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A note on pandecomposable $(v, 4, 2)$-BIBDs with subsystems

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Abstract

Rees and Stinson (On combinatorial designs with subdesigns, *Discrete Math.* 77 (1989), 259–279) proved that the necessary conditions for the existence of a pandecomposable $(v, 4, 2)$-BIBD with as a subsystem a pandecomposable $(u, 4, 2)$-BIBD are also sufficient, leaving finite open pairs of (v, u) with $v - u < 822$ and $v, u \equiv 1 \pmod 6$. In this note, we give a complete solution to the spectrum of pandecomposable $(v, 4, 2)$-BIBDs with subsystems.

1 Introduction

Let v, n, λ be positive integers. A *balanced incomplete block design* of order v with block size n and index λ, denoted (v, n, λ)-BIBD, is an ordered pair $(X, \mathcal{B})$ where X is a v-set of points, and $\mathcal{B}$ is a collection of n-subsets of X called *blocks*, such that each pair of distinct points of X occurs together in exactly λ blocks of $\mathcal{B}$. A (v, n, λ)-BIBD $(X, \mathcal{B})$ can be represented graphically as follows. Each point in X is represented by a vertex, and each block $B = \{b_1, b_2, \cdots, b_n\}$ is represented by a complete graph K_n joining the vertices $b_1, b_2, \cdots, b_n$. Since each pair of distinct points occurs in exactly λ blocks in $\mathcal{B}$, each edge belongs to exactly λ K_n's. Therefore a

(v, n, λ)-BIBD is equivalent to a complete multigraph λK_v in which the edges have been partitioned into copies of K_n (corresponding to the blocks in $\mathcal{B}$), i.e., a (v, n, λ)-BIBD is equivalent to a decomposition of λK_v into K_n's.

Let $\mathcal{G} = \{G_1, G_2, \cdots, G_\lambda\}$ be a decomposition of K_n. A *complementary decomposition* $\lambda K_v \to \mathcal{G}$ is a decomposition $\mathcal{D}$ of the complete multigraph λK_v into K_n's (i.e. a (v, n, λ)-BIBD) with the property that for each $j = 1, 2, \cdots, \lambda$ the set $\{G_j \subseteq K_n : K_n \in \mathcal{D}\}$ is a decomposition of K_v (we will refer to $\mathcal{D}$ as the root); note that this necessarily means that each $G_j \in \mathcal{G}$ contains the same number (namely $n(n-1)/2\lambda$) of edges. It is obvious that the case $\lambda = 1$ corresponds to constructing $(v, n, 1)$-BIBDs.

When $\lambda > 1$ the best-known examples of these designs are the so-called nested Steiner triple systems. A Steiner triple system STS(v) is said to be *nested* if one can add a point to each triple in the system and so obtain a $(v, 4, 2)$-BIBD. It is easy to see that a nested STS(v) is equivalent to a complementary decomposition $2K_v \to \{K_{1,3}, K_{1,3}^c\}$. The spectrum of these designs was determined by Stinson [5].

Theorem 1.1 (Stinson [5]). *There exists a nested STS(v) if and only if $v \equiv 1 (mod\ 6)$.*

Two decompositions $\mathcal{G}_1 = \{G_1^1, G_2^1, \cdots, G_\lambda^1\}$ and $\mathcal{G}_2 = \{G_1^2, G_2^2, \cdots, G_\lambda^2\}$ of K_n are said to be *distinct* if for no permutation σ on $\{1, 2, \cdots, \lambda\}$ is it true that $G_i^1 \simeq G_{\sigma(i)}^2$ for all $i = 1, 2, \cdots, \lambda$. Then a (v, n, λ)-BIBD (viewed as a decomposition $\mathcal{D}$ of $\lambda K_v \to K_n$) is *pandecomposable* if, for every decomposition $\mathcal{G}$ of K_n (with λ graphs, each with the same number of edges), there exists a complementary decomposition $\lambda K_v \to \mathcal{G}$ with $\mathcal{D}$ as its root. Therefore a pandecomposable $(v, 4, 2)$-BIBD is a $(v, 4, 2)$-BIBD (viewed as a decomposition $\mathcal{D}$ of $2K_v \to K_4$) such that for $i = 1, 2$, there exists a complementary decomposition $2K_v \to \mathcal{G}_i$ with $\mathcal{D}$ as its root, where $\mathcal{G}_1 = \{K_{1,3}, K_{1,3}^c\}$ and $\mathcal{G}_2 = \{P_4, P_4^c\}$. For example the following design is a pandecomposable $(7, 4, 2)$-BIBD [4].

