AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **36** (2006), Pages 27–38

Nonseparating vertices in tournaments with large minimum degree

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Abstract

Let T be a strongly connected tournament, and let $p \geq 1$ be an integer. We show that if $\deg_T^+(x) \geq p$ and $\deg_T^-(x) \geq p$ for all $x \in V(T)$, then T has at least $k = \min\{|V(T)|, 4p-2\}$ vertices $x_1, x_2, \ldots, x_k$ such that $T - x_i$ $(i = 1, 2, \ldots, k)$ is strongly connected. We also show that if $p \geq 2$, $|V(T)| \geq 4p$, and $\deg_T^+(x) \geq p$ and $\deg_T^-(x) \geq p$ for all $x \in V(T)$, then T has at least $4p-1$ vertices $x_1, x_2, \ldots, x_{4p-1}$ such that $T - x_i$ $(i = 1, 2, \ldots, 4p-1)$ is strongly connected. Further we show that if $p \geq 2$, $|V(T)| \geq 4p+1$, and $\deg_T^+(x) \geq p$ and $\deg_T^-(x) \geq p$ for all $x \in V(T)$, then T has at least $4p$ vertices $x_1, x_2, \ldots, x_{4p}$ such that $T - x_i$ $(i = 1, 2, \ldots, 4p)$ is strongly connected unless $p = 2$ and $|V(T)| = 4p + 1$.

1 Introduction

In this paper, we consider only simple digraphs, that is finite directed graphs without loops or multiple edges. Let $T = (V(T), E(T))$ be a tournament, i.e., a simple digraph such that for any $x, y \in V(T)$ with $x \neq y$, precisely one of xy and yx belongs to $E(T)$. For a subset X of $V(T)$, we let $\langle X \rangle = \langle X \rangle_T$ denote the tournament induced by X. For $v \in V(T)$, we let $T - v$ denote the tournament from T by deleting v; thus $T - v = \langle V(T) - \{v\} \rangle$. For disjoint subsets A and B of $V(T)$, we let $E(A, B)$ denote the set of edges joining A to B and let $e(A, B)$ denote the cardinality of $E(A, B)$. For $v \in V(T)$, we let $\deg_T^+(v) = e(\{v\}, T - \{v\})$ and $\deg_T^-(v) = e(T - \{v\}, \{v\})$. Let C be a cycle of T. For $v \in V(C)$, we denote by v^- and v^+ the predecessor and the successor of v on C, respectively, and we denote by v^+Cv the directed path from v^+ to v on C.

If T is a tournament such that any two vertices in $V(T)$ are connected by a directed path, then we say T is strongly connected.

The following theorem by J. W. Moon [2] is well-known.

Theorem A. *Each vertex of a strongly connected tournament T is contained in a cycle of length k, for $k = 3, 4, \ldots, |V(T)|$.*

By Theorem A, the following theorem is obtained immediately.

Theorem B (Lovász [1]). *Let T be a strongly connected tournament with $|V(T)| \geq 4$. Then T has two vertices x_1, x_2 such that $T - x_i$ $(i = 1, 2)$ is strongly connected.*

In addition, C. Thomassen [3] proved the following theorem:

Theorem C. *Let T be a strongly connected tournament. Set $n = |V(T)|$. Then T has three vertices x_1, x_2, x_3 such that $T - x_i$ $(i = 1, 2, 3)$ is strongly connected, unless T is isomorphic to Q_n, where Q_n is the tournament consisting of a path $v_1 v_2 \ldots v_n$ and all edges $v_i v_j$ such that $i > j + 1$.*

In this paper, we prove the following variants on Theorems B and C with large minimum degree:

Theorem 1. *Let $p \geq 2$ be an integer and set $k = min\{|V(T)|, 4p - 2\}$. Let T be a strongly connected tournament. Suppose that $\deg_T^+(x) \geq p$ and $\deg_T^-(x) \geq p$ for all $x \in V(T)$. Then T has k vertices x_1, x_2, $\ldots$, x_k such that $T - x_i$ $(i = 1, 2, \ldots, k)$ is strongly connected.*

Theorem 2. *Let $p \geq 2$ be an integer and set $k = 4p - 1$. Let T be a strongly connected tournament with $|V(T)| \geq 4p$. Suppose that $\deg_T^+(x) \geq p$ and $\deg_T^-(x) \geq p$ for all $x \in V(T)$. Then T has k vertices x_1, x_2, $\ldots$, x_k such that $T - x_i$ $(i = 1, 2, \ldots, k)$ is strongly connected.*

Theorem 3. *Let $p \geq 2$ be an integer and set $k = 4p$. Let T be a strongly connected tournament with $|V(T)| \geq 4p + 1$. Suppose that $\deg_T^+(x) \geq p$ and $\deg_T^-(x) \geq p$ for all $x \in V(T)$. In the case where $p = 2$, suppose further that $|V(T)| \geq 4p + 2$. Then T has k vertices x_1, x_2, $\ldots$, x_k such that $T - x_i$ $(i = 1, 2, \ldots, k)$ is strongly connected.*

By Theorems B and 1, we obtain the following corollary:

Corollary 4. *Let $p \geq 1$ be an integer and set $k = min\{|V(T)|, 4p - 2\}$. Let T be a strongly connected tournament with $|V(T)| \geq 4$. Suppose that $\deg_T^+(x) \geq p$ and $\deg_T^-(x) \geq p$ for all $x \in V(T)$. Then T has k vertices x_1, x_2, $\ldots$, x_k such that $T - x_i$ $(i = 1, 2, \ldots, k)$ is strongly connected.*

Theorems 1, 2 and 3 can be proved by Theorems 5, 6 and 7, respectively.

