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Trees with extremal numbers of dominating sets

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Abstract

The largest NDS (number of dominating sets) and the smallest NDS among n-vertex trees are determined. Corresponding trees (with those NDS') are characterized. NDS' in the n-vertex path P_n as well as in so-called palm trees are determined. The largest NDS' are attained among stars and some palms. $2K_2$ is the only disconnected forest with largest NDS. For $n \geq 3$, any tree T of order $\lceil \frac{n}{3} \rceil$ or $\frac{n-1}{3}$ is an induced subgraph of an n-vertex tree with smallest NDS, the n-vertex supertree with $n \geq 8$ being uniquely determined by T ifand only if $3|n$.

1 Introduction

The star $K_{1,n-1}$ and path P_n are the two extreme structures among so-called palms which are trees with at most one vertex of degree three or more. Just among those n-vertex palms there are trees whose NDS' (*numbers of dominating sets*) occupy the top positions among all trees of order n. The star is on the first position for all n and is unrivaled for $n \neq 4, 5$, the path P_n being quite close to the star in this ranking also if n is large. Recurrences and a method of "short steps" works well in establishing the ranking at the top. Recursion involving pendant stars gives a result at the other end of the ranking. All exponentially many n-vertex minimal trees (where *minimal* reads with smallest NDS) are identified. It appears, for instance, that all minimal trees on $3k$ vertices are obtained by joining two new leaves to each vertex of a tree T if T ranges over all trees of order k. Thus there is a bijection between minimal trees of order $3k$ and all k-vertex trees. No such bijection exists for remaining orders $n \geq 7$. However, for each $n \geq 3$, the removal of all leaves from minimal trees of order n gives nothing more than all trees on $\lceil \frac{n}{3} \rceil$ vertices and also, but precisely if $n \equiv 1 \pmod 3$, all trees on $\lfloor \frac{n}{3} \rfloor$ vertices.

2 Preliminaries with stars, paths, and palms

Only simple graphs G are considered. The order of G is denoted by $|G|$. In general we use the standard terminology and notation of graph theory, cf. Chartrand and Lesniak [1] or West [14]. A *dominating set* in G is a vertex subset S such that every vertex of G either is in S or is adjacent to a vertex in S. Let $\partial(G)$ stand for the *number of dominating sets* (NDS for short) in G. Given a set X of vertices, let $\partial_X(G)$ denote NDS in G with X included in each of the dominating sets; we write $\partial_x(G)$ instead if $X = \{x\}$, a singleton which includes the vertex x. On the other hand, let $\partial_{\hat{x}}(G)$ be the count of dominating sets in G which all avoid the vertex x, $\partial_{\hat{x}}(G)$ being zero exactly if x has no neighbor in G. Then the basic rule for recursively evaluating NDS in a graph G is as follows

$$\partial(G) =_x \partial_x(G) + \partial_{\hat{x}}(G) \qquad \text{for any vertex } x \text{ of } G \tag{1}$$

where $=_x$ stands for the ordinary equality symbol '=' together with requirement that what follows the symbol is the sum of two summands ∂_x and $\partial_{\hat{x}}$ in this order.

Recall that a vertex of degree 0 or 1 is called a *leaf*. Given a tree T and an integer $k \geq 2$, a *pendant* (k-)*star* with k rays in T is a subgraph $K_{1,k}$ whose central vertex is of degree k in T, too, and is adjacent to $k-1$ or k leaves of T. The *attachment vertex* $a(x)$ of a vertex x in a nontrivial tree T is defined to be the only neighbor of x if x is a leaf and the only non-leaf neighbor of x if x is the center (or central vertex) of a pendant star, otherwise $a(x)$ is not defined. The center x of a pendant star as well as the star itself are said to be *attached* to $a(x)$ if $a(x)$ exists. The following is easily seen.

$$\partial_{\hat{z}}(T) = \partial_{a(z)}(T - z) \qquad \text{if } z \text{ is a leaf of } T \text{ with } |T| \geq 2. \tag{2}$$

Given an n-vertex path P_n whose endvertices are denoted by z, z_1, let

$$\mathfrak{p}_n := \partial(P_n),\ \mathfrak{p}_n^\star := \partial_z(P_n),\ \mathfrak{p}_n^{\star\star} := \partial_{\{z,z_1\}}(P_n).$$

Proposition 1 *For the n-vertex path P_n if $\varphi = \mathfrak{p}, \mathfrak{p}^\star, \mathfrak{p}^{\star\star}$ then*

$$\varphi_n = \varphi_{n-1} + \varphi_{n-2} + \varphi_{n-3} \tag{3}$$

with initial conditions

$$\mathfrak{p}_0 = 1 = \mathfrak{p}_1,\ \mathfrak{p}_2 = 3\ \ (\varphi = \mathfrak{p}),$$
$$\mathfrak{p}_i^\star = i \ \ \textit{for} \ \ i = 0,1,2\ \ (\varphi = \mathfrak{p}^\star),$$
$$\mathfrak{p}_0^{\star\star} = 0,\ \mathfrak{p}_1^{\star\star} = 1 = \mathfrak{p}_2^{\star\star}\ \ (\varphi = \mathfrak{p}^{\star\star}).$$

By the *palm* (or *palm tree*), $P(n,D)$, with n vertices and maximum degree D where $n \geq D+1 \geq 1$ but $n = D+1$ for $D = 0,1$ we mean the path P_n if $D \leq 2$, the star $K_{1,n-1}$ if $D = n-1$, and otherwise $P(n,D)$ is the n-vertex union of a star $K_{1,D-1}$ and path P_{n-D+1} such that the center of the star coincides with an endvertex of the path. Therefore all eight trees of order up to five are palms. For each palm,

$$\partial(P(n,D)) = 2^{D-1}\mathfrak{p}^{\star}_{n-D+1} + \mathfrak{p}_{n-D}, \quad D \geq 1. \tag{4}$$

It is quite clear that NDS in the star $K_{1,n-1}$ is the maximum value of NDS, say $\mathrm{NDS}^{(n)}$, among all n-vertex trees (and n-vertex forests, too). Moreover, for $n \geq 6$, $K_{1,n-1}$ is the only n-vertex tree (forest) with that maximum NDS. Thus, if y is the central vertex of the star,

$$\mathrm{NDS}^{(n)} = \partial(K_{1,n-1}) =_y 2^{n-1} + 1. \tag{5}$$

We are going to show that the opposite number, say $\mathrm{NDS}_{(n)}$ being the minimum NDS among n-vertex trees, equals $\beta_m \cdot 5^{\lfloor n/3 \rfloor}$ provided that $n \geq 2$, $m = n \bmod 3$, and (by formula (16) below)

$$\beta_m = \begin{cases} 1 & \text{if } m = 0, \\ 9/5 & \text{if } m = 1, \\ 3 & \text{if } m = 2. \end{cases}$$

Hence, for n-vertex trees with $n \geq 2$,

$$\beta_m \cdot 5^{\lfloor n/3 \rfloor} \leq \mathrm{NDS} \leq 1 + 2^{n-1},$$

with both bounds being attainable for each n.

