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Randomly r-orthogonal $(0, f)$-factorizations of $(0, mf - m + 1)$-graphs

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Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let g, f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for every vertex x of $V(G)$. A (g, f)-factor of G is a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$ for every vertex x of $V(F)$; a (g, f)-factorization of G is a partition of $E(G)$ into edge-disjoint (g, f)-factors. Let $F = \{F_1, F_2, \ldots, F_m\}$ be a factorization of G and let H be a subgraph of G with mr edges. If F_i, $1 \leq i \leq m$, has exactly r edges in common with H, we say that F is r-orthogonal to H. In this paper it is proved that every $(0, mf - m + 1)$-graph has $(0, f)$-factorizations randomly r-orthogonal to any given subgraph with mr edges if $4r - 1 \leq f(x)$ for any $x \in V(G)$.

1 Introduction

In this paper we consider finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex x is denoted by $d_G(x)$. Let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for every vertex x of $V(G)$. Then a (g, f)-factor of G is a spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for every vertex x of $V(F)$. In particular, G is called a (g, f)-graph if G itself is a (g, f)-factor. A subgraph H of G is called an m-subgraph if H has m edges in total. A (g, f)-factorization $F = \{F_1, F_2, \ldots, F_m\}$ of a graph G is a partition of $E(G)$ into edge-disjoint (g, f)-factors $F_1, F_2, \ldots, F_m$. If $g(x) = a$ and $f(x) = b$, where a and b are non-negative integers, then a (g, f)-factorization of G is called an $[a, b]$-factorization of G. Let H be an mr-subgraph of a graph G. A (g, f)-factorization $F = \{F_1, F_2, \ldots, F_m\}$ is r-orthogonal

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to H if $|E(H) \cap E(F_i)| = r$ for $1 \le i \le m$. If for any partition $\{A_1, A_2, \ldots, A_m\}$ of $E(H)$ with $|A_i| = r$ there is a (g, f)-factorization $F = \{F_1, F_2, \ldots, F_m\}$ of G such that $A_i \subseteq E(F_i)$, $1 \le i \le m$, then we say that G has (g, f)-factorizations randomly r-orthogonal to H. Other definitions and terminologies can be found in [1].

Recently, Xu et al. [2] studied the connected factors in $K_{1,n}$-free graphs containing a (g, f)-factor. Kano [3] obtained some sufficient conditions for a graph to have $[a, b]$-factorizations. Liu [4,5] proved that every $(mg + m - 1, mf - m + 1)$-graph has a (g, f)-factorization orthogonal to a star or a matching. Lam [6] showed that every $(mg + m - 1, mf - m + 1)$-graph has a (g, f)-factorization orthogonal to km-subgraphs. Liu [7] showed that every bipartite $(mg + m - 1, mf - m + 1)$-graph has (g, f)-factorizations randomly k-orthogonal to any km-subgraph. Feng [8] proved that every $(0, mf - m + 1)$-graph has a $(0, f)$-factorization orthogonal to any given m-subgraph. Now we consider the r-orthogonal factorizations of graphs. The purpose of this paper is to prove that for any mr-subgraph H of an $(0, mf - m + 1)$-graph G, there exist $(0, f)$-factorizations of G which are randomly r-orthogonal to H, where $f(x) \ge 4r - 1$ for every $x \in V(G)$. We shall use a different technique from [4–8].

2 Preliminary results

Let S and T be two disjoint subsets of $V(G)$. We denote by $E_G(S, T)$ the set of edges with one end in S and the other in T, and by $e_G(S, T)$ the cardinality of $E_G(S, T)$. For $S \subset V(G)$ and $A \subset E(G)$, $G - S$ is a subgraph obtained from G by deleting the vertices in S together with the edges to which the vertices in S are incident, and $G - A$ is a subgraph obtained from G by deleting the edges in A, and $G[S]$ (respectively, $G[A]$) is a subgraph of G induced by S (respectively, A). For a subset X of $V(G)$, we write $f(X) = \sum_{x \in X} f(x)$ for any function f defined on $V(G)$, and define $f(\emptyset) = 0$. In particular, $d_G(X) = \sum_{x \in X} d_G(x)$.