Theorem 5. *Let $p \geq 2$ be an integer. Let T be a strongly connected tournament. Suppose that $\deg_T^+(x) \geq p$ and $\deg_T^-(x) \geq p$ for all $x \in V(T)$. Then for every $X \subset V(T)$ such that $|X| \leq \min\{|V(T)| - 1, 4p - 3\}$, there exists a cycle C such that $X \subset V(C)$ and $|V(C)| = |V(T)| - 1$.*

Theorem 6. *Let $p \geq 2$ be an integer. Let T be a strongly connected tournament with $|V(T)| \geq 4p$. Suppose that $\deg_T^+(x) \geq p$ and $\deg_T^-(x) \geq p$ for all $x \in V(T)$. Then for every $X \subset V(T)$ such that $|X| \leq 4p - 2$, there exists a cycle C such that $X \subset V(C)$ and $|V(C)| = |V(T)| - 1$.*

Theorem 7. *Let $p \geq 2$ be an integer. Let T be a strongly connected tournament with $|V(T)| \geq 4p+1$. Suppose that $\deg_T^+(x) \geq p$ and $\deg_T^-(x) \geq p$ for all $x \in V(T)$. In the case where $p = 2$, suppose further that $|V(T)| \geq 4p+2$. Then for every $X \subset V(T)$ such that $|X| \leq 4p-1$, there exists a cycle C such that $X \subset V(C)$ and $|V(C)| = |V(T)| - 1$.*

In Section 2, we prove several preliminary results, and we prove Theorems 1, 2 and 3 by Theorems 5, 6 and 7, respectively. We prove Theorems 5, 6 and 7 in Section 3. In Section 4, we discuss the sharpness of the various conditions in theorems.

2 Preliminaries

First we prove the following lemma:

Lemma 1. *Let T be a strongly connected tournament, and let q be an integer such that $1 \leq q \leq |V(T)| - 1$. Suppose that for every $X \subset V(T)$ such that $|X| \leq q$, there exists a cycle C such that $X \subset V(C)$ and $|V(C)| = |V(T)| - 1$. Then T has $q+1$ vertices $x_1, x_2, \ldots, x_{q+1}$ such that $T - x_i$ $(i = 1, 2, \ldots, q+1)$ is strongly connected.*

Proof. We set $X_0 = \{x \in V(T) | T - x \text{ is strongly connected}\}$. By way of contradiction, we assume that $|X_0| \leq q$. By the assumption of this lemma, there exists a cycle C such that $X_0 \subset V(C)$ and $|V(C)| = |V(T)| - 1$. Write $V(T) - V(C) = \{v\}$, then $v \notin X_0$. On the other hand, since C is a cycle such that $V(C) = V(T - v)$, $T - v$ is strongly connected, thus $v \in X_0$. Thus we obtain a contradiction. □

By Lemma 1, Theorems 5, 6 and 7 imply Theorems 1, 2 and 3, respectively. We use the following two lemmas in the proof of Theorems 5, 6 and 7.

Lemma 2. *Let $p \geq 1$ be an integer, and let T be a tournament. Suppose that $\deg_T^+(x) \geq p$ for all $x \in V(T)$, or $\deg_T^-(x) \geq p$ for all $x \in V(T)$. Then $|V(T)| \geq 2p+1$.*

Proof. We set $n = |V(T)|$. By symmetry, we may assume that $\deg_T^+(x) \geq p$ for all $x \in V(T)$. Then $\sum_{x \in V(T)} \deg_T^+(x) \geq np$. On the other hand, $\sum_{x \in V(T)} \deg_T^+(x) = \frac{n(n-1)}{2}$. Therefore we obtain $\frac{n(n-1)}{2} \geq np$, and hence $n \geq 2p+1$. □

Lemma 3. *Let $p \geq 1$ be an integer, and let T be a tournament. Suppose that $\deg_T^+(x) \geq p$ for all $x \in V(T)$ and there exists a vertex $x \in V(T)$ such that $\deg_T^+(x) \geq p+1$, or $\deg_T^-(x) \geq p$ for all $x \in V(T)$ and there exists a vertex $x \in V(T)$ such that $\deg_T^-(x) \geq p+1$. Then $|V(T)| \geq 2p+2$.*

Proof. We set $n = |V(T)|$. By symmetry, we may assume that $\deg_T^+(x) \geq p$ for all $x \in V(T)$ and there exists a vertex $x \in V(T)$ such that $\deg_T^+(x) \geq p+1$. Then $\sum_{x \in V(T)} \deg_T^+(x) \geq np+1$. On the other hand, $\sum_{x \in V(T)} \deg_T^+(x) = \frac{n(n-1)}{2}$. Therefore we obtain $n \geq 2p+1+\frac{2}{n}$, and hence $n \geq 2p+2$. □

3 Proof of Theorems

Let $p \geq 2$ be an integer. Let T be a strongly connected tournament such that $\deg_T^+(x) \geq p$ and $\deg_T^-(x) \geq p$ for all $x \in V(T)$. Let X be a subset of $V(T)$ such that $|X| \leq \min\{|V(T)| - 1, 4p - 1\}$. Let C be a cycle or a vertex of T such that $V(T) - V(C) - X \neq \emptyset$. We assume that we have chosen C so that $|V(C) \cap X|$ is maximal, and so that $|V(C)|$ is maximal under the condition that $|V(C) \cap X|$ is maximal. Note that $V(C) \cap X \neq \emptyset$. We set

$$X^+ = \{v \in X - V(C) | E(\{v\}, V(C)) = \emptyset\};$$

$$X^- = \{v \in X - V(C) | E(V(C), \{v\}) = \emptyset\};$$

$$Y^+ = \{v \in V(T) - X - V(C) | E(\{v\}, V(C)) = \emptyset\}; \text{and}$$

$$Y^- = \{v \in V(T) - X - V(C) | E(V(C), \{v\}) = \emptyset\}.$$

Under this notation, we prove the following claims.