Points: $0, 1, 2, 3, 4, 5, 6$.

Blocks: $\{0, 4, 2, 1\}$,$\{1, 5, 3, 2\}$,$\{2, 6, 4, 3\}$,$\{3, 0, 5, 4\}$,$\{4, 1, 6, 5\}$,$\{5, 2, 0, 6\}$,$\{6, 3, 1, 0\}$.

Here each block $\{a, b, c, d\}$ associates the graphs $K_{1,3}$ and $K_{1,3}^c$ where $K_{1,3}$ has a on one side and b, c, d on the other, and also the graphs P_4 and P_4^c where P_4 is the path $abcd$ with three edges ab, bc, cd.

The spectrum of these designs was also determined by Granville et al. [2].

Theorem 1.2 (Granville, Moisiadis and Rees [2]). *There exists a pandecomposable $(v, 4, 2)$-BIBD if and only if $v \equiv 1 (mod\ 6)$.*

A subsystem in a complementary decomposition $\lambda K_v \to \mathcal{G}$ is just a complementary decomposition $\lambda K_u \to \mathcal{G}$ for some complete multisubgraph $\lambda K_u \subseteq \lambda K_v$. In particular, the root of the subsystem (a (u,k,λ)-BIBD) is a sub-BIBD of the root of the master system (a (v,k,λ)-BIBD). In this note, we are interested in determining the spectrum of pandecomposable $(v,4,2)$-BIBDs with subsystems. Since the root of a pandecomposable $(v,4,2)$-BIBD is a $(v,4,2)$-BIBD, we have the following necessary conditions for the existence of a pandecomposable $(v,4,2)$-BIBD with a subsystem.

Lemma 1.3 *The necessary conditions for the existence of a pandecomposable* $(v,4,2)$*-BIBD with as a subsystem a pandecomposable* $(u,4,2)$*-BIBD are* $v \geq 3u+4$ *and* $u, v \equiv 1 (mod\ 6)$.

Proof. A subsystem in a pandecomposable $(v,4,2)$-BIBD is a pandecomposable $(u,4,2)$-BIBD for some complete multi-subgraph $2K_u \subseteq 2K_v$. Since this yields a $(v,4,2)$-BIBD with as a subsystem a $(u,4,2)$-BIBD a necessary condition for existence is that $v \geq 3u+1$ [3]. By Lemma 1.2, $v, u \equiv 1(mod\ 6)$ is also necessary. So this implies that $v \geq 3u+4$. The proof is completed. □

Rees and Stinson [4] discussed the existence of these designs and obtained the following result.

Lemma 1.4 (Rees and Stinson [4]). *Let* $u, v \equiv 1(mod\ 6)$*,* $v \geq 3u+4$ *and* $v-u \geq 822$*. Then there exists a pandecomposable* $(v,4,2)$*-BIBD with as a subsystem a pandecomposable* $(u,4,2)$*-BIBD.*

Note that as a corollary to Lemma 1.4 a partial solution to the spectrum of subsystems in nested Steiner triple systems was also obtained (see Corollary 6.3 in [4]). Recently, Wang and Shen [6] solved completely the spectrum of subsystems in nested Steiner triple systems.

Theorem 1.5 (Wang and Shen [6]). *There exists a nested* $STS(v)$ *with as a subsystem a nested* $STS(u)$ *if and only if* $v \geq 3u+4$ *and* $u, v \equiv 1(mod\ 6)$.

However, the spectrum of subsystems in pandecomposable $(v,4,2)$-BIBDs has not been determined completely. The purpose of the present note is to give a complete solution to the existence problem for pandecomposable $(v,4,2)$-BIBDs with subsystems.

2 Related pandecomposable $(4,2)$-GDDs

In order to solve the existence problem for pandecomposable $(v,4,2)$-BIBDs with subsystems, we need the auxiliary design of pandecomposable group divisible designs.