Let g and f be two non-negative integer-valued functions defined on $V(G)$, and C a component (i.e. a maximal connected subgraph) of $G - (S \cup T)$. If there is a vertex $x \in V(C)$ such that $g(x) \ne f(x)$, we call C a neutral component; otherwise, $g(x) = f(x)$ for all $x \in V(C)$, in which case we call C an even or odd component according to whether $e_G(T, V(C)) + f(C)$ is even or odd. We denote by $h_G(S, T)$ the number of the odd components of $G - (S \cup T)$. In 1970 Lovász [9] used the symbol $\delta_G(S, T; g, f)$ to denote the number $d_{G-S}(T) - g(T) - h_G(S, T) + f(S)$, and found that $\delta_G(S, T; g, f) = d_{G-S}(T) - g(T) - h_G(S, T) + f(S) \ge 0$ is a necessary and sufficient condition for a graph G to have a (g, f)-factor.

Lemma 2.1 (Lovász [9]) *Let G be a graph, and g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for $x \in V(G)$. Then G has a (g, f)-factor if and only if*

$$\delta_G(S, T; g, f) \ge 0$$

for any two disjoint subsets S and T of $V(G)$.

Note that if $g(x) < f(x)$ for all $x \in V(G)$ then all components of $G - (S \cup T)$ are neutral. Hence for any two disjoint subsets S and T of $V(G)$, $h_G(S,T) = 0$ provided $g(x) < f(x)$ for all $x \in V(G)$. Thus in the following $\delta_G(S,T;g,f) = d_{G-S}(T) - g(T) + f(S)$ for any two disjoint subsets S and T of $V(G)$.

Let S and T be two disjoint subsets of $V(G)$, and E_1 and E_2 be two disjoint subsets of $E(G)$. Let $D = V(G) - (S \cup T)$, and

$$E(S) = \{xy \in E(G) : x, y \in S\}, \quad E(T) = \{xy \in E(G) : x, y \in T\},$$

$$E_1^{'} = E_1 \cap E(S), \quad E_1^{"} = E_1 \cap E_G(S, D),$$
$$E_2^{'} = E_2 \cap E(T), \quad E_2^{"} = E_2 \cap E_G(T, D),$$
$$r_S(E_1) = 2|E_1^{'}| + |E_1^{"}|, \quad r_T(E_2) = 2|E_2^{'}| + |E_2^{"}|.$$

It is easily seen that $r_S(E_1) \leq d_{G-T}(S)$, $r_T(E_2) \leq d_{G-S}(T)$.

The following lemma has been obtained independently by Yuan [10] and Li [11].

Lemma 2.2 (Yuan [10]; Li [11]) *Let G be a graph, and g and f be two non-negative integer-valued functions defined on $V(G)$ such that $0 \leq g(x) < f(x)$ for all $x \in V(G)$, and let E_1 and E_2 be two disjoint subsets of $E(G)$. Then G has a (g, f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$ if and only if*

$$\delta_G(S, T; g, f) \geq r_S(E_1) + r_T(E_2)$$

for any two disjoint subsets S and T of $V(G)$.

Lemma 2.3 (Feng [8]) *Let G be a $(0, mf - m + 1)$-graph. Let f be an integer-valued function defined on $V(G)$ such that $f(x) \geq 0$, and let H be an m-subgraph of G. Then G has a $(0, f)$-factorization orthogonal to H.*

In the following, we always assume that G is a $(0, mf - m + 1)$-graph, where $m \geq 1$ is an integer. Define

$$g(x) = \max\{0, d_G(x) - ((m-1)f(x) - (m-1) + 1)\},$$

$$\triangle_1(x) = \frac{1}{m} d_G(x) - g(x),$$

$$\triangle_2(x) = f(x) - \frac{1}{m} d_G(x).$$

From the definition of $g(x)$, $\triangle_1(x)$ and $\triangle_2(x)$, we have the following lemma.

