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Two-factorisations of complete graphs of orders fifteen and seventeen

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Abstract

There are 17 non-isomorphic 2-regular graphs of order 15, and 25 non-isomorphic 2-regular graphs of order 17. Consequently, there are $\binom{17+7-1}{7}$ $= 245,157$ possible types of 2-factorisations of K_{15}, and $\binom{25+8-1}{8}$ $=$ $10,518,300$ possible types of 2-factorisations of K_{17}. We show that all except five possible types of 2-factorisations exist for K_{15}, and that all possible types of 2-factorisations exist for K_{17}. The existence or otherwise of all possible types of 2-factorisations of K_n is now settled for all $n \leq 17$.

1 Introduction

A 2-*factor* in a graph G is a 2-regular spanning subgraph, and a 2-*factorisation* of G is a set of 2-factors in G whose edge sets partition the edge set of G. A 2-factor is said to be of *type* $[m_1, m_2, \ldots, m_t]$ if it consists of t cycles of lengths $m_1, m_2, \ldots, m_t$. The order in which the cycles in a 2-factor are listed is not important, so the number of possible types of 2-factors of order n is the number of distinct ways of partitioning n into integers $m_1, m_2, \ldots, m_t$ with $3 \leq m_1, m_2, \ldots, m_t \leq n$ and $m_1 + m_2 + \ldots + m_t = n$. A 2-factorisation of a $2d$-regular graph is said to be of *type* $[\alpha_1, \alpha_2, \ldots, \alpha_d]$ if its d 2-factors are of types $\alpha_1, \alpha_2, \ldots, \alpha_d$. Again, the order in which the types of 2-factors are listed is not important, so if there are s distinct possible types of 2-factors of order n, the number of possible types of 2-factorisations of a $2d$-regular graph of order n is $\binom{d+s-1}{s}$. We shall call the problem of determining which types of 2-factorisation of a graph G exist the 2-*factorisation problem for* G.

The 2-factorisation problem for the complete graph K_n has already been settled for all $n \leq 13$, see [7, 9]. Here we settle the problem for $n = 15$ and $n = 17$. If we let

$$A^7 = [3,4] \quad B^7 = [7] \quad A^9 = [3,3,3] \quad B^9 = [4,5] \quad C^9 = [3,6] \quad D^9 = [9]$$

$$C^{11} = [3,3,5] \quad A^{15} = [3,3,3,3,3] \quad B^{15} = [3,3,4,5] \quad G^{15} = [3,5,7]$$

$$I^{15} = [5,5,5] \quad K^{15} = [4,4,7] \quad L^{15} = [7,8]$$

(this notation is chosen to match that of [1]) then for $n \leq 17$, the following table lists every (feasible) type of 2-factorisation of K_n that does not exist.

n	Types of 2-factorisations that do not exist
$3,5$	$\emptyset$
7	$[A^7, A^7, B^7]$
9	$[A^9, A^9, A^9, B^9], [A^9, A^9, A^9, C^9], [A^9, A^9, A^9, D^9], [A^9, A^9, B^9, B^9],$ $[A^9, A^9, B^9, C^9], [A^9, A^9, B^9, D^9], [A^9, A^9, C^9, D^9], [A^9, B^9, C^9, C^9],$ $[B^9, B^9, B^9, B^9]$
11	$[C^{11}, C^{11}, C^{11}, C^{11}, C^{11}]$
13	$\emptyset$
15	$[A^{15}, A^{15}, A^{15}, A^{15}, A^{15}, B^{15}], [A^{15}, A^{15}, A^{15}, A^{15}, A^{15}, G^{15}],$ $[A^{15}, A^{15}, A^{15}, A^{15}, A^{15}, I^{15}], [A^{15}, A^{15}, A^{15}, A^{15}, A^{15}, K^{15}],$ $[A^{15}, A^{15}, A^{15}, A^{15}, A^{15}, L^{15}]$
17	$\emptyset$

Table 1: Types of 2-factorisations of K_n, $n \leq 17$, that do not exist.

We have also constructed some specific families of types of 2-factorisations of K_{19} and K_{21}, see Section 5.

For complete graphs of order more than 17, relatively little is known about the 2-factorisation problem in general, although considerable progress has been made for certain special cases of the problem. The *Oberwolfach problem* asks for a 2-factorisation of the complete graph K_n in which all the 2-factors are of the same type. The Oberwolfach problem is unsolved in general, but has been completely settled in the case where all the cycles in each 2-factor are of the same length [4, 5, 10]. A survey on the Oberwolfach problem can be found in [3]. The *Hamilton-Waterloo problem* corresponds to the 2-factorisation problem for K_n in the case where two types of 2-factor are considered, see [2, 8, 11]. The case of 2-factorisations of K_n of type $[\theta, \theta, \dots, \theta, \alpha, \beta, \gamma]$, where θ is a Hamilton cycle and α, β and γ are 2-factors of any specified types is completely settled in [6].

