujisawa and Ota.

We provide new bounds on $\mathrm{wR}(n,k)$, for $k \geq 4$ and n large enough and show that determining $\mathrm{wR}(n,3)$ is asymptotically equivalent to the problem of finding the fractional packing number of monochromatic triangles in colorings of edges of complete graphs with two colors.

1 Introduction

Definition 1. The weighted Ramsey number, $\mathrm{wR}(n,k)$, is the minimum q such that there is an assignment of nonnegative real numbers (weights) to the edges of K_n with the total sum of the weights equal to $\binom{n}{2}$ and there is a Red/Blue coloring of the edges of the same K_n, such that in any complete k-vertex subgraph H of K_n, the sum of the weights on Red edges in H is at most q and the sum of the weights on Blue edges in H is at most q.

This notion was introduced by Fujisawa and Ota in [3], where the authors used the scaled version of the above definition requiring the total sum of weights to be 1; the corresponding weighted Ramsey function from [3] is $\mathrm{wR}(n,k)/\binom{n}{2}$. The main results obtained in [3] can be summarized as follows.

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Theorem 2 ([3]). *For any integers n, k, $4 \le k \le n$,*

$$\frac{1}{2}\binom{k}{2} \le \mathrm{wR}(n,k) < \frac{k^2-1}{k^2+1}\binom{k}{2}.$$

In addition, $\mathrm{wR}(5,3) = 2$ *and* $\mathrm{wR}(n,3) \ge 15/7$ *for* $n \ge 6$ *(with equality when* $n = 6$*). Moreover, the better asymptotic bound holds:* $\mathrm{wR}(n,3) \ge 110/49 - o(1)$.

The main emphasis of [3] was determining the bound on $\mathrm{wR}(n,3)$, where the authors observed a relation between weighted Ramsey numbers and the edge-disjoint packing of monochromatic triangles in a 2-colored complete graph. For an edge coloring c of a complete graph with two colors, let $\tau(c,3)$ be the largest number of edge-disjoint monochromatic triangles in c. Let

$$\tau(n,3) = \min\{\tau(c,3) : \ c \text{ is a } 2-\text{edge-coloring of } K_n\}.$$

The following was proven in [3]:

Theorem 3 ([3]).

$$\mathrm{wR}(n,3) \ge \frac{4\binom{n}{2}}{n^2 - 2\tau(n,3) + n}.$$

Together with the bound $\tau(n,3) \ge (\frac{3}{55} + o(1))n^2$ given in [2] and the upper bound of Theorem 2, the authors of [3] provide the following bound:

$$2.2448 + o(1) \le \mathrm{wR}(n,3) < 2.4. \tag{1}$$

Note that more recent better bound $\tau(n,3) \ge (\frac{1}{12.888} + o(1))n^2$ given by Keevash and Sudakov in [5], immediately improves (1) as follows.

$$2.3674 + o(1) \le \mathrm{wR}(n,3) < 2.4. \tag{2}$$

Moreover, as also observed in [3], if the value conjectured by Erdős, $\tau(n,3) = (\frac{1}{12} + o(1))n^2$, is correct, then $\mathrm{wR}(n,3)$ would be asymptotically equal to 2.4.

In Theorem 4, we treat the general case $\mathrm{wR}(n,k)$ for $k \ge 4$. We obtain better bounds and relate the weighted Ramsey problem to Turán-Ramsey type results using the regularity lemma of Szemerédi. In Theorem 5, we analyze $\mathrm{wR}(n,3)$ and related problems. We show that finding $\mathrm{wR}(n,3)$ is asymptotically equivalent to finding the fractional packing number of monochromatic triangles in 2-colored complete graphs. We choose to make the total sum of the edge-weights equal to $\binom{n}{2}$, instead of 1 as in [3], in order for easier analysis of the asymptotic behavior of $\mathrm{wR}(n,k)$. In fact, to state our main results, we use the *weighted Ramsey limit* defined as follows:

$$\mathbf{W}(k) := \lim_{n\to\infty} \mathrm{wR}(n,k).$$

We prove the existence of this limit in Section 2. Note that Theorem 2 gives that $\frac{k-1}{k}\lfloor k^2/4 \rfloor \le \mathbf{W}(k) \le 2\lfloor k^2/4 \rfloor$. Our main theorem is the following.

Theorem 4. *Let k be an integer, $k \geq 5$.*

$$1.051 \left\lfloor \frac{k^2}{4} \right\rfloor < \mathbf{W}(k) \leq 1.25 \left\lfloor \frac{k^2}{4} \right\rfloor . \tag{3}$$

If k is sufficiently large,

$$\mathbf{W}(k) > 1.059 \left\lfloor \frac{k^2}{4} \right\rfloor . \tag{4}$$

Moreover, more accurate bounds for small k can be summarized in the following table, where $U(k)$ and $L(k)$ are the upper and lower bounds on $\mathbf{W}(k)$, respectively.

k	4	5	6	7	8
$L(k)$	4.1999	6.3572	9.5197	12.7091	16.9115
$U(k)$	4.8	7.5	11.25	15	20

Note that both (3) and (4) improve the constants in the lower and upper bounds from previously known constants close to 1 and 2, respectively. We conjecture that the upper bound of $1.25\lfloor k^2/4 \rfloor$ gives the correct value for $k \geq 5$ but $1.2\lfloor k^2/4 \rfloor$ is correct for $k = 3, 4$.