Claim 1. $X - V(C) = X^+ \cup X^-$.

Proof. Assume that there exists a vertex $v \in X - V(C)$ such that $E(\{v\}, V(C)) \neq \emptyset$ and $E(V(C), \{v\}) \neq \emptyset$. Then there exist consecutive vertices v_1 and v_2 on C such that $v_1v \in E(T)$ and $vv_2 \in E(T)$. Hence there exists a cycle $C' = v_1vv_2Cv_1$ such that $V(T) - V(C') - X \neq \emptyset$, which contradicts the maximality of $|V(C) \cap X|$. □

Claim 2. $E(X^+, X^-) = \emptyset$.

Proof. Assume that $E(X^+, X^-) \neq \emptyset$. Let $u \in X^+$ and $v \in X^-$ be vertices such that $uv \in E(X^+, X^-)$. Let v_1 and v_2 be consecutive vertices on C, then $v_1u \in E(T)$ and $vv_2 \in E(T)$. Hence there exists a cycle $C' = v_1uvv_2Cv_1$ such that $V(T) - V(C') - X \neq \emptyset$, which contradicts the maximality of $|V(C) \cap X|$. □

Claim 3. $|V(T) - V(C) - X| \leq 2$. *Suppose that* $|V(T) - V(C) - X| = 2$, *then* $|Y^+| = |Y^-| = 1$.

Proof. First show that if $|V(T) - V(C) - X| \geq 2$, then $V(T) - V(C) - X = Y^+ \cup Y^-$. Assume that $V(T) - V(C) - X - Y^+ \cup Y^- \neq \emptyset$. Let $v \in V(T) - V(C) - X - Y^+ \cup Y^-$. Then there exist consecutive vertices v_1 and v_2 on C such that $v_1v \in E(T)$ and $vv_2 \in E(T)$. Hence there exists a cycle $C' = v_1vv_2Cv_1$ such that $V(T) - V(C') - X \neq \emptyset$, which contradicts the maximality of $|V(C)|$. Here we obtain if $|V(T) - V(C) - X| \geq 2$, then $V(T) - V(C) - X = Y^+ \cup Y^-$.

Now we prove Claim 3. Assume that $|Y^+| \geq 2$ or $|Y^-| \geq 2$, i.e., $|V(T) - V(C) - X| \geq 2$. Then $V(T) - V(C) - X = Y^+ \cup Y^-$. This together with Claim 1 implies $V(T) - V(C) = X^+ \cup X^- \cup Y^+ \cup Y^-$. Since T is strongly connected, $X^+ \cup Y^+ \neq \emptyset$, $X^- \cup Y^- \neq \emptyset$, and $E(X^+ \cup Y^+, X^- \cup Y^-) \neq \emptyset$. Let $u \in X^+ \cup Y^+$ and $v \in X^- \cup Y^-$ be vertices such that $uv \in E(X^+ \cup Y^+, X^- \cup Y^-)$. Let v_1 and v_2 be consecutive vertices on C, then $v_1u \in E(T)$ and $vv_2 \in E(T)$. Hence there exists a cycle $C' = v_1uvv_2Cv_1$ such that $V(T) - V(C') - X \neq \emptyset$, which contradicts the maximality of $|V(C)|$. Here we obtain $|Y^+| \leq 1$ and $|Y^-| \leq 1$. Thus $|V(T) - V(C) - X| \leq 2$, and if $|V(T) - V(C) - X| = 2$, then $|Y^+| = |Y^-| = 1$. □

Claim 4. *Suppose that* $|V(T) - V(C) - X| = 2$. *Then* $E(X^+, Y^-) = \emptyset$ *and* $E(Y^+, X^-) = \emptyset$.

Proof. Assume that $E(X^+, Y^-) \neq \emptyset$ or $E(Y^+, X^-) \neq \emptyset$. By symmetry, we may assume that $E(X^+, Y^-) \neq \emptyset$. By Claim 3, we can write $Y^- = \{v\}$. Let $u \in X^+$ such that $uv \in E(T)$. Let v_1 and v_2 be consecutive vertices on C, then $v_1 u \in E(T)$ and $vv_2 \in E(T)$. Then there exists a cycle $C' = v_1 u v v_2 C v_1$ such that $V(T) - V(C') - X \neq \emptyset$, which contradicts the maximality of $|V(C) \cap X|$. □

Claim 5. (i) *If* $X^+ \neq \emptyset$, *then* $|X^+| \geq 2p - 1$. *If* $X^- \neq \emptyset$, *then* $|X^-| \geq 2p - 1$.

(ii) *If there exists a vertex* $x \in X^+$ *such that* $E(\{x\}, V(T) - V(C) - X) = \emptyset$, *then* $|X^+| \geq 2p$. *If there exists a vertex* $x \in X^-$ *such that* $E(V(T) - V(C) - X, \{x\}) = \emptyset$, *then* $|X^-| \geq 2p$.

Proof. (i) Let $x \in X^+$. By Claims 1 through 4 and the definition of X^+, $e(\{x\}, V(T) - X^+) = e(\{x\}, Y^+) \leq 1$, and hence $\deg^+_{\langle X^+ \rangle}(x) \geq p - 1$. By applying Lemma 2 to the tournament $\langle X^+ \rangle$, we obtain $|X^+| \geq 2p - 1$. Similarly, if $X^- \neq \emptyset$, then $|X^-| \geq 2p - 1$.