Let K be a positive integer set. A *group divisible design* (GDD) with index λ is a triple $(X,\mathcal{H},\mathcal{B})$ where X is a set of points, $\mathcal{H}$ is a partition of X into subsets called *groups* or *holes*, and $\mathcal{B}$ is a collection of subsets of X called *blocks* such that any pair of distinct points from X occur together either in some group or in exactly λ blocks, but not both. A (K, λ)-GDD of type $h_1^{u_1} h_2^{u_2} \cdots h_s^{u_s}$ is a GDD with index λ in which every block has size from the set K and in which there are u_i groups of size $h_i, i = 1, 2, \cdots, s$. When $\lambda = 1$, we will write K-GDD instead of $(K, 1)$-GDD for brevity. A (v, K)-PBD is just a K-GDD of type 1^v. As with a BIBD, a $(\{n\}, \lambda)$-GDD of type $h_1 h_2 \cdots h_s$ is equivalent to a decomposition of the complete multigraph $\lambda K_{h_1, h_2, \cdots, h_s}$ into K_n's.

A pandecomposable (n, λ)-GDD of type $h_1 h_2 \cdots h_s$ is a $(\{n\}, \lambda)$-GDD of type $h_1 h_2 \cdots h_s$ (viewed as a decomposition $\mathcal{D}$ of $\lambda K_{h_1, h_2, \cdots, h_s} \to K_n$) such that, for every decomposition $\mathcal{G}$ of K_n (with λ graphs, each with the same number of edges), there exists a complementary decomposition $\lambda K_{h_1, h_2, \cdots, h_s} \to \mathcal{G}$ with $\mathcal{D}$ as its root.

For pandecomposable $(4, 2)$-GDDs, we have the following construction, which is a modification of Wilson's Fundamental Construction for GDD [1].

Lemma 2.1 (Weighting). *Let $(X, \mathcal{H}, \mathcal{B})$ be a GDD, and let $w : X \to Z^+ \cup \{0\}$ be a weight function on X. Suppose that for every block $B \in \mathcal{B}$ there exists a pandecomposable $(4, 2)$-GDD of type $\{w(x) : x \in B\}$. Then there exists a pandecomposable $(4, 2)$-GDD of type $\{\sum_{x \in H} w(x) : H \in \mathcal{H}\}$.*

As an immediate corollary to Lemma 2.1, we have

Lemma 2.2 *Suppose there exists a (v, K)-PBD, and for each $k \in K$ there exists a pandecomposable $(4, 2)$-GDD of type h^k. Then there exists a pandecomposable $(4, 2)$-GDD of type h^v.*

Proof. Give a weight h to each point of K-GDD of type 1^v (which is just a (v, K)-PBD) and apply Lemma 2.1, using pandecomposable $(4, 2)$-GDDs of type h^k as input designs. This gives the desired designs. □

To apply the above constructions, we need to find several essential pandecomposable GDDs. Let $D = (X, \mathcal{H}, \mathcal{B})$ be a pandecomposable $(4, 2)$-GDD. Let $S_{|X|}$ be the symmetric group on X and $\sigma \in S_{|X|}$ be a permutation. For each $B = \{b_1, b_2, b_3, b_4\} \in \mathcal{B}$, let $\sigma(B) = \{\sigma(b_1), \sigma(b_2), \sigma(b_3), \sigma(b_4)\}$ and $\sigma(\mathcal{B}) = \{\sigma(B) : B \in \mathcal{B}\}$. A permutation σ is called an automorphism of the design D if $\sigma(\mathcal{B}) = \mathcal{B}$. It is obvious that all automorphisms of D form a group (called an automorphism group of D). Let A be an automorphism group of D. We say that two blocks B_1,B_2 of D are in the same orbit if there an automorphism σ of A such that $\sigma(B_1) = B_2$. So the automorphism group A divides the blocks of D in disjoint orbits. If we choose one block from each orbit, the entire design D is determined and such a choice is called a *base*. In the following direct constructions, for each design, we only list the automorphism group and base blocks of the desired design.

Lemma 2.3 ([2]). *There exists a pandecomposable $(4, 2)$-GDD of type 2^4.*

Proof.

Points: $X = \{0, 1, \cdots, 7\}$.

Groups: $\{\{2i, 2i+1\} : 0 \leq i \leq 3\}$.

Automorphism group:$\langle(0)\rangle$.