For a graph G on n vertices let $\mathcal{T}_3(G)$ denote the set of triangles in G. Let each triangle be assigned a real number, called a *weight*, between 0 and 1. We say that this assignment is *proper* if, for each edge e of G, the sum of weights of triangles containing e is at most 1. The *fractional triangle packing number* of G, denoted τ^*, is the largest possible total weight of edges in a proper weight assignment. Formally, it is defined as follows.

$$\tau^*(G) = \max \sum_{T \in \mathcal{T}_3(G)} g(T) \tag{5}$$

$$\text{such that} \begin{cases} \displaystyle\sum_{\substack{T \ni e \\ T \in \mathcal{T}_3(G)}} g(T) \leq 1, & \forall e \in E(G); \\ g(T) \geq 0, & \forall T \in \mathcal{T}_3(G). \end{cases}$$

Let

$\tau^*(n,3) := \min\{\tau^*(R) + \tau^*(B) : R$ and B are color classes in a 2-edge-coloring of $K_n\}$.

Let

$$\tau^*(3) := \lim_{n \to \infty} \frac{\tau^*(n,3)}{\binom{n}{2}}, \quad \tau(3) := \lim_{n \to \infty} \frac{\tau(n,3)}{\binom{n}{2}}.$$

The fact that these limits are well-defined follows from the monotonicity and boundedness of the corresponding functions, see, for example, [5].

Theorem 5.

$$\mathbf{W}(3) = \frac{2}{1 - \tau^*(3)}.$$

Using the result of Haxell and Rödl, see [4], implying that $\tau^*(n,3) = \tau(n,3)(1+o(1))$, we have the following.

Corollary 1.

$$\mathbf{W}(3) = \frac{2}{1-\tau(3)}.$$

In Section 2 we define related linear programs and prove the correspondence between those and the original problem of finding $\mathrm{wR}(n,k)$, we also prove the existence of the weighted Ramsey limit in that section. We prove Theorem 4 in Section 3. In Section 4, we treat the case $k = 3$ and prove Theorem 5. For common graph theory notation, see, for example, [9].

2 Defining the linear programs

We formulate several problems in terms of linear programs. See [8] for the terminology. We say that a k-vertex subgraph of an edge-colored K_n is a *mono-k-subgraph* if all its edges have the same color. Let $\mathcal{T}(c;n,k)$ be a set of mono-k-subgraphs of a coloring c of K_n. Next, we define $r(c;n,k)$, which calculates the maximal total sum of nonnegative real values assigned to the edges of a 2-colored K_n such that the sum of these values on the edges of each mono-k-subgraph is at most 1. Formally, it is defined as follows.

$$r(c;n,k) = \max \sum_{e\in E(K_n)} w(e) \tag{6}$$

$$\text{such that} \quad \begin{cases} \displaystyle\sum_{e\in E(T)} w(e) \le 1, & \forall T \in \mathcal{T}(c;n,k); \\ w(e) \ge 0, & \forall e \in E(K_n) \end{cases} \tag{7}$$

Let

$$r(n,k) = \max\{r(c;n,k) : c \text{ is a Red/Blue coloring of } K_n\}.$$

The following lemma will allow us to study $\mathrm{wR}(n,k)$ using the more convenient function $r(n,k)$.

Lemma 1. *For any integers, k and n, $3 \le k \le n$,*

$$\mathrm{wR}(n,k) = \frac{\binom{n}{2}}{r(n,k)}.$$

Proof. To show an upper bound, assume that $\mathrm{wR}(n,k) > \binom{n}{2}/r(n,k)$. Thus, for any Red/Blue coloring c of K_n and any weight assignment to its edges with total sum $\binom{n}{2}$, we have that there is a mono-k-subgraph with total weight on its edges at least q, $q > \binom{n}{2}/r(n,k)$.

Consider an arbitrary Red/Blue coloring c' of K_n and a weight assignment to its edges w' such that the total weight on any mono-k-subgraph is at most 1 and the total sum of the weights on the edges of K_n is r'. Construct a new weight function w'', $w''(e) = w'(e)\binom{n}{2}/r(n,k)$. Then, with respect to w'', any mono-k-subgraph in c' has weight at most $\binom{n}{2}/r(n,k)$. The total sum of the weights on the edges of K_n is $r'\binom{n}{2}/r(n,k) > \binom{n}{2}$. Thus, $r' > r(n,k)$, a contradiction to the definition of $r(n,k)$.

To show a lower bound, assume that $\mathrm{wR}(n,k) < \binom{n}{2}/r(n,k)$. This means that there is a Red/Blue coloring c of K_n and a weight assignment w to its edges such that each mono-k-subgraph has sum of weights on its edges at most q, $q < \binom{n}{2}/r(n,k)$. Consider a new weight assignment w', $w'(e) = w(e)/q$. Then the sum of weights w' in each mono-k-subgraph of c is at most 1. Moreover, the total sum of the weights is $\binom{n}{2}/q > r(n,k)$, a contradiction to the definition of $r(n,k)$. □

Proposition 1. *Let k, ℓ, n be integers, $3 \le k \le \ell \le n$. Then*

$$\mathrm{wR}(\ell,k) \le \mathrm{wR}(n,k) \le \binom{k}{2}.$$

Proof. Note that to prove the lower bound on $\mathrm{wR}(n,k)$, it is sufficient to prove that

$$r(n,k) \le r(\ell,k)\frac{\binom{n}{2}}{\binom{\ell}{2}}.$$

Consider a Red/Blue coloring c of K_n. Let w be a weight function giving an optimal solution of (6). By adding up the sums of weights on complete ℓ-vertex subgraphs, we have that

$$r(c;n,k) \le r(c;\ell,k)\frac{\binom{n}{\ell}}{\binom{n-2}{\ell-2}} \le r(\ell,k)\frac{\binom{n}{2}}{\binom{\ell}{2}} \le r(\ell,k)\frac{\binom{n}{2}}{\binom{\ell}{2}}.$$

The upper bound is obvious by assigning weight 1 to each edge of K_n. □

Now, since function $\mathrm{wR}(n,k)$ is monotone in n, and bounded, the weighted Ramsey limit is well-defined.