(ii) Let $x \in X^+$ such that $E(\{x\}, V(T) - V(C) - X) = \emptyset$. Then by Claims 1 and 2 and the definition of X^+, $e(\{x\}, V(T) - X^+) = 0$, and hence $\deg^+_{\langle X^+ \rangle}(x) \geq p$. By applying Lemma 3 to the tournament $\langle X^+ \rangle$, we obtain $|X^+| \geq 2p$. Similarly, if there exists a vertex $x \in X^-$ such that $E(V(T) - V(C) - X, \{x\}) = \emptyset$, then $|X^-| \geq 2p$. □

Claim 6. $|V(T) - V(C) - X| = 1$.

Proof. Assume that $|V(T) - V(C) - X| \geq 2$. By Claim 3, $|V(T) - V(C) - X| = 2$ and $|Y^+| = |Y^-| = 1$. Write $Y^+ = \{y_1\}$. By Claims 1 through 4, $E(\{y_1\}, V(T)) \subset E(\{y_1\}, X^+ \cup Y^-)$. On the other hand, $|Y^-| = 1$ and $\deg^+_T(y_1) \geq p \geq 2$, and hence $E(\{y_1\}, X^+) \neq \emptyset$. Let $x_1 \in X^+$ be a vertex such that $y_1 x_1 \in E(T)$. Then by Claims 1 through 4 and the definition of X^+, $E(\{x_1\}, V(T) - V(C) - X) = \emptyset$, and hence $|X^+| \geq 2p$ by Claim 5(ii). Similarly, $|X^-| \geq 2p$. Consequently we obtain

$$
\begin{aligned}
|X| &\geq |V(C) \cap X| + |X^+| + |X^-| \\
&\geq 1 + 2p + 2p \\
&= 4p + 1,
\end{aligned}
$$

which contradicts $|X| \leq \min\{|V(T)| - 1, 4p - 1\}$. □

Claim 7. *If* $X \subset V(C)$, *then* $|V(C)| = |V(T)| - 1$.

Proof. By Claim 6, we obtain this claim immediately. □

Until the end of Claim 9, we assume that $X - V(C) \neq \emptyset$. By Claim 1, $X = X^+ \cup X^-$, and hence we may assume that $X^+ \neq \emptyset$ by symmetry. By Claim 6, $V(T) - V(C) - X$ consists of a single vertex, say y_0.

Claim 8. *Suppose that* $X^- = \emptyset$. *Then the following hold.*

(i) *There exists a path* $P = x_1x_2\cdots x_{|X^+|}y_0$ $(x_1, x_2, \ldots, x_{|X^+|} \in X^+)$, *i.e.,* $V(P) = \{y_0\} \cup X^+$ *and* P *has* y_0 *as the endvertex.*

(ii) *Suppose that* $|V(C) - X| \geq 2$, *or* $|V(C) - X| = 1$ *and* $e(\{y_0\}, V(C) \cap X) \geq 1$. *Then* $|V(C) \cap X| \geq |X^+|$.

(iii) *Suppose that* $|V(C) - X| = 1$ *and* $e(\{y_0\}, V(C) \cap X) = 0$. *Then* $|V(C) \cap X| \geq 2p$.

(iv) $|V(C) \cap X| \geq 2p - 1$.

(v) *If* $|V(T) - X| \geq 2$, *then* $E(\{y_0\}, X^+) = \emptyset$.

Proof. Note that since $X^- = \emptyset$, and since T is strongly connected, $E(\{y_0\}, V(C)) \neq \emptyset$.

(i) Let P be a path such that $V(P) \cap X^+ \neq \emptyset$ and P has y_0 as the endvertex (there exists such a path P since $E(X^+, \{y_0\}) \neq \emptyset$). We assume that we have chosen P so that $|V(P) \cap X^+|$ is maximal. In order to show $X^+ \subset V(P)$, we assume that $X^+ - V(P) \neq \emptyset$. For each vertex $x \in X^+ - V(P)$, $E(\{x\}, V(P)) = \emptyset$ by the maximality of $|V(P) \cap X^+|$. Then $E(X^+ - V(P), V(T) - (X^+ - V(P))) = \emptyset$, which contradicts the assumption that T is strongly connected. Hence $X^+ - V(P) = \emptyset$, thus $X^+ \subset V(P)$.

(ii) By the assumption of (ii), there exists $v \in V(C)$ such that $y_0v \in E(G)$ and $V(C) - X - \{v\} \neq \emptyset$, say v_0. By (i), there exists a path $P = x_1x_2\cdots x_{|X^+|}y_0$ $(x_1, x_2, \ldots, x_{|X^+|} \in X^+)$. Hence there exists a cycle $C' = Pv_0x_1$ such that $V(T) - V(C') - X \neq \emptyset$. By the maximality of $|V(C) \cap X|$, $|V(C) \cap X| \geq |V(C') \cap X| = |X^+|$.

(iii) Let $v_0 \in V(C)$ be a vertex such that $y_0v_0 \in E(T)$. By the assumption of (iii), $v_0 \notin X$. Since $\deg^-_T(v_0) \geq p \geq 2$, $E(V(C) - \{v_0\}, \{v_0\}) = E(V(C) \cap X, \{v_0\}) \neq \emptyset$. Let $v_1 \in V(C) \cap X$ such that $v_1v_0 \in E(T)$. Then $\deg^-_{\langle V(C) \cap X\rangle}(v_1) \geq p$, and $\deg^-_{\langle V(C) \cap X\rangle}(v) \geq p - 1$ for all $v \in V(C) \cap X - \{v_1\}$. By applying Lemma 3 to the tournament $\langle V(C) \cap X\rangle$, we obtain $|V(C) \cap X| \geq 2p$.