Lemma 2.4 *For all $x \in V(G)$, the following inequalities hold:*
(1) *If $m \geq 2$, then $0 \leq g(x) < f(x)$.*
(2) *If $g(x) = d_G(x) - ((m-1)f(x) - (m-1) + 1)$, then $\triangle_1(x) \geq \frac{1}{m}$.*
(3) $\triangle_2(x) \geq \frac{m-1}{m}$.

Proof (1) Note that G is a $(0, mf-m+1)$-graph, where $m \geq 2$ is an integer. Then $0 \leq mf(x)-m+1$ implies that $f(x) \geq \frac{m-1}{m}$. Note that $f(x)$ is a non-negative integer-valued function. Thus $f(x) \geq 1$.

If $g(x)=0$, then $0 \leq g(x) < f(x)$.

If $g(x) = d_G(x) - ((m-1)f(x) - (m-1)+1)$, then

$$\begin{aligned} f(x) - g(x) &= f(x) - d_G(x) + (m-1)f(x) - (m-1) + 1 \\ &= mf(x) - m + 2 - d_G(x) \\ &\geq mf(x) - m + 2 - (mf(x) - m + 1) = 1. \end{aligned}$$

Hence we find

$$0 \leq g(x) < f(x).$$

(2) If $g(x) = d_G(x) - ((m-1)f(x) - (m-1)+1)$, then

$$\begin{aligned} \Delta_1(x) &= \frac{1}{m} d_G(x) - g(x) \\ &= \frac{1}{m} d_G(x) - [d_G(x) - ((m-1)f(x) - (m-1) + 1)] \\ &= \frac{1-m}{m} d_G(x) + (m-1)f(x) - (m-1) + 1 \\ &\geq \frac{1-m}{m}(mf(x) - m + 1) + (m-1)f(x) - (m-1) + 1 \\ &= (1-m)f(x) + (m-1) - \frac{m-1}{m} + (m-1)f(x) - (m-1) + 1 \\ &= \frac{1}{m}. \end{aligned}$$

(3) We have

$$\begin{aligned} \Delta_2(x) &= f(x) - \frac{1}{m} d_G(x) \geq f(x) - \frac{1}{m}(mf(x) - m + 1) \\ &= f(x) - f(x) + \frac{m-1}{m} = \frac{m-1}{m} \end{aligned}$$

This completes the proof.

Let S and T be two disjoint subsets of $V(G)$; then

$$S_0 = \{x \mid x \in S, f(x) = 1\}, \qquad S_1 = S \setminus S_0.$$

$$T_0 = \{x \mid x \in T, g(x) = 0\}, \qquad T_1 = T \setminus T_0.$$

Hence we get that

$$S = S_0 \cup S_1, \qquad S_0 \cap S_1 = \emptyset.$$

$$T = T_0 \cup T_1, \qquad T_0 \cap T_1 = \emptyset.$$

$$r_S(E_1) = r_{S_0}(E_1) + r_{S_1}(E_1), \qquad r_T(E_2) = r_{T_0}(E_2) + r_{T_1}(E_2).$$

Lemma 2.5 *Let E_1 and E_2 be two disjoint subsets of $E(G)$, let S and T be two disjoint subsets of $V(G)$, and let S_1 and S_2 be defined as in Section 2. If*

$$\delta_G(S_1, T_1; g, f) = d_{G-S_1}(T_1) - g(T_1) + f(S_1) \geq r_{S_1}(E_1) + r_{T_1}(E_2),$$

then

$$\delta_G(S, T; g, f) = d_{G-S}(T) - g(T) + f(S) \geq r_S(E_1) + r_T(E_2).$$

Proof Note that $d_{G-S}(T_0) - g(T_0) = d_{G-S}(T_0) \geq r_{T_0}(E_2)$, and $0 \leq d_G(x) \leq mf(x) - m + 1$, and for all $x \in S_0$, $d_G(x) = 0$ or $d_G(x) = 1$. Hence we get that

$$|S_0| \geq d_G(S_0) = d_{G-T}(S_0) + e_G(S_0, T) \geq r_{S_0}(E_1) + e_G(S_0, T_1).$$