3 Proof of Theorem 4

We shall need the following definitions in this section. The classical Ramsey number, $R(i)$, is the smallest number of vertices in a complete graph such that any Red/Blue edge-coloring contains a monochromatic complete subgraph on i vertices. The Turán graph $T(n,i)$ is a complete i-partite graph on n vertices with parts of almost equal sizes (differing by at most one), its size is denoted $t(n,i)$. For a graph H, let $\mathrm{ex}(n,H)$ be the largest number of edges in an n-vertex graph which has no subgraph isomorphic to H. Turán's theorem states that $\mathrm{ex}(n,K_{i+1}) = t(n,i)$. For a complete graph on vertices $v_1, \dots, v_m$, edge-colored with a coloring c, we say that a colored complete m-partite graph with parts $V_1, \dots, V_m$ is a *balanced blow-up of* c, if the sizes of parts $V_1, \dots, V_m$ differ by at most 1 and the color of all edges between V_i and V_j is equal to $c(v_i, v_j)$, $1 \le i < j \le m$.

Finally, we shall use the following Turán-type implication of the degree form of Szemerédi's regularity lemma, see, for example [6].

Lemma 2. *For a fixed integer s, fixed $\epsilon > 0$, there is an $n_0 = n_0(s, \epsilon)$, such that for all n, $n \ge n_0$, the following holds: Let G be an n-vertex graph edge-colored with Red and Blue. If the number of edges in G is greater than $t(n, R(i)-1) + \epsilon n^2$, then G has a complete i-partite monochromatic subgraph with at least s vertices in each part.*

We will use Lemma 2 in Section 3.2 to find a lower bound on $\mathrm{wR}(n,k)$. First we show a construction which gives an upper bound on $\mathrm{wR}(n,k)$.

3.1 Upper bound on $\mathrm{wR}(n,k)$

We need to find an appropriate coloring for the edges of K_n and a weight assignment function providing a feasible solution to the linear program (6).

Let $k = 4$. Let the Red edges form a copy of $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ and let all other edges be Blue. Let each Red edge have weight 1/4 and let each Blue edge have weight 1/6. It is easy to see that this assignment satisfies constraints in (6), i.e., it is a feasible solution of that linear program. The sum of the weights on all edges is

$$r = \frac{5}{24}\binom{n}{2} + \frac{1}{24}\left\lfloor \frac{n}{2} \right\rfloor .$$

By Lemma 1,

$$\mathrm{wR}(n,4) \le \frac{\binom{n}{2}}{r} \le 4.8.$$

Now let $k \ge 5$ and $n \ge 5\lceil k/2 \rceil$. Let an edge-colored graph G on n vertices be a balanced blow-up of a 2-edge colored K_5 with no monochromatic triangles. Let G have parts $V_1, \dots, V_5$. Give the edges of G weight $\lfloor k^2/4 \rfloor^{-1}$.

Arbitrarily color the edges inside of V_i, for $i = 1, \dots, 5$. Give these edges weight 0. Since G has no monochromatic triangles, Turán's theorem implies that each mono-k-subgraph has at most $\lfloor k^2/4 \rfloor$ edges. Hence, this weight assignment gives a feasible solution to (6) with respect to constructed coloring. The total weight is $t(n,5)\lfloor k^2/4 \rfloor^{-1}$. Therefore, $\mathrm{wR}(n,k) \le \frac{\binom{n}{2}}{t(n,5)} \left\lfloor \frac{k^2}{4} \right\rfloor \le 1.25 \left\lfloor \frac{k^2}{4} \right\rfloor$.

3.2 Lower bound on $\mathrm{wR}(n,k)$

Consider a weight function w on the edges of K_n colored in Red and Blue with coloring c, such that for any mono-k-subgraph, the sum of weights on its edges is at most 1. We shall give the upper bound on the total weight on all edges in K_n by showing that one cannot have too many "heavy" edges. This will give an upper bound on $r(n,k)$ and, therefore, a lower bound on $\mathrm{wR}(n,k)$. Let $G(i)$ be a spanning graph of K_n with edges of weight strictly greater than i. Let $E(i)=E(G(i))$, $\overline{E(i)}=E(K_n)\setminus E(i)$ for all i. Then we have, for any integers $i_1\le i_2\le\cdots\le i_m$ that

$$E(K_n)=E(1/i_1)\cup\big(E(1/i_2)\setminus E(1/i_1)\big)\cup\cdots\cup\big(E(1/i_m)\setminus E(1/i_{m-1})\big)\cup\overline{E(1/i_m)}.$$

We shall consider such a partition of the edge set of K_n such that each i_j corresponds to a Turán number; i.e., $i_1=t(k,2)$, $i_2=t(k,3)$, etc.

Claim 1. $|E(1/t(k,2))|=o(n^2)$.