An n-vertex tree T is called *∂-minimal* (or *domination #-minimal*, or simply *minimal*), in symbols $T \in \mathcal{T}^{\min}_n$, if $\partial(T) = \mathrm{NDS}_{(n)}$. Let

$$\mathcal{T}^{\min} = \bigcup_n \mathcal{T}^{\min}_n.$$

All ∂-minimal trees of any order are characterized recursively.

The second and the third largest NDS' among n-vertex trees with $n \geq 9$ are attained in the palm trees $P(n,D)$ with maximum degree $D = n-3, n-2$, respectively. Thus, due to (4), these NDS' are $\frac{3}{8}\,2^n + 4 \pm 1$, respectively.

3 Counting dominating sets

We continue presenting general rules for evaluating $\partial(G)$ (which is NDS in a graph G).

$$\partial(G) = \prod_H \partial(H) \tag{6}$$

where H ranges over all (nontrivial) components of G (possibly nontrivial since $\partial(K_1) = 1$).

Lemma 2 *For any vertex x of G,*

$$\partial_{\hat{x}}(G) \leq \partial(G-x)$$

with equality if and only if x is adjacent to a leaf in G.

Proof: The inequality follows because every dominating set in G which avoids x is dominating in $G-x$. However, $G-x$ has an extra dominating set exactly if no neighbor of x in G is a leaf. It is so because the only possibility for the extra set is that it avoids all neighbors of x in G. □

By a *branch* B at a vertex x in a tree T we mean a maximal subtree which includes x and exactly one edge incident to x. On using the basic formula (1) we get the following useful rule in case G is a tree. (It is assumed that a product over the empty set is 1.)

Proposition 3 *Let T be a tree with vertex x of degree b, $b>1$, and with b branches B_i all at x. Assume that the set, $L(x)$, of leaves attached to x has ℓ elements whence $1 \leq \ell \leq b$. Then clearly*

$$\partial_x(T) = \prod_{i=1}^{b} \partial_x(B_i), \tag{7}$$

$$\begin{aligned} \partial(T) &=_x 2^\ell \partial_x(T - L(x)) + \partial(T - L(x) - x) \\ &=_x 2^\ell \prod_{i=1}^{b-\ell} \partial_x(B_i) + \prod_{i=1}^{b-\ell} \partial(B_i - x). \end{aligned} \tag{8}$$

□

Proof of Proposition 1: Let z, z_1 be the endvertices of the path P_n. The formula (8) gives

$$\mathfrak{p}_n = 2\mathfrak{p}^\star_{n-1} + \mathfrak{p}_{n-2} \tag{9}$$

and

$$\mathfrak{p}^\star_n = \mathfrak{p}^\star_{n-1} + \mathfrak{p}_{n-2} \tag{10}$$

whence

$$\mathfrak{p}_n = \mathfrak{p}^\star_n + \mathfrak{p}^\star_{n-1} \tag{11}$$

as well as $\mathfrak{p}^\star_{n-1} = \mathfrak{p}^\star_n - \mathfrak{p}_{n-2}$. Consequently $\mathfrak{p}_n = 2\mathfrak{p}^\star_n - \mathfrak{p}_{n-2}$, i.e. $\mathfrak{p}^\star_n = \frac{1}{2}(\mathfrak{p}_n + \mathfrak{p}_{n-2})$, which together with (11) gives the recurrence equation (3) for $\mathfrak{p}$. The recurrence (3) for $\mathfrak{p}^\star$ follows from (9) on using twice the equality (11), firstly in order to eliminate $\mathfrak{p}_n$, and next $\mathfrak{p}_{n-2}$. It is easily seen that $\mathfrak{p}^{\star\star}_n =_{a(z)} \mathfrak{p}^{\star\star}_{n-1} + \mathfrak{p}^\star_{n-2}$ and $\mathfrak{p}^\star_{n-2} =_{z_1} \mathfrak{p}^{\star\star}_{n-2} + \mathfrak{p}^{\star\star}_{n-3}$, which together give the recurrence (3) for $\mathfrak{p}^{\star\star}$. □

The following table will be helpful.

Remark. The characteristic equation of the recurrence (3),

$$\lambda^3 = \lambda^2 + \lambda + 1, \tag{12}$$

Table 1:

n	0	1	2	3
$\mathfrak{p}_n$	1	1	3	5
$\mathfrak{p}_n^\star$	0	1	2	3
$\mathfrak{p}_n^{\star\star}$	0	1	1	2

has exactly one real root $\lambda_0 = 1.839^+$. Standard numerical methods lead to the following general solution

$$\mathfrak{p}_n = C_1{\lambda_0}^n + C_2\rho^n \cos n\theta + C_3\rho^n \sin n\theta$$

of the recurrence (3) because the real λ_0 and non-real numbers $\rho\exp(\pm \mathrm{i}\theta)$ are the three roots of the characteristic equation (12) (with $\rho = 0.737^+$). The initial conditions for $\mathfrak{p}_n$ determine the above constants C_j, in particular $C_1 = 0.801^+$. Hence $\mathfrak{p}_n \sim C_1{\lambda_0}^n$. The bound $\mathfrak{p}_n \leq {\lambda_0}^n$ (attained for $n = 0$ only) follows by induction. The constant factor in the upper bound $\mathfrak{p}_n \leq {\lambda_0}^n$ above can be lowered for larger n's, e.g. $\mathfrak{p}_n \leq 3{\lambda_0}^{n-2}$ for $n \geq 1$, with equality for $n = 2$, and $\mathfrak{p}_n \leq 17{\lambda_0}^{n-5}$ for $n \geq 3$, with equality for $n = 5$.

The formula (4) for NDS in a palm tree follows immediately from the basic formula (8) if x therein is the vertex with maximum degree in the palm.

In order to identify palms with large and small NDS' we investigate the difference

$$\begin{aligned} d_D &:= \partial(P(n,D)) - \partial(P(n,D-1)) \qquad \text{where } D \geq 3,\ n \geq D+1, \\ &= 2^{D-2}(2\mathfrak{p}^\star_{n-D+1} - \mathfrak{p}^\star_{n-D+2}) + \mathfrak{p}_{n-D} - \mathfrak{p}_{n-D+1} \qquad \text{by (4)}. \end{aligned}$$

Hence, for large $D \geq n-3$ and $D \geq 3$, using the above table gives:

$d_D = 2^{n-3} - 2 > 0$ for $D = n-1$ if $n \geq 5$; $d_D = -2$ for $D = n-2$ if $n \geq 5$;

$d_D = 2^{n-5} - 4 > 0$ for $D = n-3$ if $n \geq 8$.