(iv) If $|V(C) - X| \neq 0$, then $|V(C) \cap X| \geq 2p - 1$ by (ii) and Claim 5(i), and (iii). If $|V(C) - X| = 0$, then $\deg^-_{\langle V(C)\rangle}(x) \geq p - 1$ for all $x \in V(C) \cap X = V(C)$. By applying Lemma 2 to the tournament $\langle V(C)\rangle$, we obtain $|V(C)| = |V(C) \cap X| \geq 2p - 1$.

(v) We assume that $E(\{y_0\}, X^+) \neq \emptyset$. Let $x_0 \in X^+$ be a vertex such that $y_0x_0 \in E(T)$, then $E(\{x_0\}, V(T) - V(C) - X) = \emptyset$, and hence $|X^+| \geq 2p$ by Claim 5(ii). Since $V(C) - X \neq \emptyset$ by the assumption of (v), (ii) or (iii) holds. Hence $|X| = |V(C) \cap X| + |X^+| \geq 4p$, which contradicts $|X| \leq \min\{4p - 1, |V(T)| - 1\}$. □

Claim 9. *Suppose that* $|V(T) - X| \geq 2$. *Then* $|V(C) - X| \geq 1$, $|V(C)| \geq 3$, *and* $X^- = \emptyset$.

Proof. By Claim 6, we have $|V(C) - X| \geq 1$. This together with $|V(C) \cap X| \geq 1$ implies $|V(C)| \geq 3$. Let $v_0 \in V(C) - X$. Assume that $X^- \neq \emptyset$. Since T is strongly connected, $E(X^+, \{y_0\}) \neq \emptyset$ and $E(\{y_0\}, X^-) \neq \emptyset$ by Claim 2. Let $x_1 \in X^+$ such that $x_1y_0 \in E(T)$, and let $x_2 \in X^-$ such that $y_0x_2 \in E(T)$. Hence there exists a cycle $C' = v_0^- x_1 y_0 x_2 v_0^+ C v_0^-$ such that $V(T) - V(C') - X \neq \emptyset$, which contradicts the maximality of $|V(C) \cap X|$. □

Proof of Theorem 5.

Let p and T be as in Theorem 5. Let X be a subset of $V(T)$ such that $|X| \leq \min\{|V(T)| - 1, 4p - 3\}$, and C, X^+, X^- be as in the paragraph preceding the statement of Claim 1. In order to obtain $X \subset V(C)$, we assume that $X - V(C) \neq \emptyset$. By Claim 1, we may assume $X^+ \neq \emptyset$ by symmetry. By Claim 5 (i), $|X^+| \geq 2p - 1$. In the case where $X^- \neq \emptyset$, $|X^-| \geq 2p - 1$ by Claim 5 (i). Hence $|X| = |X \cap V(C)| + |X^+| + |X^-| \geq 1 + 2p - 1 + 2p - 1 = 4p - 1$, which contradicts the definition of X. In the case where $X^- = \emptyset$, $|X| = |X \cap V(C)| + |X^+| \geq 4p - 2$ by Claim 8 (iv), which also contradicts the definition of X. Here we obtain $X \subset V(C)$, and hence $|V(C)| = |V(T)| - 1$ by Claim 7. This completes the proof of Theorem 5. □

Proof of Theorem 6.

Let p and T be as in Theorem 6. Let X be a subset of $V(T)$ such that $|X| \leq 4p - 2$, and C, X^+, X^- be as in the paragraph preceding the statement of Claim 1. In order to obtain $X \subset V(C)$, we assume that $X - V(C) \neq \emptyset$. By Claim 1, we may assume that $X^+ \neq \emptyset$ by symmetry. By Claim 6, $|V(T) - V(C) - X| = 1$. Let y_0 be as in the paragraph preceding the statement of Claim 8. Since $|V(T)| \geq 4p$, $|V(T) - X| \geq 2$, and hence $X^- = \emptyset$ and $|V(C)| \geq 3$ by Claim 9. We write the cycle $C = v_1v_2 \ldots v_lv_1 (l \geq 3)$. By Claim 8(v), there exist two vertices v_i, $v_j \in V(C)$ $(1 \leq i < j \leq l)$ such that y_0v_i, $y_0v_j \in E(T)$. By Claim 9, $|V(C) - X| \geq 1$. This together with $|V(C) \cap X| \geq 1$ implies that one of the following holds (subscripts of the letter v are to be read modulo l):

(1) $|\{v_i, \ldots, v_{j-1}\} \cap X| \geq 1$ and $|\{v_j, \ldots, v_{i-1}\} \cap (V(T) - X)| \geq 1$; or

(2) $|\{v_j, \ldots, v_{i-1}\} \cap X| \geq 1$ and $|\{v_i, \ldots, v_{j-1}\} \cap (V(T) - X)| \geq 1$.

By symmetry, we may assume that (1) holds. By Claim 8 (i), there exists a path $P = x_1x_2 \cdots x_{|X^+|}y_0$ $(x_1, x_2, \ldots, x_{|X^+|} \in X^+)$. Hence there exists a cycle $C' = Pv_iCv_{j-1}x_1$ such that $V(T) - V(C') - X \neq \emptyset$ and $|V(C') \cap X| \geq |X^+| + 1$. By Claim 5(i), $|X^+| \geq 2p - 1$. By the maximality of $|V(C) \cap X|$, $|V(C) \cap X| \geq |V(C') \cap X| \geq |X^+| + 1 \geq 2p$. Consequently, $|X| = |V(C) \cap X| + |X^+| \geq 4p - 1$, which contradicts the definition of X. Here we obtain $X \subset V(C)$, and hence $|V(C)| = |V(T)| - 1$ by Claim 7. This completes the proof of Theorem 6. □

Proof of Theorem 7.

