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The average degree in a vertex-magic graph

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1 Introduction

A *graph* $\mathcal{G}$ is a set $\mathcal{V} \cup \mathcal{E}$, where $\mathcal{V}$ is any finite non-empty set, and $\mathcal{E}$ is a set of unordered pairs of elements of $\mathcal{V}$. The elements of $\mathcal{V}$ and $\mathcal{E}$ are the *vertices* and *edges*, respectively, of $\mathcal{G}$. We say that u and v are the *end-points* of the edge $\{u, v\}$, and that this edge *joins* u and v. Every edge has exactly two (distinct) vertices as its end-points, and there is at most one edge joining any two vertices. We let E, G and V be the cardinalities of $\mathcal{E}$, $\mathcal{G}$ and $\mathcal{V}$, respectively, so that $G = E + V$. A *labelling* of a graph $\mathcal{G}$ is a bijection $\lambda : \{1, 2, \ldots, G\} \to \mathcal{G}$. The graph $\mathcal{G}$ is said to be *vertex-magic* if there is a labelling λ, and an integer k, called the *magic-constant* for λ, such that for every vertex v,

$$\lambda(v) + \sum_{v \in e, e \in \mathcal{E}} \lambda(e) = k. \tag{1}$$

We call the left-hand side of (1) the *weight* of v and we denote it by $\omega(v)$; this is the sum of the labels of v and all of the edges that end at v. The definition of a vertex-magic labelling of a graph was introduced in [3], but we refer the reader to [2] and [4] for a survey of various types of labellings of graphs. In [2], Gallian writes "Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments." The purpose of this paper is to establish several general results on vertex-magic graphs.

First we recall that for any connected graph $\mathcal{G}$, $V - 1 \leq E \leq \frac{1}{2}V(V-1)$, where these bounds are best possible (equality occurs on the left if and only if $\mathcal{G}$ is a tree, and on the right if and only if $\mathcal{G}$ is a complete graph). The *degree* $d(v)$ of a vertex

v is the number of edges that contain v. Throughout, *and without further mention*, we shall assume that $\mathcal{G}$ has no isolated vertices; that is, $d(v) \geq 1$ for each v. Our first result shows that if a vertex-magic graph $\mathcal{G}$ has a vertex of small degree, then it has relatively few edges; for example, if $\mathcal{G}$ has at least one vertex of degree one, then $E < 2V$. Note that this shows how a *local feature* of a vertex-magic graph can influence its *global structure*.

Theorem 1. *Let $\mathcal{G}$ be a vertex-magic graph with magic constant k, and let $d_0 = \min_v d(v)$. Then $k \leq (1 + d_0)(G - d_0/2)$, and $E < (1 + d_0)V$.*

There is another way to express the last inequality in Theorem 1. If we count the ends of all of the edges in $\mathcal{G}$, we see that $\sum_v d(v) = 2E$. Thus the average value of $d(v)$ is $2E/V$, so that *in any vertex-magic graph, the average degree is less than $2 + 2d_0$, where d_0 is the smallest degree.*

Next we consider vertex-magic graphs that have a vertex of degree one. It is known that if a vertex-magic tree $\mathcal{T}$ has q internal vertices (of degree greater than one) and p ends (vertices of degree one) then $p \leq 2q$ or, equivalently, $q \geq V/3$ ([4], page 91). Thus, *in any vertex-magic tree, at least one third of the vertices must have degree at least two.* As $V = E + 1$ in any tree, it is not clear whether V or E is the appropriate term in any general form of this inequality. We shall now derive a similar inequality that applies to *all* vertex-magic graphs with a vertex of degree one, and we shall see that it is E rather than V that arises in the inequality. As $E \geq V - 1$ we obtain essentially the same inequality as for trees, but in general, E may be significantly larger than V. Note, however, that E cannot be too much larger that V for Theorem 1 implies that for a vertex-magic graph with a vertex of degree one, $E < 2V$.