Otherwise, on using twice both (3) for $\varphi = \mathfrak{p}, \mathfrak{p}^\star$, each time in order to eliminate the largest subscript, we get

$$\begin{aligned} d_D &= 2^{D-2}\mathfrak{p}^\star_{n-D-2} - 2\mathfrak{p}_{n-D-2} - \mathfrak{p}_{n-D-3} - \mathfrak{p}_{n-D-4}, \\ &= (2^{D-2} - 3)\mathfrak{p}^\star_{n-D-2} - 2\mathfrak{p}^\star_{n-D-3} - \mathfrak{p}^\star_{n-D-4}, \qquad \text{by (11) and (3)}, \end{aligned} \tag{13}$$

for $D \geq 3$ and $n \geq D+4$.

Summarizing, due to monotonicity of the function $\mathfrak{p}^\star$, we see that $d_D > 0$ for $5 \leq D \leq n-3$ or for $D = n-1$ with $n \geq 5$. On the other hand, $d_D < 0$ if $D = n-2 \geq 3$ as well as if $D = 3$ and $n \geq 5$; and if $D = 4$ and $n \geq 9$.

Proposition 4 *For $n \geq 9$,* NDS *on the path P_n is strictly between* NDS*' on palm trees $P(n,D)$ with $D = 5,6$. In fact,*

$$\begin{aligned} \Delta_{n,5} &:= \partial(P(n,5)) - \partial(P_n) < 0 \qquad \text{for } n \geq 7, \\ \Delta_{n,6} &:= \partial(P(n,6)) - \partial(P_n) > 0 \qquad \text{for } n \geq 9. \end{aligned}$$

Proof: Because $P_n = P(n, D)$ with $D = 2$,

$$\begin{aligned}\Delta_{n,5} &= d_5 + d_4 + d_3\\ &= -\mathfrak{p}^\star_{n-5} - \mathfrak{p}^\star_{n-6} + 2\mathfrak{p}^\star_{n-7} - 3\mathfrak{p}^\star_{n-8} - \mathfrak{p}^\star_{n-9}, \qquad \text{by (13), for } n \geq 9,\end{aligned}$$

which is negative because the function $\mathfrak{p}^\star$ is nonnegative and strictly increasing. Similarly,

$$\begin{aligned}\Delta_{n,6} &= d_6 + \Delta_{n,5}\\ &= -\mathfrak{p}^\star_{n-5} - \mathfrak{p}^\star_{n-6} + 2\mathfrak{p}^\star_{n-7} + 10\mathfrak{p}^\star_{n-8} - 3\mathfrak{p}^\star_{n-9} - \mathfrak{p}^\star_{n-10} \qquad \text{for } n \geq 10.\end{aligned}$$

Hence using recurrence (3) for $\varphi = \mathfrak{p}^\star$ four times, each time to decrease the largest subscript at $\mathfrak{p}^\star$, gives

$$\Delta_{n,6} = 4\mathfrak{p}^\star_{n-10} + 6\mathfrak{p}^\star_{n-11} > 0 \qquad \text{for } n \geq 11.$$

The lacking inequalities for $n = 7, 8, 9, 10$ follow from (4) and recurrence (3). □

The concluding result on NDS in palm trees follows.

Proposition 5 NDS, *as a function of an integer D among n-vertex palms $P(n, D)$ with $n \geq 9$, has exactly three strong local extremes: local minima at $D = 4, n-2$ and local maximum at $D = n-3$. The global maximum is at $D = n-1$, only in the star if $n \geq 6$.* □

4 Domination #-minimal trees

Recall that a tree T is called ∂-minimal if NDS in T, $\partial(T)$, is the smallest possible among all trees of order $|T|$. Let $\mathcal{T}^m$ denote the *subclass* of $\mathcal{T}^{\min}$ comprising ∂-minimal trees of order $n \equiv m \pmod 3$, $m = 0, 1, 2$. The next subclass, with fixed n, is denoted by $\mathcal{T}^{\min}_n$ or $\mathcal{T}^m_n$.

Table 2:

n	1	2	3	4	5	6	7	8	9	10
$\lvert\mathcal{T}^{\min}_n\rvert$	1	1	1	2	1	1	3	2	1	8

Lemma 6 *Let z be a leaf in a nontrivial tree T. Then*

$$\partial_z(T) - \partial(T - z) \geq 0$$

with equality if and only if there is another leaf, y, attached to the neighbor, $a(z)$, of z in T, $y \neq z$.

Proof: The inequality is clear. To avoid trivialities, assume that $|T| \geq 3$ and suppose that T does not have any such leaf y. Then $|T| > 3$ and T has a dominating set, say S, which includes z and avoids both $a(z)$ and all neighbors of $a(z)$ different from z. Hence S is not dominating in $T - z$. This shows that the existence of y is necessary for the equality to hold. Sufficiency is evident. □

Corollary 7 *If z is one of two leaves with a common neighbor in a tree T then $\partial_z(T) = \partial(T - z)$.* □

From now on, for a tree T with $|T| \geq 2$, *k-addition* (or ad_k) stands for a local augmentation which is the operation $T \mapsto \mathrm{ad}_k(T)$ of adding to T a pendant k-star with k new vertices and k new edges, $k \geq 2$, so that a vertex of T which is adjacent to a leaf of T (and is not a leaf if $|T| \geq 3$) is identified with a leaf of the k-star.

Corollary 7 and Proposition 3 imply the following.

Corollary 8 *If a tree $\widetilde{T} = ad_k(T)$ where $k \geq 2$ then $\partial(\widetilde{T}) = (2^{k-1} + 1)\,\partial(T)$.* □

Theorem 9 *Every pendant star in a minimal tree of order $n \geq 5$ is attached to a vertex which has a leaf as a neighbor.*

Proof: Let T' be a minimal tree in which a pendant star has exactly ℓ leaves, $\ell \geq 1$, and has central vertex, x, attached to the vertex $a(x)$ nonadjacent to any leaf of T'. Let $T = T' + xy - \{xa(x)\}$ where y is a vertex of $T' - x$ adjacent to a leaf of T'. Let H be a nontrivial component of $T - x$. Then H contains vertices y and $a(x)$. Let B and B' be branches both at x in T and T', respectively. Then $H = B - x = B' - x$. Hence and due to formula (8),

$$\partial(T') - \partial(T) = 2^{\ell}(\partial_x(B') - \partial_x(B)) > 0$$

(because $\partial_x(B') > \partial(H) = \partial_x(B)$ by Lemma 6 and Corollary 7, respectively), a contradiction. □

