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On groupies in graphs

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Abstract

A non-isolated vertex of a graph is called a *groupie* if the average degree of the vertices adjacent to it is larger than or equal to the average degree of all vertices in the graph. An isolated vertex is a groupie only if the graph has no edges. Mackey (1996) proved that any graph with at least two vertices contains at least two groupies. In this note we show that if a graph has a vertex of degree at least two, then the graph contains two distinct groupies with a common neighbour.

1 Introduction

A non-isolated vertex of a graph G is called a *groupie* if the average degree of the vertices adjacent to it is large than or equal to the average degree of the vertices in G. An isolated vertex is a groupie only if all vertices of G are isolated.

The concept of groupie was first used by Ajtai, Komlós, and Szemerédi [1] to obtain an upper bound for the Ramsey number $R(3,k)$. Later, in [2], Bertram, Erdős, Horák, Širáň, and Tuza conjectured that there exist at least two groupies in any graph with at least two vertices and proved this in some special cases. This conjecture was completely settled by Mackey in [3].

In this note, using a technique similar to the one in [3], we show that if G has three or more vertices and has no isolated vertices, then G contains a pair of distinct groupies with a common neighbour. In particular, if G has two groupies at distance at least three, then G has at least three groupies.

2 Results

Let G be a graph with vertex set V. Let $\mathcal{G}$ be the set of groupies in G and let $g = |\mathcal{G}|$. For $v \in V$, let d_v be the degree of v and g_v be the number of groupies adjacent to v. Let $\overline{d}$ be the average degree of G, that is, $\overline{d} = \sum_{v\in V} d_v/|V|$. Let V^+ and V^- denote the sets of vertices with degree greater than $\overline{d}$ and less than $\overline{d}$ respectively. For $u, v \in V$, we denote by $d_G(u, v)$ the distance between u and v in G. Finally, for $u, v \in V$, let

$$\chi(u,v) = \begin{cases} 1, & \text{if } u \text{ is adjacent to } v; \\ 0, & \text{otherwise.} \end{cases}$$

Following a similar proof of the main theorem in [3], we have:

Theorem 2.1. *If G has a non-groupie of positive degree, then*

$$g > [\sum_{v\in V}(d_v - \overline{d})^2 + \sum_{v\in V^+}(g - g_v)(d_v - \overline{d})]/\sum_{v\in V^+}(d_v - \overline{d}).$$

Proof. Let $l = \sum_{v\in V}(d_v - \overline{d})^2$. Since $\sum_{v\in V}(d_v - \overline{d}) = 0$, we have

$$l = \sum_{v\in V} d_v(d_v - \overline{d}) - \overline{d}\sum_{v\in V}(d_v - \overline{d}) = \sum_{v\in V} d_v(d_v - \overline{d}). \tag{1}$$

Replacing d_v with $(d_v - g_v) + g_v$ in (1), we have

$$l = l_1 + l_2, \tag{2}$$

where $l_1 = \sum_{v\in V}(d_v - g_v)(d_v - \overline{d})$ and $l_2 = \sum_{v\in V} g_v(d_v - \overline{d})$.

For l_1, since $d_v - g_v$ is the number of non-groupies adjacent to v, we can write $d_v - g_v = \sum_{u\in V\backslash\mathcal{G}} \chi(u, v)$ which shows that

$$l_1 = \sum_{v\in V}\sum_{u\in V\backslash\mathcal{G}} \chi(u,v)(d_v - \overline{d}). \tag{3}$$

Now we interchange the summation in (3) to get

$$l_1 = \sum_{u\in V\backslash\mathcal{G}} (\sum_{v\in V} \chi(u,v)d_v - \overline{d}\sum_{v\in V}\chi(u,v)). \tag{4}$$

By the fact that $\sum_{v\in V} \chi(u, v)) = d_u$, (4) becomes

$$l_1 = \sum_{u\in V\backslash\mathcal{G}} (\sum_{v\in V} \chi(u,v)d_v - \overline{d}d_u). \tag{5}$$

Note also that $\sum_{v\in V} \chi(u, v)d_v$ is the sum of the degrees of the vertices adjacent to u. If $u \in V\backslash\mathcal{G}$ with positive degree, then $\sum_{v\in V} \chi(u, v)d_v < d_u\overline{d}$. Since we have assumed that G has non-groupies of positive degree, that is, there exists $u \in V \backslash \mathcal{G}$ with positive degree, (5) implies that

$$l_1 < 0. \tag{6}$$

For l_2, note that for any $v \in V^-$, $d_v \leq \overline{d}$, we have $g_v(d_v - \overline{d}) \leq 0$. Therefore,

$$l_2 \leq \sum_{v \in V^+} g_v(d_v - \overline{d}). \tag{7}$$

Replacing g_v by $(g_v - g) + g$ in (7), we obtain

$$l_2 \leq \sum_{v \in V^+} (g_v - g)(d_v - \overline{d}) + g \sum_{v \in V^+} (d_v - \overline{d}). \tag{8}$$

Combining (2), (6) and (8), we have

$$\sum_{v \in V} (d_v - \overline{d})^2 < \sum_{v \in V^+} (g_v - g)(d_v - \overline{d}) + g \sum_{v \in V^+} (d_v - \overline{d}). \tag{9}$$

Note that G cannot be regular, otherwise there does not exist a non-groupie of positive degree. Thus V^+ is non-empty and we may divide (9) by $\sum_{v \in V^+} (d_v - \overline{d})$ to obtain the result. □

The technique used to prove Corollary 2.1 is the same as for Corollary 1 in [3].