Base blocks:

$$\begin{array}{llll} \{0,2,7,4\}, & \{1,3,6,5\}, & \{2,1,5,7\}, & \{3,0,4,6\}, \\ \{4,2,6,1\}, & \{5,3,7,0\}, & \{6,0,5,2\}, & \{7,1,4,3\}. \end{array}$$

□

Lemma 2.4 *There exists a pandecomposable* $(4, 2)$*-GDD of type* 2^7.

Proof.

Points: $X = \{0, 1, \cdots, 13\}$.

Groups: $\{\{2i, 2i+1\} : 0 \leq i \leq 6\}$.

Automorphism group: $\langle(0\ 1)(2\ 12\ 3\ 13)(4\ 9\ 5\ 8)(6\ 10\ 7\ 11)\rangle$.

Base blocks:

$$\begin{array}{llll} \{0,2,4,6\}, & \{2,1,4,12\}, & \{2,10,5,8\}, & \{4,7,13,9\}, \\ \{4,10,1,6\}, & \{6,2,9,1\}, & \{6,11,12,3\}. & \end{array}$$

□

Lemma 2.5 *There exists a pandecomposable* $(4, 2)$*-GDD of type* 2^{10}.

Proof.

Points: $X = \{0, 1, \cdots, 19\}$.

Groups: $\{\{i, i+10\} : 0 \leq i \leq 9\}$.

Automorphism group: $\langle(0\ 4\ 8\ 12\ 16)(1\ 5\ 9\ 13\ 17)(2\ 6\ 10\ 14\ 18)(3\ 7\ 11\ 15\ 19)\rangle$.

Base blocks:

$$\begin{array}{llll} \{0,1,2,3\}, & \{0,4,6,8\}, & \{0,5,9,18\}, & \{1,3,4,19\}, \\ \{1,14,9,6\}, & \{1,15,17,8\}, & \{2,8,5,13\}, & \{2,10,6,3\}, \\ \{2,15,7,1\}, & \{3,12,15,6\}, & \{3,16,4,17\}, & \{3,19,14,8\}. \end{array}$$

□

Lemma 2.6 *There exists a pandecomposable* $(4, 2)$*-GDD of type* 2^{19}.

Proof.

Points: $X = \{0, 1, \cdots, 37\}$.

Groups: $\{\{i, i+19\} : 0 \le i \le 18\}$.

Automorphism group: $\langle(0\ 2\ \cdots\ 36)(1\ 3\ \cdots\ 37)\rangle$.

Base blocks:

$$\begin{array}{llll} \{0,1,2,4\}, & \{0,3,5,9\}, & \{0,6,11,23\}, & \{0,7,25,30\}, \\ \{0,10,37,16\}, & \{0,12,20,24\}, & \{1,4,18,11\}, & \{1,7,23,37\}, \\ \{1,9,19,8\}, & \{1,10,26,13\}, & \{1,22,35,12\}, & \{1,24,6,15\}. \end{array}$$

□

Lemma 2.7 *There exists a pandecomposable* $(4, 2)$*-GDD of type* 6^6*.*

Proof.

Points: $X = \{0, 1, \cdots, 35\}$.

Groups: $\{\{i, i+5, \cdots, i+25\} : 0 \le i \le 4\} \cup \{\{30, 31, 32, 33, 34, 35\}\}$.

Automorphism group: $\langle(0\ 2\ \cdots\ 28)(1\ 3\ \cdots\ 29)(30\ 32\ 34)(31\ 33\ 35)\rangle$.

Base blocks:

$$\begin{array}{llll} \{0,2,6,14\}, & \{0,7,35,26\}, & \{0,9,33,22\}, & \{0,13,1,29\}, \\ \{0,18,32,27\}, & \{1,10,29,23\}, & \{1,14,8,5\}, & \{1,17,25,32\}, \\ \{1,20,13,33\}, & \{1,34,12,28\}, & \{30,0,1,2\}, & \{31,0,3,7\}. \end{array}$$

□

Now we may use Lemma 2.2 to get a certain class of pandecomposable $(4, 2)$-GDDs.

Lemma 2.8 *There exists a pandecomposable* $(4, 2)$*-GDD of type* 2^u *for* $u \equiv 1(mod\ 3)$ *and* $u \ge 4$*.*

Proof. From Lemmas 2.3-2.6, there are pandecomposable $(4, 2)$-GDDs of types $2^4, 2^7, 2^{10}$, 2^{19}. For $u \in \{m : m \ge 4, m \equiv 1(mod\ 3)\}$, it is known that there is a $(u, \{4, 7, 10, 19\})$-PBD [1, III.3, Table 3.17]. Then the conclusion follows from Lemma 2.2.