Let p and T be as in Theorem 7. Let X be a subset of $V(T)$ such that $|X| \leq 4p - 1$, and C, X^+, X^- be as in the paragraph preceding the statement of Claim 1. In order to obtain $X \subset V(C)$, we assume that $X - V(C) \neq \emptyset$. By Claim 1, we may assume that $X^+ \neq \emptyset$ by symmetry. By Claim 5(i), $|X^+| \geq 2p - 1$. By

Claim 6, $|V(T) - V(C) - X| = 1$. Let y_0 be as in the paragraph preceding the statement of Claim 8. Since $|V(T)| \geq 4p + 1$, $|V(T) - X| \geq 2$, and hence $X^- = \emptyset$ and $|V(C)| \geq 3$ by Claim 9. We write the cycle $C = v_1v_2 \dots v_lv_1$ $(l \geq 3)$. By Claim 8(iv), $|V(C) \cap X| \geq 2p - 1 \geq 3$. By Claim 9, $|V(C) - X| \geq 1$. Now we divide into the following two cases.

Case 1. $p \geq 3$.

By Claim 8(v), there exist three vertices v_{i_1}, v_{i_2}, $v_{i_3} \in V(C)$ $(1 \leq i_1 < i_2 < i_3 \leq l)$ such that $y_0v_{i_j} \in E(T)$ $(j = 1, 2, 3)$. Since $|V(C) \cap X| \geq 3$ and $|V(C) - X| \geq 1$, one of the following holds (subscripts of the letter v are to be read modulo l):

(1) $|\{v_{i_1}, \dots, v_{i_3-1}\} \cap X| \geq 2$ and $|\{v_{i_3}, \dots, v_{i_1-1}\} \cap (V(T) - X)| \geq 1$; or

(2) $|\{v_{i_2}, \dots, v_{i_1-1}\} \cap X| \geq 2$ and $|\{v_{i_1}, \dots, v_{i_2-1}\} \cap (V(T) - X)| \geq 1$; or

(3) $|\{v_{i_3}, \dots, v_{i_2-1}\} \cap X| \geq 2$ and $|\{v_{i_2}, \dots, v_{i_3-1}\} \cap (V(T) - X)| \geq 1$.

By arguing as in the proof of Theorem 6, there exists a cycle C' such that $V(T) - V(C') - X \neq \emptyset$ and $|V(C') \cap X| \geq |X^+| + 2$. Then $|V(C) \cap X| \geq |V(C') \cap X| \geq |X^+| + 2 \geq 2p + 1$, and hence $|X| = |V(C) \cap X| + |X^+| \geq 4p$, which contradicts the definition of X.

Case 2. $p = 2$.

By Claim 8(v), there exist two vertices v_i, $v_j \in V(C)$ $(1 \leq i < j \leq l)$ such that y_0v_i, $y_0v_j \in E(T)$.

First we consider the case where $j-i \geq 2$ and $i+l-j \geq 2$. Since $|V(C) \cap X| \geq 3$ and $|V(C) - X| \geq 1$, one of the following holds (subscripts of the letter v are to be read modulo l):

(1) $|\{v_i, \dots, v_{j-1}\} \cap X| \geq 2$ and $|\{v_j, \dots, v_{i-1}\} \cap (V(T) - X)| \geq 1$; or

(2) $|\{v_j, \dots, v_{i-1}\} \cap X| \geq 2$ and $|\{v_i, \dots, v_{j-1}\} \cap (V(T) - X)| \geq 1$.

By arguing as in the proof of Theorem 6, there exists a cycle C' such that $V(T) - V(C') - X \neq \emptyset$ and $|V(C') \cap X| \geq |X^+| + 2$. Then $|V(C) \cap X| \geq |V(C') \cap X| \geq |X^+| + 2 \geq 2p + 1$, and hence $|X| = |V(C) \cap X| + |X^+| \geq 4p$, which contradicts the definition of X.

Now we consider the case where $j - i = 1$ or $i + l - j = 1$. We may assume that $j - i = 1$ by symmetry. Assume for the moment that there exists a path Q such that the beginning of Q is v_i or v_j, $V(Q) \subset V(C)$, $|V(Q) \cap X| \geq 2$, and $V(C) - V(Q) - X \neq \emptyset$. By Claim 8 (i), there exists a path $P = x_1x_2 \cdots x_{|X^+|}y_0$ $(x_1, x_2, \dots, x_{|X^+|} \in X^+)$. Hence there exists a cycle $C' = PQx_1$ such that $V(T) - V(C') - X \neq \emptyset$ and $|V(C') \cap X| \geq |X^+| + 2$. Therefore $|X| \geq 4p$ by arguing as in the preceding paragraph, which contradicts the definition of X. Assume now that $V(C) - V(Q) - X = \emptyset$ for any path Q such that

$$\text{the beginning of } Q \text{ is } v_i \text{ or } v_j,\ V(Q) \subset V(C), \text{ and } |V(Q) \cap X| \geq 2. \qquad (*)$$