Corollary 2.1. *If G has a non-groupie of positive degree, then*

$$g > 1 + \frac{\sum_{v \in V^+} (g - g_v)(d_v - \overline{d})}{\sum_{v \in V^+} (d_v - \overline{d})}.$$

Proof. Let $t = \sum_{v \in V} (d_v - \overline{d})^2 / \sum_{v \in V^+} (d_v - \overline{d})$. Since $\sum_{v \in V^+} (d_v - \overline{d}) = \sum_{v \in V^-} (\overline{d} - d_v)$, we have

$$t = \frac{\sum_{v \in V^+} (d_v - \overline{d})^2}{\sum_{v \in V^+} (d_v - \overline{d})} + \frac{\sum_{v \in V^-} (d_v - \overline{d})^2}{\sum_{v \in V^-} (\overline{d} - d_v)}.$$

Applying the Cauchy inequality to the numerators of the right hand side yields

$$t \geq \frac{\sum_{v \in V^+} (d_v - \overline{d})}{|V^+|} + \frac{\sum_{v \in V^-} (\overline{d} - d_v)}{|V^-|},$$

which is

$$t \geq \left(\frac{\sum_{v \in V^+} d_v}{|V^+|} - \overline{d}\right) + \left(\overline{d} - \frac{\sum_{v \in V^-} d_v}{|V^-|}\right) = \frac{\sum_{v \in V^+} d_v}{|V^+|} - \frac{\sum_{v \in V^-} d_v}{|V^-|}.$$

Note that in the last expression the first term is greater than or equal to the least integer larger than $\overline{d}$ while the second term is less than or equal to the largest integer less than $\overline{d}$. This implies that $t \geq 1$. The result now follows by applying Theorem 2.1. □

Corollary 2.2. *If G has a non-groupie of positive degree, then there exist $u, v \in \mathcal{G}$ such that u and v are both adjacent to w for some vertex w in G.*

Proof. Suppose that it were not true, i.e. for any $v \in V$, the number of groupies adjacent to v were less than 2. This would imply that $g_v \leq 1$ for all $v \in V$. In particular,

$$g_v \leq 1 \text{ for all } v \in V^+. \tag{10}$$

Then by (10) and Corollary 2.1,

$$g > 1 + \frac{\sum_{v \in V^+}(g-1)(d_v - \overline{d})}{\sum_{v \in V^+}(d_v - \overline{d})} = g,$$

which is absurd. □

Corollary 2.3. *If G has a vertex of degree at least 2, then there exist distinct $u, v \in \mathcal{G}$ such that u and v are both adjacent to w for some vertex w in G.*

Proof. Since G has a vertex of positive degree, all the groupies must be of positive degree. Now if all the vertices in G of positive degree are groupies, then we are finished since the vertex of degree at least 2 is adjacent to two distinct groupies. If G has a non-groupie of positive degree, Corollary 2.3 follows from Corollary 2.2. □

Corollary 2.4. *If $|\mathcal{G}| = 2$, then one of the following cases occurs:*

1. *$G = 2K_1$;*
2. *$G = K_2 \cup mK_1$ where $m \geq 0$;*
3. *$d_G(u, v) \leq 2$ where $\mathcal{G} = \{u, v\}$.*

Proof. If $\mathcal{G}$ contains an isolated vertex, then all the vertices in G are isolated, that is, $G = tK_1$. Note that the number of groupies in tK_1 is t, which implies that $t = 2$ since we have assumed $|\mathcal{G}| = 2$. Therefore $G = 2K_1$ in this case.

Now suppose that $\mathcal{G}$ does not contain an isolated vertex. Then all the groupies must be of positive degree. We consider two cases:

If G does not have a vertex of degree at least 2, then G must be of the form $tK_2 \cup mK_1$ where $t \geq 1$ and $m \geq 0$. Note that the number of groupies in $tK_2 \cup mK_1$ is $2t$, which implies that $t = 1$ since we have assumed that $|\mathcal{G}| = 2$. Therefore, $G = K_2 \cup mK_1$ in this case.

If G has a vertex of degree at least 2, then by Corollary 2.3, $\mathcal{G} = \{u, v\}$, and u and v are both adjacent to w for some vertex w in G. This shows that $d_G(u, v) \leq 2$. □

From Corollary 2.4 we have:

Corollary 2.5. *Suppose G has 3 or more vertices. If there exist $u, v \in \mathcal{G}$ such that $d_G(u,v) > 2$, then we have $|\mathcal{G}| \geq 3$.*

Proof. We prove the result by contradiction, by assuming $|\mathcal{G}| < 3$. Since there exist $u, v \in \mathcal{G}$ such that $d_G(u,v) > 2$, we have $|\mathcal{G}| \geq 2$. Hence $|\mathcal{G}| = 2$. We can assume that $\mathcal{G} = \{u, v\}$. Then by Corollary 2.4, one of the following cases happens:

1. $G = 2K_1$;
2. $G = K_2 \cup mK_1$ where $m \geq 0$;
3. $d_G(u,v) \leq 2$ where $\mathcal{G} = \{u, v\}$.

Note that it is impossible for G to be $2K_1$ since we have assumed that G has three or more vertices. Note also that it is also impossible for cases 2 or 3 to happen, since in both cases $d_G(u,v) \leq 2$ for all $u, v \in \mathcal{G}$, which contradicts our assumption that $d_G(u,v) > 2$ for some $u, v \in \mathcal{G}$. □

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