3 Conclusions

In this section, we shall give a complete solution to the existence problem for pandecomposable $(v, 4, 2)$-BIBDs with subsystems. Now we give our main construction. It is a variant of the Filling in Holes Construction in [4]. So, we state the following construction without proof.

Lemma 3.1 (Filling in Holes). *Suppose there exists a pandecomposable* $(4, 2)$*-GDD of type* $h_1 h_2 \cdots h_s$*, and for* $1 \le i \le s-1$ *there exists a pandecomposable* $(h_i+\varepsilon, 4, 2)$*-BIBD with as a subsystem a pandecomposable* $(\varepsilon, 4, 2)$*-BIBD. Suppose there exists a pandecomposable* $(h_s + \varepsilon, 4, 2)$*-BIBD. Then there exists a pandecomposable* $(v, 4, 2)$*-BIBD with as a subsystem a pandecomposable* $(u, 4, 2)$*-BIBD, where* $v = \sum_{1 \le i \le s} h_i + \varepsilon$ *and* $u = h_s + \varepsilon$.

We are now in a position to show the main result of this note.

Theorem 3.2 *There exists a pandecomposable* $(v, 4, 2)$*-BIBD with as a subsystem a pandecomposable* $(u, 4, 2)$*-BIBD if and only if* $v \ge 3u + 4$ *and* $u, v \equiv 1 (mod\ 6)$.

Proof. By Lemma 1.3, we need only to show the sufficiency. From [7], we have a $((v+1)/2, K_{1(3)} \cup \{((u+1)/2)^*\})$-PBD for $v, u \equiv 1(mod\ 6)$, $v \ge 3u+4$ and $(v, u) \ne (37, 7)$ where $K_{1(3)} = \{m : m \ge 4, m \equiv 1(mod\ 3)\}$. This PBD is equivalent to a $K_{1(3)}$-GDD with a group of size $(u-1)/2$ and the other group sizes $\equiv 0(mod\ 3)$. Give a weight 2 to each point of the GDD and apply Lemma 2.1, using pandecomposable $(4, 2)$-GDDs of type 2^m, $m \in K_{1(3)}$ from Lemma 2.8 as input designs. This gives a pandecomposable $(4, 2)$-GDD with a group of size $u - 1$ and the other group sizes $\equiv 0(mod\ 6)$. Applying Lemma 3.1 with $\varepsilon = 1$, we can get a pandecomposable $(v, 4, 2)$-BIBD with as a subsystem a pandecomposable $(u, 4, 2)$-BIBD which exists by Theorem 1.2. For $(v, u) = (37, 7)$, applying Lemma 3.1 with $\varepsilon = 1$ to a pandecomposable $(4, 2)$-GDD of type 6^6 from Lemma 2.7, we get a pandecomposable $(37, 4, 2)$-BIBD with as a subsystem a pandecomposable $(7, 4, 2)$-BIBD which exists by Theorem 1.2. This completes the proof. □

As an immediate corollary to Theorem 3.2, we can also obtain Theorem 1.5.

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References

[1] C.J. Colbourn and J.H. Dinitz (Editors), *The CRC handbook of combinatorial designs*, CRC Press, Boca Raton, FL, 1996.

[2] A. Granville, A. Moisiadis and R. Rees, On complementary decompositions of the complete graph, *Graphs and Combinatorics* 5 (1989), 57–61.

[3] R. Rees and C.A. Rodger, Subdesigns in complementary path decompositions and incomplete two-fold designs with block size four, *Ars Combin.* 35 (1993), 117–122.

[4] R. Rees and D.R. Stinson, On combinatorial designs with subdesigns, *Discrete Math.* 77 (1989), 259–279.

[5] D.R. Stinson, The sepectrum of nested Steiner triple systems, *Graphs and Combinatorics* 1 (1985), 189–191.

[6] J. Wang and H. Shen, Doyen-Wilson theorem for nested Steiner triple systems, *J. Combin. Des.* 12 (2004), 389–403.

[7] J. Wang and H. Shen, Existence of $(v, K_{1(3)} \cup \{w^*\})$-PBDs and its applications, preprint.

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