2 General Strategy

Naturally, our strategy involves finding numerous distinct types of 2-factorisations of various graphs. To determine the existence or otherwise of a particular type of 2-factorisation of a graph G, we use a computer search, based on recursion and backtracking. A number of fairly obvious techniques are used to reduce redundancy within the searches. In particular, canonical orderings are placed on the vertices within each cycle in each 2-factor, and on the 2-factors within each 2-factorisation.

For some of the larger searches, performance was greatly enhanced by including some randomness in the search.

The number of possible types of 2-factorisations of K_{15} and K_{17} is quite large, $245,157$ and $10,518,300$ respectively. We will make use of *circulants* to reduce the number of 2-factorisations for which we need to search. The *circulant* $C(n,S)$ is the graph with vertex set $\mathbb{Z}_n$ and edge set given by $\{x,y\} \in E(C(n,S))$ if and only if $x-y \in S$ or $y-x \in S$ (calculations being done in $\mathbb{Z}_n$). We will always have $S \subseteq \{1,2,\ldots,\lfloor (n-1)/2 \rfloor\}$ for any circulant $C(n,S)$.

We write K_{15} as the edge-disjoint union of $G_1 = C(15,\{1,2,4\})$ and $G_2 = C(15,\{3,5,6,7\})$, find all distinct types of 2-factorisations of G_1 and G_2, combine each type of 2-factorisation of G_1 with each type of 2-factorisation of G_2 to produce 2-factorisations of K_{15}, and then search directly for any missing types of 2-factorisations of K_{15}. A similar strategy is used for K_{17}. We write K_{17} as the edge-disjoint union of two copies of $G = C(17,\{1,2,4,8\})$ (note that $C(17,\{1,2,4,8\}) \cong C(17,\{3,5,6,7\})$), find all distinct types of 2-factorisations of G, combine these types in pairs to produce 2-factorisations of K_{17}, and then search directly for any missing types of 2-factorisations of K_{17}.

3 The complete graph of order fifteen

Since there are 17 non-isomorphic 2-regular graphs of order 15, there are $\binom{17+3-1}{3} = 969$ possible types of 2-factorisations of $C(15,\{1,2,4\})$. We have computationally constructed 788 of these types and verified that the remaining 181 do not exist. Those which exist (and a list of those which don't) are available on the web, see [1]. There are $\binom{17+4-1}{4} = 4845$ possible types of 2-factorisations of $C(15,\{3,5,6,7\})$. We have computationally constructed 4793 of these types, and verified that the remaining 52 do not exist; see [1].

It is straightforward, with the aid of a computer, to check which types of 2-factorisations of K_{15} can be obtained by combining an existing 2-factorisation of $C(15,\{1,2,4\})$ with an existing 2-factorisation of $C(15,\{3,5,6,7\})$. It turns out that all except 2954 of the $245,157$ possible types of 2-factorisation of K_{15} can be obtained in this manner. We have computationally constructed 2949 of these remaining types, and verified that the other 5 do not exist, see [1]. The 5 types of 2-factorisations of K_{15} which do not exist are listed in Table 1. The non-existence of these 5 types of 2-factorisations of K_{15} was observed in [8].

4 The complete graph of order seventeen

Since there are 25 non-isomorphic 2-regular graphs of order 17, there are $\binom{25+4-1}{4} = 20,475$ possible types of 2-factorisations of $C(17,\{1,2,4,8\})$. We have computationally constructed $20,460$ of these types and verified that the remaining 15 do not exist, see [1].

It is straightforward, with the aid of a computer, to check which types of 2-factorisations of K_{17} can be obtained by combining pairs of existing 2-factorisations of $C(17, \{1, 2, 4, 8\})$. It turns out that all except 480 of the $10, 518, 300$ possible types of 2-factorisation of K_{17} can be obtained in this manner. We have computationally constructed all of these remaining 480 possible types of 2-factorisations of K_{17}; see [1].

5 Conclusions

In this section we list and discuss a few questions on 2-factorisations of graphs. The first question has undoubtedly been considered by many people and is mentioned in [7].

(1) Is it true that there exists an N such that for all odd $n \geq N$, every possible type of 2-factorisation of K_n exists? If so, does $N = 17$?