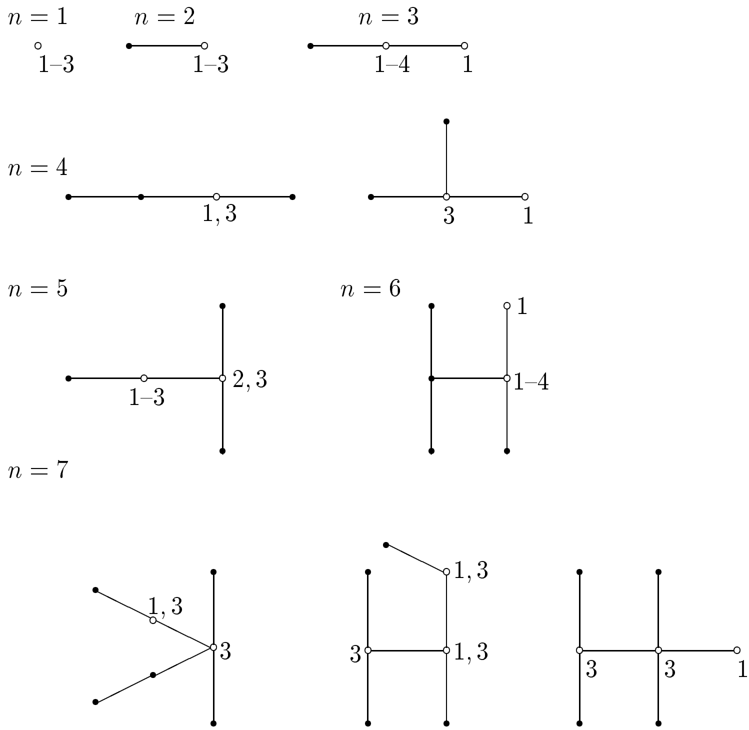

Fig. 1. All small ∂-minimal trees and possible local augmentations

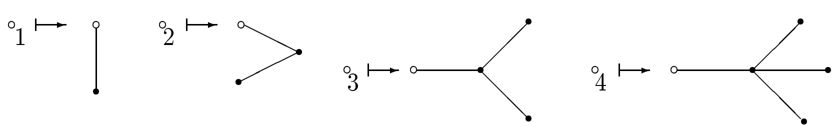

Fig. 2. Local augmentations

The list of all ∂-minimal trees of order up to 10 has been found by inspection. Numbers of those trees are given in Table 2. Trees themselves of order up to seven are depicted in Fig. 1. We have noted that larger ∂-minimal trees are obtainable by so-called local augmentations, the augmentations being encoded in Fig. 2. These are four k-vertex augmentations (i.e. k new vertices are added, $1 \leq k \leq 4$). Just such augmentations are indicated in Fig. 1 by using numbers k at some vertices. Each (up to isomorphism) of such augmentations is indicated there if it results in a larger ∂-minimal tree.

Fig. 1 is a data base which can be used for determining the rules of producing larger ∂-minimal trees from smaller ones. It can be seen that the lacking ∂-minimal trees of orders 8–10 are obtainable by applying 2- and 3-additions only. In fact, it appears that 1-vertex augmentations can be neglected.

In the sequel, if $\widetilde{T}$ has a pendant k-star S_k $(= K_{1,k})$ with $k = 2, 3$ and $|\widetilde{T}| \geq k+2$, the symbol $\widetilde{T} - S_k$ denotes the tree resulting from $\widetilde{T}$ on removing the center of S_k together with incident edges and the resulting isolated vertices.

Let M_n denote the n-vertex caterpillar with $n = 3k+1$, $k \geq 1$, which is obtainable from a path P on $k+1$ vertices by attaching two new leaves to every inner vertex of P and a single new leaf to each endvertex of P. Then $M_4 = P_4$ and

$$M_n \text{ is of the form } \quad \mathrm{ad}_2(M_n - S_2) \quad \text{where } \; M_n - S_2 \in \mathcal{T}^2. \tag{14}$$

Note that from Fig. 1 we get

$$\mathcal{T}_2^{\min} = \{P_2\}, \qquad \mathcal{T}_3^{\min} = \{P_3\}, \qquad \mathcal{T}_4^{\min} = \{P_4, K_{1,3}\}. \tag{15}$$

Let $\mathrm{ad}_k[\mathcal{T}]$ stand for the image of a class $\mathcal{T}$ under the operation ad_k. The main result of this section follows.

Theorem 10 *For* $n \geq 2$

$$\mathrm{NDS}_{(n)} = \begin{cases} 5^{n/3} & \text{if } 3 \mid n, \\ 9 \cdot 5^{(n-4)/3} & \text{if } n \equiv 1 \pmod 3, \\ 3 \cdot 5^{(n-2)/3} & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{16}$$

For $n \geq 5$*, all n-vertex ∂-minimal trees are obtainable recursively by* 3- *or possibly* 2-*additions only, namely for* $m = n \bmod 3$,

$$\mathcal{T}_n^m = \begin{cases} \mathrm{ad}_3[\mathcal{T}_{n-3}^m] & \text{for } m = 0, 2, \\ \mathrm{ad}_3[\mathcal{T}_{n-3}^1] \cup \{M_n\} & \text{for } m = 1. \end{cases} \tag{17}$$

The construction (15), (17) of minimal trees is made more clear by the following auxiliary observation.

Proposition 11 *Let* $T \in \mathcal{T}_n^{\min}$ *and* $m = n \bmod 3$*. Then, for* $n \geq 5$*, T has two or more pendant k-stars, all k-stars with* $k = 3$ (*which is the case if* $m = 0$) *unless possibly one with* $k = 2$ *for* $m = 2$*, or otherwise* $m = 1$ *and possibly either one is with* $k = 4$ *or either one or two is/are with* $k = 2$.

Remark 1. If $\widetilde{T} = \mathrm{ad}_4(T)$ for a $T \in \mathcal{T}^0$ (whence $\widetilde{T} \in \mathcal{T}^1$ and $|\widetilde{T}| \geq 7$) then ad_4 can be replaced by ad_3 because there exists a pendant 3-star, S_3, of $\widetilde{T}$ whence $\widetilde{T}$ can be $\mathrm{ad}_3(\widetilde{T} - S_3)$ where $\widetilde{T} - S_3 \in \mathcal{T}^1$.

Remark 2. If $\widetilde{T}$ is obtained by a 2-addition from a $T \in \mathcal{T}^0 \cup \mathcal{T}^2$ then $\widetilde{T}$ can be obtained from a $T_2 := \widetilde{T} - S_3$ by the 3-addition unless $\widetilde{T} = M_n$ with m $(= n \bmod 3) =$ 1, $n \geq 4$ (and $T \in \mathcal{T}^2$).