Indeed, let G be a monochromatic subgraph of $G(1/t(k,2))$. G has no subgraph isomorphic to $K_{\lfloor k/2\rfloor,\lceil k/2\rceil}$ since otherwise this subgraph will have weight greater than 1. Therefore, using the fact that $\mathrm{ex}(n;K_{\lceil k/2\rceil,\lfloor k/2\rfloor})\le cn^{2-2/k}$ (see, for example, Chapter 6 in [1]) we have the desired result.

Claim 2. $|E(1/t(k,i))|\le(1+o(1))\left(1-\frac{1}{R(i)-1}\right)\binom{n}{2}$, for all $i\ge 3$.

Assume the opposite, then Lemma 2 implies that $G(1/t(k,i))$ has a monochromatic complete i-partite subgraph with at least k vertices in each part. Thus, $G(1/t(k,i))$ has a monochromatic copy, T, of $T(k,i)$. Since the weight of each edge in T is greater than $1/t(k,i)$, the total weight on this subgraph is greater than 1, a contradiction. This proves Claim 2.

Now, we are ready to write down the expression of the total weight on edges of K_n giving an upper bound on $r(n,k)$. Since each edge has weight at most 1,

$$\begin{aligned}
r(n,k) &\le |E(1/t(k,2))|\cdot 1+\sum_{i=2}^{k-1}\left(|E(1/t(k,i+1))|-|E(1/t(k,i))|\right)\frac{1}{t(k,i)}\\
&\quad+\left(\binom{n}{2}-|E(1/t(k,k))|\right)\frac{1}{t(k,k)}\\
&=\left(1-\frac{1}{t(k,2)}\right)|E(1/t(k,2))|+\sum_{i=3}^{k}\left(\frac{1}{t(k,i-1)}-\frac{1}{t(k,i)}\right)|E(1/t(k,i))|\\
&\qquad+\frac{1}{t(k,k)}\binom{n}{2}\\
&\le o(n^2)+(1+o(1))\sum_{i=3}^{k}\left(\frac{1}{t(k,i-1)}-\frac{1}{t(k,i)}\right)\left(1-\frac{1}{R(i)-1}\right)\binom{n}{2}\\
&\qquad+\frac{1}{t(k,k)}\binom{n}{2}\\
&=o(n^2)+\left(\frac{1}{t(k,2)}-\sum_{i=3}^{k}\frac{1}{R(i)-1}\left[\frac{1}{t(k,i-1)}-\frac{1}{t(k,i)}\right]\right)\binom{n}{2}\\
&=(1+o(1))\frac{\binom{n}{2}}{t(k,2)}\left(1-\sum_{i=3}^{k}\frac{1}{R(i)-1}\left[\frac{t(k,2)}{t(k,i-1)}-\frac{t(k,2)}{t(k,i)}\right]\right).
\end{aligned}\tag{8}$$

Thus, for $j \in \{3, \dots, k\}$, the following holds

$$r(n,k) \leq (1+o(1)) \frac{\binom{n}{2}}{t(k,2)} \left(1 - \sum_{i=3}^{j} \frac{1}{R(i)-1} \left[\frac{t(k,2)}{t(k,i-1)} - \frac{t(k,2)}{t(k,i)}\right]\right). \qquad (9)$$

Let us denote the terms used in summation as follows.

$$\alpha(k,i) \stackrel{\text{def}}{=} \frac{t(k,2)}{t(k,i-1)} - \frac{t(k,2)}{t(k,i)}.$$

Let $UR(i)$ be an upper bound on $R(i)-1$. For $j = \min\{k, 8\}$, denote the expression given in parentheses in (9), divided by $t(k,2)$, by $c(k)$. That is,

$$c(k) \stackrel{\text{def}}{=} \frac{1}{t(k,2)} \left(1 - \sum_{i=3}^{\min\{k,8\}} \frac{1}{UR(i)} \alpha(k,i)\right).$$

Then, we have that for any $j \leq k$,

$$r(n,k) \leq (1+o(1)) \binom{n}{2} c(k). \qquad (10)$$

We use the values of $UR(i)$, $i = 3, \dots, 8$, provided by Appendix 4 and the values of $\alpha(k,i)$ provided by Appendix 4. For $k = 3, \dots, 8$, the values of $\alpha(k,i)$ are found by looking them up in a table. There is a general lower bound for $\alpha(k,i)$ when $k \geq 9$, which gives a better result for larger k. We summarize the upper bounds on $c(k)$ in the following table.

k	4	5	6	7	8	≥ 9	large enough
$c(k)$	0.2381	$\frac{0.9438}{t(5,2)}$	$\frac{0.9454}{t(6,2)}$	$\frac{0.9442}{t(7,2)}$	$\frac{0.9461}{t(8,2)}$	$\frac{0.95143}{t(k,2)}$	$\frac{0.9441}{t(k,2)}$

Using the exact values on Turán numbers, in Appendix 4, and the fact that $\text{wR}(n,k) \geq \frac{1}{c(k)}(1+o(1))$, we conclude the proof of the lower bound of $\text{wR}(n,k)$ for $k \geq 4$.

4 Equivalence of fractional packing and constraint weigh assignment in graphs with respect to 3-vertex subgraphs

Let G be a graph. We define $\tau(G)$ to be the triangle packing number; i.e., the size of the largest edge-disjoint family of triangles in G. Its fractional relaxation is $\tau^*(G)$, as defined in the introduction. Let $\mathcal{T}(G)$, $\tilde{\mathcal{T}}(G)$, $\mathcal{T}_3(G)$, be the sets of: all induced 3-vertex subgraphs of G, all 3-vertex subgraphs of G, all complete 3-vertex subgraphs of G, respectively. Observe that $\mathcal{T}_3(G) \subseteq \mathcal{T}(G) \subseteq \tilde{\mathcal{T}}(G)$.