Theorem 2. *Let $\mathcal{G}$ be a vertex-magic graph with a vertex of degree one, and let q be the number of vertices of degree at least two. Then $q > E/3 \geq (V - 1)/3$.*

2 The proof of Theorem 1

Throughout this section we let $\mathcal{G}$ be a vertex-magic graph with the magic constant k arising from the labelling λ, and we use the notation $\mathcal{E}$, $\mathcal{V}$, E, V, G and ω as above. We begin with two simple inequalities that exist for any vertex-magic graph. If we sum both sides of (1) over each vertex, we obtain

$$Vk = \sum_{v \in \mathcal{V}} \lambda(v) + 2 \sum_{e \in \mathcal{E}} \lambda(e) = (1 + \cdots + G) + \sum_{e \in \mathcal{E}} \lambda(e).$$

As the sum of the labels on the edges is at least $1 + \cdots + E$, and at most $(V + 1) + \cdots + (V + E)$, we see that

$$G(G + 1) + E(E + 1) \leq 2Vk \leq G(G + 1) + E(E + 1) + 2EV. \tag{2}$$

This double inequality, which is well-known, gives upper and lower bounds for the magic constant k in terms of E and V, and it restricts k to at most $E+1$ possible values.

Suppose now that $\mathcal{G}$ has a vertex of degree d, and apply (1) to that vertex. Then k is the sum of $d+1$ distinct numbers taken from $1, 2, \ldots, G$, so that

$$k \leq G + (G-1) + \cdots + (G-d) = (d+1)(G-d/2). \tag{3}$$

If we take d to be the smallest degree, namely d_0, this gives the first inequality in Theorem 1. We remark that there are many circumstances in which the bound in (3) is better than the upper bound for k given in (2). Indeed, the bound in (3) depends linearly in E for a given V, whereas the dependence on E in (2) is quadratic. We shall illustrate this point with some examples later.

Next, we combine (2) and (3) and obtain

$$G(G+1) + E(E+1) \leq V(d+1)(2G-d),$$

and as $G = E + V$ this implies that

$$2E^2 + 2E - 2dVE \leq (2d+1)V^2 - (d^2+d+1)V,$$

and hence

$$(2E+1-dV)^2 - (dV-1)^2 \leq 2[(2d+1)V^2 - (d^2+d+1)V].$$

This simplifies to give

$$(2E+1-dV)^2 \leq (d^2+4d+2)V^2 - 2V(d+1)^2 + 1 < (d+2)^2V^2, \tag{4}$$

from which it follows that

$$2E < 2E+1 < dV + (d+2)V = (2d+2)V.$$

The second inequality in Theorem 1 now follows if we take d to be the minimal degree d_0. □

We remark that our proof of Theorem 1 yields slightly more than is given in Theorem 1. For example, if $d_0 = 1$ then (4) yields

$$(2E+1-V)^2 \leq (7V-1)(V-1) < 7V^2,$$

so that $E < \frac{1}{2}(1+\sqrt{7})V = 1.8229\cdots V$.

3 The proof of Theorem 2

The inequality (3) shows that if $\mathcal{G}$ has a vertex of degree one then $k \leq 2G-1$. More generally, suppose that $\mathcal{G}$ has exactly p vertices, say $u_1, \ldots, u_p$, of degree one, and q, where $q = V - p$, vertices of degree at least two, say $v_1, \dots v_q$. Let e_j be the unique edge ending at u_j. As an edge cannot have both ends of degree one (for this would violate the assumption that $\mathcal{G}$ is vertex magic) we see that $e_1, \dots, e_p$ are distinct. Thus

$$pk = \sum_{j=1}^{p} [\lambda(u_j) + \lambda(e_j)] \leq G + (G-1) + \cdots (G - [2p-1]).$$

We deduce that if $\mathcal{G}$ has p vertices of degree one, then

$$k \leq 2G - (2p-1) = 2E + 2q + 1. \tag{5}$$

Next, we obtain a lower bound on k. By considering the q vertices $v_1, \dots, v_q$ of degree at least two, we see that

$$qk = \sum_{j=1}^{q} \omega(v_j). \tag{6}$$

Now each term $\lambda(v_j)$ occurs in the sum on the right of (6), as does $\lambda(e)$ for every edge e (otherwise there would be an edge whose two end-points are of degree one). Thus the sum on the right of (6) certainly includes $E + q$ distinct numbers taken from $1, 2, \dots, G$. Thus