Proof of Theorem 10: Proceed by induction. By using Proposition 1 for paths, formula (5) for stars, or the recursive formula in Corollary 8, we first show that RHS of (16) is equal to $\partial(T)$ for each initial tree T in (15) as well as for each remaining T in Fig. 1. So is $\partial(\widetilde{T})$ for each n-vertex tree $\widetilde{T}$ belonging to RHS of (17), which follows from Corollary 8 by the induction hypothesis. For instance, if $m = 1$ then $\widetilde{T} = \mathrm{ad}_k(T)$ with $k = 3$ or 2, see formula (14), whence $T \in \mathcal{T}^{4-k}_{n-k}$ and consequently $\partial(\widetilde{T}) = (2k-1)\partial(T) = 9 \cdot 5^{\frac{n-4}{3}}$ can be seen for both k, as is required. Thus RHS of (16) is the exact value of $\mathrm{NDS}_{(n)}$ for $n \le 7$ only and an upper bound otherwise. Take now any $\widetilde{T} \in \mathcal{T}^{\min}_n$ with $|\widetilde{T}| = n \ge 8$. Then $\widetilde{T}$ has two or more pendant stars because $\partial(K_{1,n-1})$ is too large. Due to Theorem 9, $\widetilde{T}$ is obtained from a smaller tree T by applying an operation ad_k for some $k \ge 2$. Moreover, if $\widetilde{T} = \mathrm{ad}_k(T)$ then T is of order $n-k$ and, by Corollary 8, T is minimal, $T \in \mathcal{T}^{\min}_{n-k}$. Furthermore, $k = 4$ for $m = 1$ is possible since then, by Corollary 8, $\partial(\widetilde{T}) = 9 \cdot \partial(T) = 9 \cdot 5^{\frac{n-4}{3}}$ because $T \in \mathcal{T}^0_{n-4}$. However, $k = 4$ can be omitted, cf Remark 1 above. Thus $k = 3$ for $m = 0$, 2 and $k = 2$, 3, or 4 for $m = 1$. In each of remaining cases if $T \in \mathcal{T}^{\min}_{n-k}$ then $\partial(\widetilde{T}) > \mathrm{NDS}_{(n)}$ can be seen. For example, if $m = 0$ (whence $\widetilde{T} \in \mathcal{T}^0$) but $k = 2$ (whence $T \in \mathcal{T}^1$), then, by Corollary 8, $\partial(\widetilde{T}) = 3\partial(T) = 3 \cdot 9 \cdot 5^{\frac{n-6}{3}} = \frac{27}{25} \cdot 5^{\frac{n}{3}} > 5^{\frac{n}{3}}$. Thus the proof is complete. □

Corollary 12 *Let* $T \in \mathcal{T}^{\min}_n$ *and* $m = n \bmod 3$. *Then* $\mathrm{ad}_k(T) \in \mathcal{T}^{\min}_{n+k}$ *if and only if* $m = 0$ *and* $k \in \{2, 3, 4\}$, $m = 2$ *and* $k \in \{2, 3\}$, *or* $m = 1$ *and* $k = 3$. □

The construction of minimal trees described in Theorem 10 and Proposition 11 implies the following structural properties of the set of leaves. Let T be an n-vertex minimal tree, $T \in \mathcal{T}^{\min}_n$, and let $m = n \bmod 3$, $n \ge 4$. Then each vertex is either a leaf or a neighbor of a leaf in T and the set of leaves is split into 2-element subsets but possibly η subsets of cardinality 1 or 3 with $\eta \le 2$ and such that leaves in each of subsets have one private neighbor (which implies that different subsets have different neighbors). Moreover, for $m = 0$, $\eta = 0$ (and the number of leaves is even). If $m = 2$, the number of leaves is odd, $\eta = 1$, and the unique exceptional subset is a singleton. For $m = 1$, if the number of leaves is odd, then $\eta = 1$ and the exceptional subset has 3 leaves. Otherwise the number of leaves is even, $\eta = 2$, and both exceptional subsets are singletons.

Thus, given any $T \in \mathcal{T}^{\min}_n$ with $n \ge 3$, removal of all leaves from T gives a unique tree, say $\widetilde{T}$, of order $\lceil \frac{n}{3} \rceil$ or possibly $(n-1)/3$. Conversely, attaching new leaves to each vertex of a small tree $\widetilde{T}$ can give a member of $\mathcal{T}^{\min}_n$. This is made precise below.

Theorem 13 *Assume that* $k \ge 1$. *The value of* m, $m = 0, 1, 2$, *differentiates between three cases:*

(0) *The class* $\mathcal{T}^{\min}_{3k}$ *is obtainable from the class of all* k*-vertex trees* $\widetilde{T}$ *by the following operation: join two new leaves to every vertex of* $\widetilde{T}$.

(2) *The class* $\mathcal{T}^{\min}_{3k+2}$ *(containing trees with* $2k+1$ *leaves) is obtainable from the class of all* $(k+1)$*-vertex trees* $\widetilde{T}$ *by joining one new leaf to a vertex, say* y, *of* $\widetilde{T}$ *and two new leaves to each of remaining vertices of* $\widetilde{T}$.

(1) *A tree $T \in \mathcal{T}^{\min}_{3k+1}$ is obtainable either from a $(k+1)$-vertex tree $\widetilde{T}$ or from a k-vertex tree $\widetilde{T}'$. Namely, if the number of leaves in T is even, i.e., $2k$, then T is obtainable from $\widetilde{T}$ by choosing two vertices in $\widetilde{T}$ and joining one new leaf to each of them, and then joining two new leaves to each of remaining vertices in $\widetilde{T}$. If T has an odd number of leaves, which is necessarily $2k+1$, T is obtainable from $\widetilde{T}'$ by joining three leaves to any one vertex of $\widetilde{T}'$ and by joining two new leaves to each of remaining vertices of $\widetilde{T}'$.* □

Corollary 14 *The cardinality of $\mathcal{T}^{\min}_n$ is exponential because it is not less than the number of all trees on $\lceil \frac{n}{3} \rceil$ vertices, the lower bound being sharp precisely if $3 \mid n$ and $n \geq 6$.* □

5 Domination #-maximal forests and trees

A forest (or tree) F is called *∂-maximal* (or *domination #-maximal*) if NDS in F is the largest possible among all forests (trees) of order $|F|$. By inspection we can see that five paths P_n, two stars $K_{1,k}$ with $k = 3, 4$, and the disconnected forest $2K_2$ make up the family of ∂-maximal forests of order $n \leq 5$. The forest $2K_2$ appears to be unique in general.

Theorem 15 *$2K_2$ is the only disconnected forest which is ∂-maximal.*

It is enough to show the following.