In order to establish the equivalence we want, we need to define the following graph invariants:

$$r(G) = \min \sum_{T \in \mathcal{T}(G)} t(T) \qquad (11)$$
$$\text{such that} \begin{cases} \sum_{\substack{T \ni e \\ T \in \mathcal{T}(G)}} t(T) \geq 1, & \forall e \in E(G); \\ t(T) \geq 0, & \forall T \in \mathcal{T}(G) \end{cases}$$

$$\tilde{r}(G) = \min \sum_{T \in \tilde{\mathcal{T}}(G)} \tilde{t}(T) \qquad (12)$$
$$\text{such that} \begin{cases} \sum_{\substack{T \ni e \\ T \in \tilde{\mathcal{T}}(G)}} \tilde{t}(T) = 1, & \forall e \in E(G); \\ \tilde{t}(T) \geq 0, & \forall T \in \tilde{\mathcal{T}}(G) \end{cases}$$

We prove the following in Appendix 4.

Lemma 3. *Let G be a graph, then $r(G) = \tilde{r}(G)$.*

Lemma 4. *Let G be a graph on $n \geq 3$ vertices with $e(G)$ edges and fractional triangle packing number $\tau^*(G)$. Then,*

$$\frac{1}{2}e(G) - \frac{1}{2}\tau^*(G) \leq r(G) \leq \frac{1}{2}e(G) - \frac{1}{2}\tau^*(G) + \left\lfloor \frac{n}{2} \right\rfloor .$$

Proof. Let $\tilde{t}^*$ be an optimal solution of (12). We shall construct a feasible solution of (5), giving a lower bound on τ^*. Let $g(T) = \tilde{t}^*(T)$ if $T \in \mathcal{T}_3(G)$ and let $g(T) = 0$ otherwise. Observe first that

$$e(G) = \sum_{e \in E(G)} \sum_{\substack{T \ni e \\ T \in \tilde{\mathcal{T}}(G)}} \tilde{t}^*(T)$$

Note that each member of $\mathcal{T}_3(G)$ appears in three sums of the form $\sum_{T \ni e, T \in \mathcal{T}_3(G)} \tilde{t}^*(T)$. In addition, each member of $\tilde{\mathcal{T}}(G) \setminus \mathcal{T}_3(G)$ appears in at most two of the sums of the form $\sum_{T\ni e, T\in\tilde{\mathcal{T}}(G)\setminus\mathcal{T}_3(G)}\tilde{t}^*(T)$. Thus,

$$\begin{aligned}
e(G) &= \sum_{e\in E(G)}\sum_{\substack{T\ni e\\ T\in\tilde{\mathcal{T}}(G)}}\tilde{t}^*(T)\\
&= \sum_{e\in E(G)}\sum_{\substack{T\ni e\\ T\in\mathcal{T}_3(G)}}\tilde{t}^*(T)+\sum_{e\in E(G)}\sum_{\substack{T\ni e\\ T\in\tilde{\mathcal{T}}(G)\setminus\mathcal{T}_3(G)}}\tilde{t}^*(T)\\
&= \sum_{e\in E(G)}\sum_{\substack{T\ni e\\ T\in\mathcal{T}_3(G)}}g(T)+\sum_{e\in E(G)}\sum_{\substack{T\ni e\\ T\in\tilde{\mathcal{T}}(G)\setminus\mathcal{T}_3(G)}}\tilde{t}^*(T)\\
&\leq 3\sum_{T\in\mathcal{T}_3(G)}g(T)+2\sum_{T\in\tilde{\mathcal{T}}(G)}\left(\tilde{t}^*(T)-g(T)\right)\\
&\leq \sum_{T\in\mathcal{T}_3(G)}g(T)+2\sum_{T\in\tilde{\mathcal{T}}(G)}\tilde{t}^*(T)\\
&\leq \tau^*(G)+2r(G)
\end{aligned}$$

As a result, $r(G)\geq\frac{1}{2}e(G)-\frac{1}{2}\tau^*(G)$.

For the other direction, let g^* be an optimal solution of (5). We shall construct $\tilde{t}$, a feasible solution of (12), from g^* via the following algorithm. For a weight function, ν, defined on $\tilde{\mathcal{T}}(G)$, we define the *deficiency of an edge e with respect to ν*, to be $\text{def}(\nu,e)=1-\sum_{T\ni e,T\in\tilde{\mathcal{T}}(G)}\nu(T)$. We say that an edge is *underweight* with respect to ν if $\text{def}(\nu,e)>0$.