If $\delta_G(S_1, T_1; g, f) \geq r_{S_1}(E_1) + r_{T_1}(E_2)$, then

$$\begin{aligned}
\delta_G(S, T; g, f) &= f(S) + d_{G-S}(T) - g(T) \\
&= f(S_1) + |S_0| + d_{G-S}(T_1) + d_{G-S}(T_0) - g(T_1) \\
&\geq f(S_1) + r_{S_0}(E_1) + e_G(S_0, T_1) + d_{G-S}(T_1) + r_{T_0}(E_2) - g(T_1) \\
&= f(S_1) + r_{S_0}(E_1) + d_{G-S_1}(T_1) + r_{T_0}(E_2) - g(T_1) \\
&= \delta_G(S_1, T_1; g, f) + r_{S_0}(E_1) + r_{T_0}(E_2) \\
&\geq r_{S_1}(E_1) + r_{T_1}(E_2) + r_{S_0}(E_1) + r_{T_0}(E_2) \\
&= r_S(E_1) + r_T(E_2),
\end{aligned}$$

completing the proof.

3 Main result and proof

In this section, we are going to prove our main theorem.

Theorem 1 *Let $m \geq 3$ and $r \geq 1$ be integers, and let G be a $(0, mf - m + 1)$-graph, and let f be an integer-valued function defined on $V(G)$ such that $4r - 1 \leq f(x)$, and let H be an mr-subgraph of G. Then G has $(0, f)$-factorizations randomly r-orthogonal to H.*

Proof According to Lemma 2.3, the theorem holds for $r = 1$. In the following, we consider $r \geq 2$. Let E_1 be an arbitrary subset of $E(H)$ with $|E_1| = r$. Put $E_2 = E(H) \backslash E_1$. Then $|E_2| = (m-1)r$. For any two disjoint subsets $S \subseteq V(G)$ and $T \subseteq V(G)$, let $g(x)$, $E_1^{'}$, $E_1^{"}$, $E_2^{'}$, $E_2^{"}$, $r_S(E_1)$, $r_T(E_2)$, S_0, S_1, T_0 and T_1 be defined as in Section 2. It follows instantly from the definitions of $r_S(E_1)$ and $r_T(E_2)$ that

$$r_{S_1}(E_1) \leq \min\{2r, r|S_1|\},$$

$$r_{T_1}(E_2) \leq \min\{2(m-1)r, (m-1)r|T_1|\}.$$

For S_1 and T_1, we find that

$$\begin{aligned}\delta_G(S_1,T_1;g,f) &= d_{G-S_1}(T_1)-g(T_1)+f(S_1)\\ &= \frac{1}{m}d_G(T_1)-g(T_1)+f(S_1)-\frac{1}{m}d_G(S_1)\\ &\quad+\frac{m-1}{m}d_{G-S_1}(T_1)+\frac{1}{m}d_{G-T_1}(S_1)\\ &= \Delta_1\ (T_1)+\ \Delta_2\ (S_1)+\frac{m-1}{m}d_{G-S_1}(T_1)+\frac{1}{m}d_{G-T_1}(S_1).\end{aligned}$$

By Lemma 2.4, we have

$$\begin{aligned}\delta_G(S_1,T_1;g,f) &= d_{G-S_1}(T_1)-g(T_1)+f(S_1)\\ &\geq \frac{1}{m}|T_1|+\frac{m-1}{m}|S_1|+\frac{m-1}{m}d_{G-S_1}(T_1)\\ &\quad+\frac{1}{m}d_{G-T_1}(S_1).\end{aligned}\tag{3.1}$$

Now we prove that the following inequality holds:

$$\delta_G(S_1,T_1;g,f)\geq r_{S_1}(E_1)+r_{T_1}(E_2).$$

Now let us distinguish among four cases.

Case 1. If $S_1=\emptyset$, $T_1=\emptyset$, then $r_{S_1}(E_1)=0$ and $r_{T_1}(E_2)=0$.