It seems reasonable to suggest that such an N exists, and perhaps that $N = 17$. The results for $n \leq 17$, in particular $n = 15$, suggest some likely candidates if one wishes to search for non-existent types of 2-factorisations of K_{19} and K_{21}. We have computationally constructed a 2-factorisation of K_{19} of type $[A^{19}, A^{19}, \ldots, A^{19}, \alpha]$ where $A^{19} = [3, 3, 3, 3, 3, 4]$ and α is any one of the 39 non-isomorphic 2-regular graphs of order 19, see [1]. We have also computationally constructed a 2-factorisation of K_{21} of type $[A^{21}, A^{21}, \ldots, A^{21}, \alpha]$ where $A^{21} = [3, 3, 3, 3, 3, 3, 3]$ and α is any one of 57 of the 60 non-isomorphic 2-regular graphs of order 21. The three 2-factors α for which we have not been able to ascertain the existence or otherwise of a 2-factorisation of type $[A^{21}, A^{21} \ldots, A^{21}, \alpha]$ are $\alpha = [3, 3, 3, 3, 3, 6]$, $[3, 3, 3, 6, 6]$ and $[3, 3, 3, 3, 9]$. A 2-factorisation of type $[A^{21}, A^{21}, \ldots, A^{21}, [21]]$ was found by Mariusz Meszka, see [11].

When n is even, one can ask an analogous question for $K_n - I$, the complete graph of order n with the edges of a perfect matching removed.

(2) Does there exist an N such that for all even $n \geq N$, every possible type of 2-factorisation of $K_n - I$ exists?

If it exists, N is at least 14, since there is no 2-factorisation of $K_{12} - I$ in which each 2-factor is of type $[3, 3, 3, 3]$, see [3].

Although the existence of N for questions (1) and (2) seems likely, the answer to the following question is less clear.

(3) For which k does there exist an N_k such that for all $n \geq N_k$, there exists a k-regular graph $G_{k,n}$ of order n for which every type of 2-factorisation of $G_{k,n}$ exists? What is the smallest such k (if one exists)? Does there exist such a k with $N_k = k + 1$?

We have obtained the following interesting results which perhaps shed some light, though not much, on this question in the case $k = 8$. For all integers n in the range $9 \leq n \leq 17$, we have computationally checked the existence of all possible types of 2-factorisations of $C(n, \{1, 2, 3, 4\})$. The results, which can be found in [1], are:

- For $n = 9$, $C(n, \{1, 2, 3, 4\}) = K_n$ and 26 of 35 possible types of 2-factorisation of $C(n, \{1, 2, 3, 4\})$ exist.

- For $n = 10$, $C(n, \{1, 2, 3, 4\}) = K_n - I$ and all 70 types of 2-factorisation of $C(n, \{1, 2, 3, 4\})$ exist.

- For $n = 11$, all 126 types of 2-factorisation of $C(n, \{1, 2, 3, 4\})$ exist.

- For $n = 12$, all except 4 of 495 possible types of 2-factorisation of $C(n, \{1, 2, 3, 4\})$ exist. The four types which do not exist are

$$[[3,3,3,3],[3,3,3,3],[3,3,3,3],[4,4,4]],$$
$$[[3,3,3,3],[3,3,3,3],[3,3,3,3],[3,4,5]],$$
$$[[3,3,3,3],[3,3,3,3],[3,3,3,3],[3,3,6]],$$
$$[[3,3,3,3],[3,3,3,3],[3,3,3,3],[12]].$$

- For $n = 13$, all 715 types of 2-factorisation of $C(n, \{1, 2, 3, 4\})$ exist.

- For $n = 14$, all 1820 types of 2-factorisation of $C(n, \{1, 2, 3, 4\})$ exist.

- For $n = 15$, all except 33 of 4845 possible types of 2-factorisation of $C(n, \{1, 2, 3, 4\})$ exist. (The non-existing types are listed in [1].)

- For $n = 16$, all except 3 of 10,626 possible types of 2-factorisation of $C(n, \{1, 2, 3, 4\})$ exist. The three types which do not exist are

$$[[3,3,3,3,4],[3,3,3,3,4],[3,3,3,3,4],[4,4,4,4]],$$
$$[[3,3,3,3,4],[3,3,3,3,4],[3,3,3,3,4],[3,4,4,5]],$$
$$[[3,3,3,3,4],[3,3,3,3,4],[4,4,4,4],[3,3,3,7]].$$

- For $n = 17$, all except 1 of 20,475 possible types of 2-factorisation of $C(n, \{1, 2, 3, 4\})$ exist. The type which does not exist is

$$[[3,3,3,3,5],[3,3,3,3,5],[3,3,3,3,5],[4,4,4,5]].$$

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