Initialization. Let

$$\tilde{t}(T)=\tilde{t}_0(T)=\begin{cases}g^*(T), & T\in\mathcal{T}_3(G);\\ 0, & \text{otherwise.}\end{cases}\tag{13}$$

Iteration. Let U be a set of underweight edges with respect to $\tilde{t}$. Since g^* is optimal, the edges in U do not have triangles. Let

$$U=\left(\{e_1,e_1'\}\cup\cdots\cup\{e_m,e_m'\}\right)\cup\left(\{e_{m+1},\ldots,e_u\}\right),$$

such that $e_1,e_1',\ldots,e_m,e_m',e_{m+1},\ldots,e_u$ are distinct edges; e_i,e_i' are adjacent, $i=1,\ldots,m$, and m is as large as possible. Let $T_i\in\tilde{\mathcal{T}}(G)$ be a subgraph with two edges e_i,e_i', and assume also that $\text{def}(\tilde{t},e_i)<\text{def}(\tilde{t},e_i')$, for $i=1,\ldots,m$. Let

$$\tilde{t}'(T)=\begin{cases}\text{def}(\tilde{t},e_i), & \text{if } T=T_i;\\ \tilde{t}(T), & \text{otherwise.}\end{cases}$$

Note that

$$\sum_{T\in\tilde{\mathcal{T}}(G)}\tilde{t}'(T)=\sum_{T\in\tilde{\mathcal{T}}(G)}\tilde{t}(T)+\sum_{i=1}^{m}\text{def}(\tilde{t},e_i).$$

Moreover, $\mathrm{def}(\tilde{t}', e_i) = 0$, and $\mathrm{def}(\tilde{t}', e_i') = \mathrm{def}(\tilde{t}, e_i') - \mathrm{def}(\tilde{t}, e_i)$, for $i = 1, \ldots, q$. For all other edges, the deficiencies are not changed. Let $\tilde{t}(T) = \tilde{t}'(T)$, $T \in \tilde{\mathcal{T}}$.

Termination. Stop when the set of edges that are underweight, with respect to $\tilde{t}$, is a matching, $\{e_1, \ldots, e_q\}$. Note that $q \leq \lfloor n/2 \rfloor$. Let $T_i \in \tilde{\mathcal{T}}(G)$ be a graph formed by a single edge e_i and a single vertex v, $i = 1, \ldots, q$. Let $\mathcal{T}_1 = \{T_1, \ldots, T_q\}$. Let $\tilde{t}(T_i) := \mathrm{def}(e_i) \leq 1$, for $i = 1, \ldots, q$, let $\mathcal{T}_2(G)$ be the set of three-vertex, 2-edge-subgraphs of G.

We have that

$$\sum_{T \in \tilde{\mathcal{T}}(G)} \tilde{t}(T) \leq \sum_{T \in \mathcal{T}_3(G)} g^*(T) + \sum_{T \in \mathcal{T}_2(G)} \tilde{t}(T) + \sum_{T \in \mathcal{T}_1(G)} \tilde{t}(T).$$

Note that, for a fixed edge e of G,

$$\sum_{\substack{T \in \mathcal{T}_2(G) \\ e \in E(T)}} \tilde{t}(T) \leq \mathrm{def}(\tilde{t}_0, e).$$

We also have, since each $T \in \mathcal{T}_2(G)$ contains exactly two edges, that

$$\begin{aligned} 2 \sum_{T \in \mathcal{T}_2(G)} \tilde{t}(T) &= \sum_{T \in \mathcal{T}_2(G)} \sum_{e \in E(T)} \tilde{t}(T) = \sum_{e \in E(G)} \sum_{\substack{T \ni e \\ T \in \mathcal{T}_2}} \tilde{t}(T) \\ &\leq \sum_{e \in E(G)} \mathrm{def}(\tilde{t}_0, e) = \sum_{e \in E(G)} \left[1 - \sum_{\substack{T \ni e \\ T \in \mathcal{T}_3(G)}} g^*(T) \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{T \in \tilde{\mathcal{T}}(G)} \tilde{t}(T) &\leq \sum_{T \in \mathcal{T}_3(G)} \tilde{t}(T) + \sum_{T \in \mathcal{T}_2(G)} \tilde{t}(T) + \sum_{T \in \mathcal{T}_1(G)} \tilde{t}(T) \\ &\leq \sum_{T \in \mathcal{T}_3(G)} g^*(T) + \frac{1}{2} \sum_{e \in E(G)} \left[1 - \sum_{\substack{T \ni e \\ T \in \mathcal{T}_3(G)}} g^*(T) \right] + \sum_{T \in \mathcal{T}_1(G)} 1 \\ &\leq \frac{1}{2} e(G) - \frac{1}{2} \sum_{T \in \mathcal{T}_3(G)} g^*(T) + \left\lfloor \frac{n}{2} \right\rfloor \\ &= \frac{1}{2} e(G) - \frac{1}{2} \tau^*(G) + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

Recall from (11) that r computes a minimum. Therefore

$$r(G) \leq \frac{1}{2} e(G) - \frac{1}{2} \tau^*(G) + \left\lfloor \frac{n}{2} \right\rfloor.$$

□

Appendix

A Proof of Lemma 3

A.1 $r(G) \leq \tilde{r}(G)$

Let $\tilde{t}$ be a feasible solution of (12). For each $T \in \mathcal{T}(G)$, let $t(T) = \sum_{S \subseteq T, S \in \tilde{\mathcal{T}}(G)} \tilde{t}(S)$. This ensures that, for all $e \in E(G)$,

$$\sum_{T \ni e, T \in \mathcal{T}(G)} t(T) \geq 1.$$

Since any $S \in \tilde{\mathcal{T}}(G)$ is in a unique $T \in \mathcal{T}(G)$, we have

$$\sum_{T \in \mathcal{T}(G)} t(T) = \sum_{T \in \tilde{\mathcal{T}}(G)} \tilde{t}(T).$$

Since both linear programs compute a minimum, if $\tilde{t}^*$ is an optimal solution to (12) and t^* is the corresponding solution to (11) as computed above, then

$$r(G) \leq \sum_{T \in \mathcal{T}(G)} t^*(T) = \sum_{T \in \tilde{\mathcal{T}}(G)} \tilde{t}^*(T) = \tilde{r}(G).$$