It is easily seen that

$$\delta_G(S_1,T_1;g,f)\geq 0=r_{S_1}(E_1)+r_{T_1}(E_2).$$

Case 2. If $S_1=\emptyset$, $T_1\neq\emptyset$, then $r_{S_1}(E_1)=0$.

By the definition of T_1, it is easy to see that $g(x)\geq 1$ for all $x\in T_1$.

Note that $g(x)=\max\{0,d_G(x)-((m-1)f(x)-(m-1)+1)\}$. For all $x\in T_1$, we have

$$g(x)=d_G(x)-((m-1)f(x)-(m-1)+1)\geq 1.$$

Thus, we get

$$\begin{aligned}d_G(x) &\geq (m-1)f(x)-(m-1)+2\\ &\geq (m-1)(4r-1)-(m-1)+2\\ &= 4mr-4r-2m+4\end{aligned}\tag{3.2}$$

for all $x\in T_1$.

By (3.1) and (3.2), we get that

$$\begin{aligned}
\delta_G(S_1, T_1; g, f) &\geq \frac{m-1}{m} d_G(T_1) \\
&\geq \frac{m-1}{m}(4mr - 2m - 4r + 4)|T_1| \\
&= (m-1)r|T_1| + \frac{m-1}{m}((3m-4)r - 2m + 4)|T_1| \\
&\geq (m-1)r|T_1| + \frac{m-1}{m}(6m - 8 - 2m + 4)|T_1| \\
&\geq (m-1)r|T_1| \geq r_{T_1}(E_2) = r_{S_1}(E_1) + r_{T_1}(E_2).
\end{aligned}$$

Case 3. If $S_1 \neq \emptyset$, $T_1 = \emptyset$, then $r_{T_1}(E_2) = 0$.
Thus, we have

$$\begin{aligned}
\delta_G(S_1, T_1; g, f) &= d_{G-S_1}(T_1) - g(T_1) + f(S_1) \\
&= f(S_1) \geq (4r-1)|S_1| \\
&\geq r|S_1| \geq r_{S_1}(E_1) = r_{S_1}(E_1) + r_{T_1}(E_2).
\end{aligned}$$

Case 4. $S_1 \neq \emptyset$, $T_1 \neq \emptyset$.
Note that $d_{G-T_1}(S_1) \geq r_{S_1}(E_1)$. In view of (3.1) and (3.2), we get that

$$\begin{aligned}
\delta_G(S_1, T_1; g, f) &\geq \frac{1}{m}|T_1| + \frac{m-1}{m}|S_1| + \frac{m-1}{m} d_{G-S_1}(T_1) + \frac{1}{m} d_{G-T_1}(S_1) \\
&= \frac{1}{m}|T_1| + \frac{m-1}{m}(d_{G-S_1}(T_1) + |S_1|) + \frac{1}{m} d_{G-T_1}(S_1) \\
&\geq \frac{1}{m}|T_1| + \frac{1}{m} d_{G-T_1}(S_1) + \frac{m-1}{m} d_G(x) \qquad (x \in T_1) \\
&\geq \frac{1}{m}|T_1| + \frac{1}{m} d_{G-T_1}(S_1) \\
&\quad + \frac{m-1}{m}(4mr - 4r - 2m + 4). \qquad (3.3)
\end{aligned}$$

Case 4.1. $|T_1| = 1$.
Thus we have $r_{T_1}(E_2) \leq \min\{2(m-1)r, (m-1)r|T_1|\} = (m-1)r$. By (3.3), we get that

$$\delta_G(S_1, T_1; g, f) \geq \frac{1}{m}|T_1| + \frac{1}{m} d_{G-T_1}(S_1) + \frac{m-1}{m}(4mr - 4r - 2m + 4)$$