Lemma 16 *Let F be a forest with two or more components such that $F \neq 2K_2$. Assume that a spanning supergraph F^+ of F is obtained by adding one edge or two edges to F as follows. If $F = sK_2$ with $s \geq 3$ then two edges are added so that F^+ has P_6 as a component. Otherwise exactly one edge, $e = xy$, is added such that endvertices x, y are in different components of F and there is no leaf among neighbors of x (or y) in F (e.g., x is a leaf in a component $\neq K_2$). Then $\partial(F^+) > \partial(F)$.*

Proof: Assume that $F = sK_2$ with $s \geq 3$. Then by (3) and (6), $\partial(F^+) > \partial(F)$ because $\partial(P_6) = 31 > \partial(3K_2) = 3^3$. Consider $F \neq sK_2$ with $s \geq 2$. It is enough to show that F^+ has a dominating set, S^+, which does not dominate F. Note that there is a set, $\tilde{S}$, which is dominating in $F - x$ and includes neither x nor any neighbor of x. Then, for any fixed $\tilde{S}$, the set $S^+ := \tilde{S} \cup \{y\}$ is dominating in F^+ but not in F. □

By the *root* of a star or of a path we mean any of its leaves. By the *root of a palm* $P(n, D)$ with $2 < D < n-1$ we mean the only leaf whose neighbor is of degree 2. Given a palm $P(n, D)$, consider vertices x, z and r where x is of maximum degree D, z is a leaf incident to x, and r is a root; $z \neq r$ if $2 < D < n-1$ and $n \geq 5$. Due to (1)

$$\partial_r(P(n, D)) =_x 2^{D-1}\mathfrak{p}^{\star\star}_{n-D+1} + \mathfrak{p}^{\star}_{n-D}, \tag{18}$$

$$\partial_z(P(n, D)) =_x 2^{D-2}\mathfrak{p}^{\star}_{n-D+1} + \mathfrak{p}_{n-D}.$$

Lemma 17 *Given a palm $P(n,D)$ with $n \geq 5$ and $2 < D < n-1$, let r and z be respectively the root and another leaf of the palm. Then $\partial_r(P(n,D)) > \partial_z(P(n,D))$.*

Proof: The two preceding formulas give

$$\begin{aligned} \partial_r(P(n,D)) - \partial_z(P(n,D)) &= (2^{D-2}-1)\mathfrak{p}^\star_{n-D-1} \\ &> 0 \qquad \text{due to (10) and Table 1.} \end{aligned}$$ □

Using (4) and (18) we get the following table of the largest possible NDS' attained at corresponding palms $P(n,D)$.

Table 3: For $n \geq 6$

D	$\partial(P(n,D))$	$\partial_r(P(n,D))$
$n-1$	$2^{n-1}+1$	$2^{n-2}+1$
$n-2$	$3 \cdot 2^{n-3}+3$	$2^{n-2}+2$
$n-3$	$3 \cdot 2^{n-3}+5$	$2^{n-2}+3$
$n-4$	$11 \cdot 2^{n-5}+9$	$7 \cdot 2^{n-5}+6$

Proposition 18 *Table 4 found by inspection presents n-vertex trees T at which the largest possible NDS' are attained; $P_{n,D}$ therein stands for $P(n,D)$, $z = r$, and $2 \leq n \leq 9$.* □

Table 4:

n	2	3	4	5	6		7		8		9		
T	P_2	P_3	P_4	P_5	P_6	$P_{6,5}$	P_7	$P_{7,6}$	P_8	$P_{8,7}$	$P_{9,6}$	$P_{9,8}$	P_9
$\partial_z(T)$	2	3	6	11	20	17	37	33	68	65	131	129	125
$\partial(T)$	3	5	9	17	31	33	57	65	105	129	197	257	193

Remark. Part (b) of the following theorem is an auxiliary result which helps to prove the main part (a). The following general observation is clear.

Proposition 19 *For any vertex x of an n-vertex tree T,*

$$\partial_x(T) \leq 2^{n-1}, \tag{19}$$

the equality being attained exactly when $T = K_{1,n-1}$, $n \geq 1$, and x is of maximum degree in T.

Theorem 20 (a) *The four largest* NDS' *in descending order among n-vertex trees T are attained precisely at $T = K_{1,n-1}$ (for $n \geq 6$), $P(n,n-3)$ (for $n \geq 9$), $P(n,n-2)$ (for $n \geq 9$) and $P(n,n-4)$ (for $n \geq 10$), for $n = 9$ the 4th largest value,* 193*, being attained at the path P_9 only.*

(b) *If z is a leaf of T, the four largest values of $\partial_z(T)$ in descending order for $n \geq 9$ are attained precisely at palm trees $P(n, D)$ with $D = n-3, n-2, n-1, n-4$ provided that z is the root r of the palm, with two exceptions: the fourth value is attained only at P_9 if $n = 9$ and at P_{10} as well as $P(10, 6)$ if $n = 10$.*

Proof: Proceed by induction on n. Using the tree diagrams in Harary [3] for $n \leq 10$ we can see that all data in Table 4 are correct. In fact, the largest value of $\partial(T)$ is attained at $T = P_n$ for $2 \leq n \leq 5$ and at $T = K_{1,n-1}$ for $n \geq 4$; the second largest value at $T = P_n$ for $n = 6, 7, 8$ only. If z is a leaf of T, the largest value of $\partial_z(T)$ is attained at $T = P_n$ for $2 \leq n \leq 8$; the second largest value at $T = P(n, n-3)$ for $n = 6, 7, 8$ only, and at $T = S(K_{1,3})$ (the subdivision of the claw $K_{1,3}$) for $n = 7$, z therein being necessarily a root of T if T is a palm. Furthermore, what is stated in theorem for $n \leq 10$ can similarly be verified. Moreover, there is no other extremal tree. (Here and in the rest of this proof, *extremal tree* is a tree with NDS being equal to the NDS in any tree listed in Theorem.)

Passing on to the next step of induction we take any $n > 10$ and an n-vertex tree T.

Part (a)**.** Assume that T is different from each of would-be extremal palms $P(n, n-i)$, $i = 1, 2, 3, 4$. Let y be the center of a pendant star in T with k rays where k is as small as possible, $2 \leq k \leq \frac{n}{2}$, and let B stand for the unique non-K_2 branch of T at y. From Proposition 3,

$$\partial(T) =_y 2^{k-1}\partial_y(B) + \partial(B - y). \tag{20}$$

Let n_B, $n_B = n - k + 1$, be the order of the branch B. Consider three cases.