A.2 $\tilde{r}(G) \leq r(G)$

Let t be a minimal feasible solution of (11). We shall create a feasible solution, $\tilde{t}$, of (12) by redistributing the weights on $\mathcal{T}(G)$ to $\tilde{\mathcal{T}}(G)$, such that the total weights on edges become equal to one. For any weight function ν on $\tilde{\mathcal{T}}(G)$, define the *excess of an edge e with respect to ν* to be $\mathrm{exc}(\nu, e) = \left(\sum_{S \ni e, S \in \tilde{\mathcal{T}}(G)} \nu(S)\right) - 1$. We call an edge e *overweight* with respect to ν if $\mathrm{exc}(\nu, e) > 0$. We define $\tilde{t}$ via the following algorithm.

Initialization. Let

$$\tilde{t}(T) = \begin{cases} t(T), & T \in \mathcal{T}(G); \\ 0, & \text{otherwise.} \end{cases}$$

Observe that the total weight is as follows:

$$\sum_{T \in \tilde{\mathcal{T}}(G)} \tilde{t}(T) = \sum_{T \in \mathcal{T}(G)} \tilde{t}(T) = \sum_{T \in \mathcal{T}(G)} t(T).$$

Iteration. Consider some $T \in \tilde{\mathcal{T}}(G)$ containing an overweight edge with respect to $\tilde{t}$ such that $\tilde{t}(T) > 0$. Note that as long as edges with positive excess exist, such a T exists as well.

If T contains e as its only overweight edge, then let

$$\begin{aligned}\tilde{t}'(T) &= \tilde{t}(T) - \min\left\{\tilde{t}(T), \mathrm{ex}(\tilde{t}, e)\right\},\\ \tilde{t}'(T \setminus e) &= \tilde{t}(T \setminus e) + \min\left\{\tilde{t}(T), \mathrm{exc}(\tilde{t}, e)\right\}.\end{aligned}$$

For all other $S \in \tilde{\mathcal{T}}(G)$, let $\tilde{t}'(S) = \tilde{t}(S)$.

Let T contain two overweight edges, e and e', such that $\mathrm{exc}(\tilde{t}, e') \leq \mathrm{exc}(\tilde{t}, e)$. If $\tilde{t}(T) > \mathrm{exc}(\tilde{t}, e)$, then let

$$\begin{aligned}\tilde{t}'(T) &= \tilde{t}(T) - \mathrm{exc}(\tilde{t}, e)\\ \tilde{t}'(T \setminus e) &= \tilde{t}(T \setminus e) + \mathrm{exc}(\tilde{t}, e) - \mathrm{exc}(\tilde{t}, e')\\ \tilde{t}'(T \setminus (e \cup e')) &= \tilde{t}(T \setminus (e \cup e')) + \mathrm{exc}(\tilde{t}, e').\end{aligned}$$

Otherwise (i.e, if $\tilde{t}(T) \leq \mathrm{exc}(\tilde{t}, e)$), let

$$\begin{aligned}\tilde{t}'(T) &= 0\\ \tilde{t}'(T \setminus e) &= \tilde{t}(T \setminus e)\\ \tilde{t}'(T \setminus (e \cup f)) &= \tilde{t}(T \setminus (e \cup f)) + \tilde{t}(T)\end{aligned}$$

For all other $S \in \tilde{\mathcal{T}}(G)$, let $\tilde{t}'(S) = \tilde{t}(S)$.

Finally, T cannot have three overweight edges because t was minimal and $\mathrm{exc}(\tilde{t}, e) \leq \mathrm{exc}(t, e)$ for all $e \in G$.

Clearly, the total weight does not change:

$$\sum_{T \in \tilde{\mathcal{T}}(G)} \tilde{t}'(T) = \sum_{T \in \tilde{\mathcal{T}}(G)} \tilde{t}(T).$$

Moreover, $0 \leq \mathrm{exc}(\tilde{t}', f) \leq \mathrm{exc}(\tilde{t}, f)$ for all $f \in E(G)$ and $\sum_{e \in E(G)} \mathrm{exc}(\tilde{t}', e) < \sum_{e \in E(G)} \mathrm{exc}(\tilde{t}, e)$.

Set $\tilde{t}(T) := \tilde{t}'(T)$ for all $T \in \tilde{\mathcal{T}}(G)$.

Termination. Stop if $\mathrm{exc}(\tilde{t}, e) = 0$ for all $e \in E(G)$.

To see that the process terminates, observe that at each iteration of this procedure, we either reduce the number of overweight edges or we both (1) reduce the sum $\sum_{e \in G} \mathrm{ex}(\tilde{t}, e)$ by at least $m(\tilde{t}) := \min\{\tilde{t}(T) : \tilde{t}(T) > 0\}$ and (2) ensure that each $\tilde{t}'(T)$ will either remain the same, be zero or increase by at least $m(\tilde{t})$. So, $m(\tilde{t}') \geq m(\tilde{t})$. Therefore, each iteration of the algorithm will decrease $\sum_{e \in G} \mathrm{ex}(\tilde{t}, e)$ by a fixed amount until the number of overweight edges decreases.