Let $Q_0 = v_iCv_{i-1}$ (throughout the end of this paragraph, subscripts of the letter v are to be read modulo l). Since $|X| \le 4p-1$ and $|V(T)| \ge 4p+2$, $|V(T)-X| \ge 3$, and hence $|V(Q_0) \cap (V(T)-X)| \ge 2$. Since $V(C)-V(Q)-X = \emptyset$ for any path Q satisfying $(*)$, there exists an integer m with $i+2 \le m \le l+i-2$ such that $v_{i+1(=j)}, \cdots, v_m \in V(T)-X$ and $v_{m+1}, \cdots, v_i \in X$. Set $Z_1 = \{v_{i+1}, \cdots, v_m\}$ and $Z_2 = \{v_{m+1}, \cdots, v_{i-1}\}$. Since $(\{v_m\} \cup Z_2) \cap \{v_i, v_j\} = \emptyset$, $E(\{y_0\}, \{v_m\} \cup Z_2) = \emptyset$. Then

$$E(X^+ \cup \{y_0\}, \{v_m\}) = \emptyset, \tag{1}$$

and

$$E(X^+ \cup \{y_0\}, Z_2) = \emptyset. \tag{2}$$

Since $V(C)-V(Q)-X = \emptyset$ for any path Q satisfying $(*)$, we also obtain

$$E(\{v_i, \dots, v_{m-2}, v_{m+1}\}, \{v_m\}) = \emptyset, \tag{3}$$

and

$$E(\{v_i, \dots, v_{m-1}\}, Z_2) = \emptyset. \tag{4}$$

By (1) and (3), $e(Z_2 - \{v_{m+1}\}, \{v_m\}) \ge 1$. Let $w_0 \in Z_2 - \{v_{m+1}\}$ such that $w_0v_m \in E(T)$. By (2) and (4), $\deg^-_{\langle Z_2 \rangle}(w_0) \ge p$ and $\deg^-_{\langle Z_2 \rangle}(w) \ge p-1$ for all $w \in Z_2 - \{w_0\}$. By applying Lemma 3 to the tournament $\langle Z_2 \rangle$, $|Z_2| \ge 2p$. Therefore $|X| = |Z_2 \cup \{v_i\}| + |X^+| \ge 4p$, which contradicts the definition of X.

Consequently, we obtain $X \subset V(C)$, and hence $|V(C)| = |V(T)|-1$ by Claim 7. This completes the proof of Theorem 7. □

4 Examples

In this section, we discuss the sharpness of the various conditions in theorems.

Proposition 1. *Let $p \ge 2$ be an integer. There exists a strongly connected tournament T with $|V(T)| = 4p-1$ such that $deg^+_T(x) \ge p$ and $deg^-_T(x) \ge p$ for all $x \in V(T)$, but T has no $4p-1$ vertices $x_1, x_2, \dots, x_{4p-1}$ such that $T - x_i$ $(i = 1, 2, \dots, 4p-1)$ is strongly connected; that is, there exists a subset X of $V(T)$ having $4p-2$ vertices such that $X - V(C) \ne \emptyset$ for every cycle C such that $|V(C)| \le |V(T)|-1$.*

Proof. Let $p \ge 2$ be an integer. Let T_1 and T_2 be tournaments having $2p-1$ vertices such that $V(T_m) = \{v_1^m, v_2^m, \dots, v_{2p-1}^m\}$ and $E(T_m) = \{v_i^m v_j^m \mid 1 \le i \le 2p-1, i+1 \le j \le i+p-1,\ j$ is to be read modulo $2p-1\}$ $(m = 1, 2)$. We define a tournament T having $4p-1$ vertices by

$$\begin{aligned} V(T) =& V(T_1) \cup V(T_2) \cup \{w_0\}, \\ E(T) =& E(T_1) \cup E(T_2) \\ & \cup \{vw_0 \mid v \in V(T_2)\} \ \cup\ \{w_0v \mid v \in V(T_1)\} \\ & \cup \{uv \mid u \in V(T_1), v \in V(T_2)\}. \end{aligned}$$

Then T has the desired properties. To see this, set $X = V(T_1) \cup V(T_2)$, then $X - V(C) \neq \emptyset$ for every cycle C such that $|V(C)| \leq |V(T)| - 1$. □

Proposition 2. *Let $p \geq 2$ be an integer. There exists a strongly connected tournament T with $|V(T)| = 4p$ such that $deg_T^+(x) \geq p$ and $deg_T^-(x) \geq p$ for all $x \in V(T)$, but T has no $4p$ vertices $x_1, x_2, \dots, x_{4p}$ such that $T - x_i$ $(i = 1, 2, \dots, 4p)$ is strongly connected; that is, there exists a subset X of $V(T)$ having $4p - 1$ vertices such that $X - V(C) \neq \emptyset$ for every cycle C such that $|V(C)| \leq |V(T)| - 1$.*

Proof. Let $p \geq 2$ be an integer. Let T_1 and T_2 be tournaments having $2p-1$ vertices such that $V(T_m) = \{v_1^m, v_2^m, \dots, v_{2p-1}^m\}$ and $E(T_m) = \{v_i^m v_j^m \mid 1 \leq i \leq 2p-1, i+1 \leq j \leq i+p-1,\ j$ is to be read modulo $2p-1\}$ $(m = 1, 2)$. We define a tournament T by

$$\begin{aligned} V(T) =& V(T_1) \cup V(T_2) \cup \{w_1, w_2\}, \\ E(T) =& E(T_1) \cup \{vw_1 \mid v \in V(T_1)\} \cup \{w_2 v \mid v \in V(T_1)\} \\ & \cup E(T_2) \cup \ \{vw_2 \mid v \in V(T_2)\} \cup \{w_1 v \mid v \in V(T_2)\} \\ & \cup \{uv \mid u \in V(T_1), v \in V(T_2)\} \ \cup \ \{w_1 w_2\}. \end{aligned}$$

Then T has the desired properties. To see this, set $X = V(T_1) \cup V(T_2) \cup \{w_1\}$, then $X - V(C) \neq \emptyset$ for every cycle C such that $|V(C)| \leq |V(T)| - 1$. □

Propositions 1 and 2 imply that the cardinality of the set of nonseparating vertices in Theorems 1 and 2 is sharp, and that the bound on the order T is best possible in Theorems 2 and 3.