Case $a1$: $k \geq 3$, $n_B \geq 6$, $B_i := P(n_B, n_B - i)$, T_i stands for T with $B = B_i$ where $i = 1, 2, 3, 4$, and y is a root of B. Then, by (20) and Table 3 (with $r = y$), we have

$$\partial(T_1) =_y 2^{k-1}\partial_y(K_{1,n_B-1}) + \partial(K_{1,n_B-2}) = 2^{n-2} + 2^{k-1} + 2^{n-k-1} + 1,$$

$$\partial(T_2) =_y 2^{k-1}\partial_y(P(n_B, n_B-2)) + \partial(P(n_B-1, n_B-2)) = 2^{n-2} + 2^{k} + 2^{n-k-1} + 1,$$

$$\begin{aligned}\partial(T_3) &=_y 2^{k-1}\partial_y(P(n_B, n_B-3)) + \partial(P(n_B-1, n_B-3)) \\ &= 2^{n-2} + 3\cdot 2^{k-1} + 3\cdot 2^{n-k-3} + 3,\end{aligned}$$

$$\begin{aligned}\partial(T_4) &=_y 2^{k-1}\partial_y(P(n_B, n_B-4)) + \partial(P(n_B-1, n_B-4)) \\ &= 7\cdot 2^{n-5} + 6\cdot 2^{k-1} + 3\cdot 2^{n-k-3} + 5,\end{aligned}$$

$$\partial(P(n, n-4)) - \partial(T_1) = 3\cdot 2^{n-5} - 2^{k-1} - 2^{n-k-1} + 8 > 0,$$

$$\partial(P(n, n-4)) - \partial(T_2) = 3\cdot 2^{n-5} - 2^{k} - 2^{n-k-1} + 8 > 0,$$

$$\partial(P(n, n-4)) - \partial(T_3) = 3\cdot 2^{n-5} - 3\cdot 2^{k-1} - 3\cdot 2^{n-k-3} + 6 > 0$$

$$\partial(P(n, n-4)) - \partial(T_4) = 2^{n-3} - 3\cdot 2^{k} - 3\cdot 2^{n-k-3} + 4 > 0$$

because $n > 10$ and $3 \leq k \leq n/2$. Thus no T_i is extremal.

Case $a2$: $k = 2$, $B = P(n_B, n_B - i)$ for $i = 2, 3$ or 4, and y is not a root of B, or otherwise $B \neq P(n_B, n_B - i)$ for $i = 1, 2, 3, 4$ only. Since $n_B = n - 1 \geq 10$, using induction hypothesis and Table 3 gives

$$\partial_y(B) < 7\cdot 2^{n-6} + 6, \qquad \partial(B - y) \leq 11\cdot 2^{n-7} + 9$$

(because $|B-y| = n-2$) whence, by (20), $\partial(T) < 39 \cdot 2^{n-7} + 21$. Therefore, by Table 3,

$$\partial(P(n, n-4)) - \partial(T) > 5 \cdot 2^{n-7} - 12 > 0.$$

Case $a3$: $k \geq 3$, $B \neq P(n_B, n_B - i)$ for $i = 1, 2, 3, 4$, and each pendant star of T has k or more rays. Let $n_B \geq 10$, $n_B = n - k + 1$. Then, by the induction hypothesis and Table 3,

$$\partial_y(B) < 7 \cdot 2^{n_B - 5} + 6, \qquad \partial(B - y) \leq 11 \cdot 2^{n_B - 6} + 9$$

whence, by (20), $\partial(T) < 7 \cdot 2^{n-5} + 11 \cdot 2^{n-k-5} + 3 \cdot 2^k + 9$. Therefore

$$\partial(P(n, n-4)) - \partial(T) > 2^{n-k-5}(2^{k+1} - 11) + 2^{n-4} - 3 \cdot 2^k > 0, \text{ since } n \geq 9 + k \geq 12.$$

Otherwise, $n_B \leq 9$ and $k = n - n_B + 1 \geq 12 - n_B$. Hence $k \geq 6, 5$ if respectively $n_B \leq 6, 7$. Therefore there is no required B of order $n_B \leq 7$. By inspection of the Harary list in [3] we find all possible trees B with $n_B = 8, 9$. Let $n_B = 8$. Then $k \geq 4$ and there are four trees B of which one is good if $k = 5$, too. For $n_B = 9$, $k \geq 3$ and there are 23 trees B of which one is good for k up to 6. The sharp upper bounds on $\partial_y(B)$ and $\partial(B - y)$ are respectively 54 and 53 if $n_B = 8$, on the other hand, 116 and 101 if $n_B = 9$ (the four bounds being attained at distinct trees B). Since $n = k + n_B - 1$, using formula (20) and Table 3 gives

$$\partial(P(n, n-4)) - \partial(T) \geq 11 \cdot 2^{k + n_B - 6} + 9 - \begin{cases} 54 \cdot 2^{k-1} + 53 > 0 & \text{if } n_B = 8,\ k \geq 4, \\ 116 \cdot 2^{k-1} + 101 > 0 & \text{if } n_B = 9,\ k \geq 3. \end{cases}$$

Part (b). Assume that either T is different from extremal trees, $T \neq P(n, n-i)$, $i = 1, 2, 3, 4$, or otherwise z is not the root, r, of $T = P(n, n-i)$ with $i = 2, 3$, or 4 ($i \neq 1$). In the latter case $\partial_z(T) = \partial_z(P(n, n-i))$ which, due to Lemma 17 and Tables 3 and 1, is too small because it is smaller than any value in the last column in Table 3. In the former case, assume that $\partial_z(T)$ is the largest possible. Let x be the only neighbor of z, let ℓ be the number of leaves attached to x, and let the degree $\deg_T x = t + \ell$, $t \geq 1$, $\ell \geq 1$. If $\ell > 1$ then, due to Corollary 7, $\partial_z(T) = \partial(T - z) \leq 11 \cdot 2^{n-6} + 9$ by induction hypothesis and Table 3. For the same reason as above, $\partial_z(T)$ is too small. Therefore $\ell = 1$. Let $B_1, B_2, \ldots, B_t$ denote the t non-K_2 branches of T at x. Let n_i be the order of B_i with $\sum_{i=1}^t n_i = n + t - 2$ and $n_1 \geq n_2 \geq \ldots \geq n_t \geq 3$. By Proposition 3,

$$\partial(T) =_x 2\partial_x(T - z) + \partial(T - \{z, x\}).$$

Hence and from (1)

$$\begin{aligned} \partial_z(T) &= \partial(T) - \partial_{\hat{z}}(T) = \partial(T) - \partial_x(T - z) \qquad \text{by (2)} \\ &= \partial_x(T - z) + \partial(T - \{z, x\}). \end{aligned} \tag{21}$$

We are going to show that $\partial_z(T)$ is too small. Assume that $t = 1$. Then x is a leaf in the subgraph $T - z$. Hence, by induction hypothesis and from Table 3,

$\partial_x(T-z) \leq 7 \cdot 2^{n-6}+6$ (because x can be the root in $T-z = P(n-1,(n-1)-4)$) and the second summand $\partial(T-\{z,x\}) \leq 3 \cdot 2^{n-5}+5$ because $T-\{z,x\}$ cannot be the star on $n-2$ vertices (since $T \neq P(n,n-i)$ with $i=2,3$). Therefore $\partial_z(T) \leq (7/2+3)2^{n-5}+11 < \partial_r(P(n,n-4))$ as claimed.