Concluding the proof. At the end of this procedure, we have a feasible solution $\tilde{t}_0$ of (12) such that $\sum_{T \in \tilde{\mathcal{T}}(G)} \tilde{t}_0(T) = \sum_{T \in \mathcal{T}(G)} t(T)$. Since both linear programs compute a minimum, if t^* is an optimal solution to (11) and $\tilde{t}_0^*$ is the corresponding solution to (12) as computed above, then

$$\tilde{r}(G) \leq \sum_{T \in \tilde{\mathcal{T}}(G)} \tilde{t}_0^*(T) = \sum_{T \in \mathcal{T}(G)} t^*(T) = r(G).$$

B Bounds on Ramsey numbers

i	3	4	5	6	7	8
$UR(i)$	5	17	48	164	539	1869

Figure 1: Known values for $UR(i)$, an upper bound on $R(i)-1$, from [7].

C Turán numbers

The Turán number $t(k,i)$, for $k \geq 3$ and $i = 2,\dots,k$, can be computed exactly to be

$$t(k,i) = \frac{k^2}{2}\left(\frac{i-1}{i}\right) - \frac{i}{2}\left(\left\lceil\frac{k}{i}\right\rceil - \frac{k}{i}\right)\left(\frac{k}{i} - \left\lfloor\frac{k}{i}\right\rfloor\right).$$

Clearly, $t(k,i) \leq \frac{k^2}{2}\left(\frac{i-1}{i}\right)$ and

$$t(k,i) \geq \frac{k^2}{2}\left(\frac{i-1}{i}\right) - \frac{1}{2i}\left\lfloor\frac{i^2}{4}\right\rfloor \geq \frac{k^2}{2}\left(\frac{i-1}{i}\right) - \frac{i}{8}.$$

Figure 2 gives the exact values for small Turán numbers.

		k					
	$t(k,i)$	3	4	5	6	7	8
	2	2	4	6	9	12	16
	3	3	5	8	12	16	21
	4		6	9	13	18	24
i	5			10	14	19	25
	6				15	20	26
	7					21	27
	8						28

Figure 2: Turán numbers, $t(k,i)$, $k \leq 8$.

The number $\alpha(k,i)$ is used in Section 3.2. Recall that for $3 \leq i \leq k$,

$$\alpha(k,i) = \frac{t(k,2)}{t(k,i-1)} - \frac{t(k,2)}{t(k,i)}.$$

Figure 3 gives exact values for $\alpha(k,i)$ for small values of k.

$\alpha(k,i)$		$k=3$	4	5	6	7	8
i	3	1/3	1/5	1/4	1/4	1/4	5/21
	4		2/15	1/12	3/52	1/12	2/21
	5			1/15	9/182	2/57	2/75
	6				3/70	3/95	8/325
	7					1/35	8/351
	8						4/189

Figure 3: Values of $\alpha(k,i)$, $k \leq 8$.

For $k \geq 9$, we determine a lower bound on $\alpha(k,i)$.

$$\begin{aligned}
\alpha(k,i) &\geq \frac{\left\lfloor \frac{k^2}{4} \right\rfloor}{\frac{k^2}{2}\left(\frac{i-2}{i-1}\right)} - \frac{\left\lfloor \frac{k^2}{4} \right\rfloor}{\frac{k^2}{2}\left(\frac{i-1}{i}\right) - \frac{i}{8}} \\
&= \frac{2}{k^2}\left\lfloor \frac{k^2}{4} \right\rfloor \left(\frac{i-1}{i-2} - \frac{i}{i-1} \frac{1}{1 - \frac{i^2}{4k^2(i-1)}} \right) \\
&\geq \frac{2}{k^2}\left\lfloor \frac{k^2}{4} \right\rfloor \left(\frac{i-1}{i-2} - \frac{i}{i-1} \left(1 + \frac{1}{4k-5} \right) \right) \\
&\geq \frac{2}{k^2}\left\lfloor \frac{k^2}{4} \right\rfloor \left(\frac{1}{(i-1)(i-2)} - \frac{i}{i-1} \frac{1}{4k-5} \right). \qquad (14)
\end{aligned}$$

Substituting (14) into (10), we obtain the following for $k \geq 9$:

$$\begin{aligned}
r(n,k) &\leq (1+o(1))\binom{n}{2}c(k) \\
&\leq (1+o(1))\frac{\binom{n}{2}}{t(k,2)}\left(1 - \sum_{i=3}^{8} \frac{1}{UR(i)}\alpha(n,k)\right) \\
&\leq (1+o(1))\frac{\binom{n}{2}}{t(k,2)}\left(1 - \sum_{i=3}^{8} \frac{1}{UR(i)} \frac{2}{k^2}\left\lfloor \frac{k^2}{4} \right\rfloor \frac{1}{(i-1)(i-2)} \right. \\
&\qquad \left. + \sum_{i=3}^{8} \frac{1}{UR(i)} \frac{2}{k^2}\left\lfloor \frac{k^2}{4} \right\rfloor \frac{i}{i-1}\frac{1}{4k-5}\right) \\
&\leq (1+o(1))\frac{\binom{n}{2}}{t(k,2)}\left(1 - \frac{2}{k^2}\left\lfloor \frac{k^2}{4} \right\rfloor 0.11191 + \frac{2}{k^2}\left\lfloor \frac{k^2}{4} \right\rfloor \frac{0.41457}{4k-5}\right) \\
&\leq (1+o(1))\frac{\binom{n}{2}}{t(k,2)}\left(.94405 + \frac{0.05596}{k^2} + \frac{0.20729}{4k-5}\right) \qquad (15)
\end{aligned}$$

The expression in (15) given in parentheses is bounded above by 0.9515 for all $k \geq 9$ and bounded above by 0.9441 for k large enough.

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