$$\begin{aligned}
&= \frac{1}{m}|T_1| + \frac{1}{m}d_{G-T_1}(S_1) + (m-1)r \\
&\quad + \frac{(m-1)(3mr-4r-2m+4)}{m} \\
&= \frac{1}{m}d_{G-T_1}(S_1) + \frac{2r(m-1)}{m} + (m-1)r + \frac{1}{m}|T_1| \\
&\quad + \frac{(m-1)(3mr-6r-2m+4)}{m} \\
&\geq \frac{1}{m}r_{S_1}(E_1) + \frac{m-1}{m}r_{S_1}(E_1) + r_{T_1}(E_2) \\
&\quad + \frac{(m-1)((3m-6)r-2m+4)+2}{m} \\
&\geq r_{S_1}(E_1) + r_{T_1}(E_2) + \frac{(m-1)(2(3m-6)-2m+4)+2}{m} \\
&\geq r_{S_1}(E_1) + r_{T_1}(E_2) + \frac{(m-1)(4m-4)+2}{m} \\
&\geq r_{S_1}(E_1) + r_{T_1}(E_2).
\end{aligned}$$

Case 4.2. $|T_1| \geq 2$.

Thus we have $r_{T_1}(E_2) \leq \min\{2(m-1)r, (m-1)r|T_1|\} = 2(m-1)r$. By (3.3), we get that

$$\begin{aligned}
\delta_G(S_1, T_1; g, f) &\geq \frac{1}{m}|T_1| + \frac{1}{m}d_{G-T_1}(S_1) + \frac{m-1}{m}d_G(x) \qquad (x \in T_1) \\
&\geq \frac{2}{m} + \frac{1}{m}r_{S_1}(E_1) + \frac{(m-1)(4mr-4r-2m+4)}{m} \\
&= \frac{1}{m}r_{S_1}(E_1) + \frac{2r(m-1)}{m} + 2(m-1)r \\
&\quad + \frac{(m-1)(4mr-4r-2m+4)}{m} - \frac{2r(m-1)}{m} \\
&\quad -2(m-1)r + \frac{2}{m} \\
&\geq \frac{1}{m}r_{S_1}(E_1) + \frac{m-1}{m}r_{S_1}(E_1) + r_{T_1}(E_2) \\
&\quad + \frac{(m-1)(2mr-6r-2m+4)+2}{m} \\
&\geq r_{S_1}(E_1) + r_{T_1}(E_2) + \frac{(m-1)(2(2m-6)-2m+4)+2}{m} \\
&= r_{S_1}(E_1) + r_{T_1}(E_2) + \frac{(m-1)(2m-8)+2}{m} \\
&\geq r_{S_1}(E_1) + r_{T_1}(E_2) - \frac{2}{3} \qquad \text{(since } m \geq 3 \text{ is an integer)} \\
&> r_{S_1}(E_1) + r_{T_1}(E_2) - 1.
\end{aligned}$$

According to the integrity of $\delta_G(S_1, T_1; g, f)$, we get that

$$\delta_G(S_1, T_1; g, f) \geq r_{S_1}(E_1) + r_{T_1}(E_2).$$

For S_1 and T_1, we always have

$$\delta_G(S_1, T_1; g, f) \geq r_{S_1}(E_1) + r_{T_1}(E_2).$$

By Lemma 2.5, for any two disjoint subsets S and T of $V(G)$, we have

$$\delta_G(S, T; g, f) \geq r_S(E_1) + r_T(E_2).$$

In view of Lemma 2.2, G has a (g, f)-factor F_1 such that $E_1 \subseteq E(F_1)$ and $E_2 \cap E(F_1) = \emptyset$. By the definition of $g(x)$, clearly, F_1 is also a $(0, f)$-factor of G. Set $G' = G - E(F_1)$. By the definition of $g(x)$, we have

$$\begin{aligned} 0 \leq d_{G'}(x) &= d_G(x) - d_{F_1}(x) \leq d_G(x) - g(x) \\ &\leq (m-1)f(x) - (m-1) + 1. \end{aligned}$$

Hence G' is a $(0, (m-1)f - (m-1) + 1)$-graph. Let $H' = G[E_2]$. By the induction hypothesis, G' has $(0, f)$-factorizations randomly r-orthogonal to H'. Thus G has $(0, f)$-factorizations randomly r-orthogonal to H. This completes the proof.

Remark 3.1 *In the proof of Theorem 1, it is required that $f(x) \geq 4r - 1$ for all $x \in V(G)$. We do not know whether the condition can be improved.*

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