Consider the remaining case that $t>1$. We are going to reduce it to the preceding case $t=1$. Firstly, both the summands in (21) are products:

$$\partial_x(T-z) = \prod_{i=1}^{t} \partial_x(B_i) \qquad \text{by Proposition 3,}$$

$$\partial(T-\{z,x\}) = \prod_{i=1}^{t} \partial(B_i - x) \qquad \text{by formula (6).}$$

Let $\tau = t-1$. Assume that T' is obtained from T by replacing two branches B_t and B_τ by a single tree B'_τ which is made a branch at x in T'. In order to have T' of order n we find a tree B'_τ of order $n'_\tau := n_t + n_\tau - 1$ (≥ 5) and such that x is a leaf of B'_τ with $\partial_x(B'_\tau)$ being as large as possible. Therefore B'_τ is among trees, actually palms, in Table 4 if $n'_\tau \leq 8$ or otherwise, due to induction hypothesis, B'_τ is the palm tree of order $n'_\tau \geq 9$ with $D = n'_\tau - 3$, cf. Table 3. Then $x=r$, the root of B'_τ, due to Lemma 17.

In the following table $\pi' = \varphi'_{n_t} \cdot \varphi'_{n_\tau}$ and $\pi'' = \varphi''_{n_t} \cdot \varphi''_{n_\tau}$ where φ'_{n_j} and φ''_{n_j} stand for the largest possible values of $\partial_x(B_j)$ and $\partial(B_j - x)$, respectively, $j = t, \tau$, the largest values being taken from Table 4 (if $n_j \leq 9$) or Table 3. For all pairs of small orders $3 \leq n_t \leq n_\tau \leq 8$, which appear in Table 5, $\partial_z(T') > \partial_z(T)$ because

$$\partial_r(B'_\tau) > \pi' \text{ (strongly) and} \qquad \partial(B'_\tau - r) \geq \pi''$$

which follows from Table 5. Similarly we proceed in cases $n_t \leq 9 \leq n_\tau$ and $9 < n_t$. □

6 Concluding remarks

The method of "short steps" (mentioned in the Introduction) is best seen in Sect. 3 where it establishes monotonicity and local extrema of the NDS among palm trees. This contributes decisively to the inductive proof in the preceding section that a few largest NDS' are attained at palms with maximum degree $\Delta = n-1, n-3, n-2, n-4$ provided that the order, n, is not too small.

Our characterization of domination #-minimal trees in terms of attaching new leaves to all vertices of any tree suggests similar form of Ravindra's description of well-covered trees. Recall that a tree T is called *well-covered* if each maximal independent (vertex) set in T is of maximum cardinality. T is known to be well-covered [9] if and only if T has a perfect matching consisting of pendant edges. Thus there is a bijection $\tilde{T} \mapsto T$ from trees $\tilde{T}$ on k vertices onto the family of well-covered trees T on $2k$ vertices. Namely, attaching a single new leaf to each vertex of a tree $\tilde{T}$ defines such a bijection.

Table 5:

n_t	n_τ	π'	π''	B'_τ	$\partial_r(B'_\tau)$	$\partial(B'_\tau - r)$	$B'_\tau - r$
3	3	$3 \cdot 3$	$3 \cdot 3$	P_5	11	9	P_4
	4	$3 \cdot 6$	$3 \cdot 5$	P_6	20	17	P_5
	5	$3 \cdot 11$	$3 \cdot 9$	P_7	37	31	P_6
	6	$3 \cdot 20$	$3 \cdot 17$	P_8	68	57	P_7
	7	$3 \cdot 37$	$3 \cdot 33$	$P(9,6)$	131	99	$P(8,6)$
	8	$3 \cdot 68$	$3 \cdot 65$	$P(10,7)$	259	195	$P(9,7)$
4	4	$6 \cdot 6$	$5 \cdot 5$	P_7	37	31	P_6
	5	$6 \cdot 11$	$5 \cdot 9$	P_8	68	57	P_7
	6	$6 \cdot 20$	$5 \cdot 17$	P_9	125	105	P_8
	7	$6 \cdot 37$	$5 \cdot 33$	$P(10,7)$	259	195	$P(9,7)$
	8	$6 \cdot 68$	$5 \cdot 65$	$P(11,8)$	515	387	$P(10,8)$
5	5	$11 \cdot 11$	$9 \cdot 9$	$P(9,6)$	131	99	$P(8,6)$
	6	$11 \cdot 20$	$9 \cdot 17$	$P(10,7)$	259	195	$P(9,7)$
	7	$11 \cdot 37$	$9 \cdot 33$	$P(11,8)$	515	387	$P(10,8)$
	8	$11 \cdot 68$	$9 \cdot 65$	$P(12,9)$	1027	771	$P(11,9)$
6	6	$20 \cdot 20$	$17 \cdot 17$	$P(11,8)$	515	387	$P(10,8)$
	7	$20 \cdot 37$	$17 \cdot 33$	$P(12,9)$	1027	771	$P(11,9)$
	8	$20 \cdot 68$	$17 \cdot 65$	$P(13,10)$	2051	1539	$P(12,10)$
7	7	$37 \cdot 37$	$33 \cdot 33$	$P(13,10)$	2051	1539	$P(12,10)$
	8	$37 \cdot 68$	$33 \cdot 65$	$P(14,11)$	4099	3075	$P(13,11)$
8	8	$68 \cdot 68$	$65 \cdot 65$	$P(15,12)$	8195	6147	$P(14,12)$

Recursive characterization of trees with some extremal domination properties has recently been considered in a number of papers, see e.g. Haynes and Henning [4, 5], Henning [6, 7], Fischermann and Volkmann [2].

Investigations presented in this paper were stimulated by characterizations of trees with maximum number either of maximal independent sets [15, 10, 16] or of maximum path-factors [11, 12]. The latter problem resulted in a constrained sharp binomial inequality established by the second author in cooperation with A. Schinzel, cf [12]. The inequality gives a solution to Tomescu's number-theoretical problem [13] of maximizing a product of binomial coefficients, the simplest possible case of the problem which consists in maximizing a product of natural numbers whose sum is a constant being solved in Moon and Moser [8].

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