$$qk \geq 1 + 2 + \cdots + (E+q) = \tfrac{1}{2}(E+q)(E+q+1). \tag{7}$$

If we now eliminate k from (5) and (7) we obtain the inequality

$$3q^2 + (2E+1)q \geq E^2 + E.$$

Thus

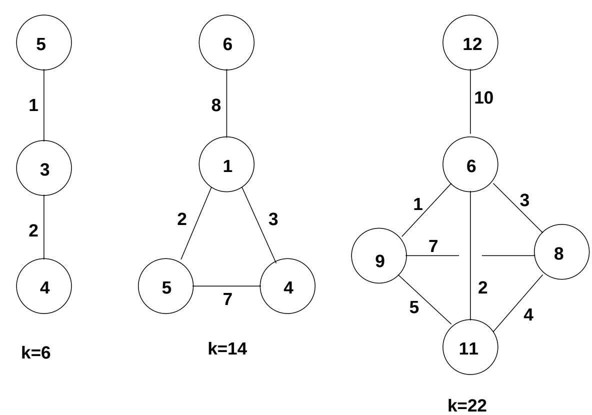
$$\left(q + \frac{2E+1}{6}\right)^2 \geq \frac{16E^2 + 16E + 1}{36} > \left(\frac{4E+1}{6}\right)^2,$$

and Theorem 2 follows immediately. □

4 Some examples

The ideas used in this paper are already implicit in the literature, but here we have deliberately avoided referring to particular classes of graphs. Nevertheless, our results do apply to classes of graphs that have already been studied. For example, the inequalities (5) and (7) are sufficient to show that if a disjoint union of t stars with $n_1, \dots, n_t$ rays, respectively, is vertex-magic then (5) and (7) lead to the inequality $N^2 + N(1-2t) \leq 3t^2 + t$, where $N = \sum_j n_j$. This inequality occurs on the first line of page 88 in [4]; thus (5) and (7) contain a proof of Theorem 3.13 in [4], namely that *if a disjoint union of stars is vertex-magic then the average number of rays in these stars is less than three.* This result is best possible as, for any t, there exist a vertex-magic disjoint union of t stars with exactly $3t-1$ edges ([4], page 89). Note, however, that this result is about the average number of rays in the star; the average degree in a union of stars is always less than two (because $E < V$).

Given a graph $\mathcal{G}$ we form a new graph, which we call a *pendant* $P(\mathcal{G})$, by attaching a single edge to a vertex of $\mathcal{G}$ (and, of course, a vertex of degree one at the other end of the edge). The complete graph K_n is the graph with n vertices and all possible $n(n-1)/2$ edges, and K_n is vertex-magic if and only if $n \neq 2$ ([2], Theorem 3.1, p.66 and Theorem 3.17, p.93). Here, we shall show that the pendant $P(K_n)$ is vertex-magic if and only if $n = 2, 3, 4$. As observed earlier, this illustrates the impact of having a vertex of low degree in a graph. First, as $P(K_1) = K_2$, $P(K_1)$ is not vertex-magic. Let us now assume that $P(K_n)$ is vertex-magic. As $d = 1$, $V = n+1$ and $E = 1 + n(n-1)/2$, (4) yields an inequality for n which is easily seen to be satisfied only for $n \leq 4$. Thus if $n \neq 2, 3, 4$ then $P(K_n)$ is not vertex-magic. The figure shows that for $n = 2, 3, 4$ $P(K_n)$ is vertex-magic. We note that for the graph $P(K_4)$, the well-known inequalities (2) yield $22 \leq k \leq 28$. By contrast, the inequality (3) yields $k \leq 23$.



As another class of examples consider the graph formed by supending one of the five Platonic solids in the manner described above. First, the graph of a suspended tetrahedron is vertex-magic, for this is $P(K_4)$. For a suspended octahedron, $E = 13$ and $V = 7$, and as this implies that $2V(2G-1) < G(G+1)+E(E+1)$, we see that the suspended octahedron is not vertex-magic. Likewise, for the suspended icosahedron we have $V = 13$ and $E = 31$. As $E > 2V$ this too is not vertex-magic. The reader may like to consider whether or not the suspended cube and dodecahedron are vertex-magic.

It is known that if a wheel with n spokes is vertex-magic then $n \leq 11$ ([4], Theorem 3.6). However, a calculation shows that if a suspended wheel with n spokes is vertex-magic, then $n \leq 8$. Finally, Theorem 1 shows that in any vertex-magic graph $\mathcal{G}$ the degree d of a vertex cannot be too small compared with the degrees of the other vertices. In [1] the author showed that we also have the inequality

$$(d+2)^2 \leq \frac{14E^2+16E+4}{V} < \frac{36E^2}{V} = (3\bar{d})^2 V,$$

where $\bar{d}$ is the average degree. Thus, with a given V, the degree of a vertex cannot be too large compared with the degrees of the other vertices.

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