Proposition 3. *Let $p \geq 2$ be an integer. There exist infinitely many strongly connected tournaments T with $|V(T)| \geq 4p+1$ such that $deg_T^+(x) \geq p$ and $deg_T^-(x) \geq p$ for all $x \in V(T)$, but T has no $4p+1$ vertices $x_1, x_2, \dots, x_{4p+1}$ such that $T - x_i$ $(i = 1, 2, \dots, 4p+1)$ is strongly connected; that is, there exists a subset X of $V(T)$ having $4p$ vertices such that $X - V(C) \neq \emptyset$ for every cycle C such that $|V(C)| \leq |V(T)| - 1$.*

Proof. Let $p \geq 2$ be an integer, and $l \geq 1$ be an integer. Let T_1 and T_2 be tournaments having $2p$ vertices such that $V(T_m) = \{v_1^m, v_2^m, \dots, v_{2p}^m\}$ and $E(T_m) = \{v_i^m v_j^m \mid 1 \leq i \leq p, i+1 \leq j \leq i+p\} \cup \{v_i^m v_j^m \mid p+1 \leq i \leq 2p, i+1 \leq j \leq i+p-1,\ j$ is to be read modulo $2p\}(m = 1, 2)$, and let T_3 be a tournament having l vertices $v_1^3, v_2^3, \dots, v_l^3$ with $E(T_3) = E_1 \cup E_2$, where

$$E_1 = \begin{cases} \{v_i^3 v_{i+1}^3 \mid 1 \leq i \leq l-1\} & (l \geq 2) \\ \emptyset & (l = 1); \end{cases}$$

$$E_2 = \begin{cases} \{v_i^3 v_j^3 \mid 3 \leq i \leq l, 1 \leq j \leq i-2\} & (l \geq 3) \\ \emptyset & (l \leq 2). \end{cases}$$

We define a tournament T by

$$\begin{aligned} V(T) =& V(T_1) \cup V(T_2) \cup V(T_3), \\ E(T) =& E(T_1) \cup E(T_2) \cup E(T_3) \\ & \cup \{v_1^3 v \mid v \in \{v_1^1, \ldots, v_p^1\}\} \cup \{v v_1^3 \mid v \in \{v_{p+1}^1, \ldots, v_{2p}^1\}\} \\ & \cup \{v_l^3 v \mid v \in \{v_1^2, \ldots, v_p^2\}\} \cup \{v v_l^3 \mid v \in \{v_{p+1}^2, \ldots, v_{2p}^2\}\} \\ & \cup \{uv, vw \mid u \in V(T_2), v \in V(T_3) - \{v_1^3, v_l^3\}, w \in V(T_1)\} \\ & \cup \{uv \mid u \in V(T_2), v \in V(T_1)\} \cup E_3, \end{aligned}$$

where

$$E_3 = \begin{cases} \{v_l^3 v \mid v \in V(T_1)\} \cup \{v v_1^3 \mid v \in V(T_2)\} & (l \geq 2) \\ \emptyset & (l = 1). \end{cases}$$

Then T has the desired properties. To see this, set $X = V(T_1) \cup V(T_2)$, then $|X| = 4p$, and $X - V(C) \neq \emptyset$ for every cycle C such that $|V(C)| \leq |V(T)| - 1$. □

Proposition 4. *Let $p = 2$. There exists a strongly connected tournament T with $|V(T)| = 4p + 1$ such that $\deg_T^+(x) \geq p$ and $\deg_T^-(x) \geq p$ for all $x \in V(T)$, but T has no $4p$ vertices $x_1, x_2, \ldots, x_{4p}$ such that $T - x_i$ $(i = 1, 2, \ldots, 4p)$ is strongly connected; that is, there exists a subset X of $V(T)$ having $4p - 1$ vertices such that $X - V(C) \neq \emptyset$ for every cycle C such that $|V(C)| \leq |V(T)| - 1$.*

Proof. We define a tournament T having $4p + 1 = 9$ vertices $v_1, v_2, \ldots, v_9$ by

$$\begin{aligned} E(T) = & \{v_1 v_2, v_2 v_3, v_3 v_1\} \cup \{v_7 v_8, v_8 v_9, v_9 v_7\} \\ & \cup \{v_5 v_i, v_i v_6 | 7 \leq i \leq 9\} \cup \{v_6 v_5\} \\ & \cup \{v_i v_4 | 1 \leq i \leq 3, 7 \leq i \leq 9\} \cup \{v_4 v_5, v_4 v_6\} \\ & \cup \{v_i v_j \mid 5 \leq i \leq 9, 1 \leq j \leq 3\}. \end{aligned}$$

Then T has the desired properties. To see this, set $X = \{v_i | 1 \leq i \leq 3, 6 \leq i \leq 9\}$, then $|X| = 4p - 1 = 7$, and $X - V(C) \neq \emptyset$ for every cycle C such that $|V(C)| \leq |V(T)| - 1$. □

Proposition 3 implies that the cardinality of the set of nonseparating vertices in Theorems 3 is sharp. Proposition 4 shows that for $p = 2$ and $|V(T)| = 4p - 1$, a result like Theorem 3 no longer holds.

Acknowledgments

I would like to thank Professor Yoshimi Egawa for the help he gave to me during the preparation of this paper. I am also grateful to the referees for helpful comments and valuable suggestions toward the improvement of this paper.

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(Received 20 Oct 2004; revised 11 May 2006)