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Edge-fault-tolerant properties of hypercubes and folded hypercubes *

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Abstract

This paper shows that for any two vertices u and v of the hypercube Q_n with at most $n-2$ faulty edges there exists a fault-free uv-path of length l with $d_{Q_n}(u,v)+2 \le l \le 2^n-1$ and $2|(l-d_{Q_n}(u,v))$, and that every non-faulty edge of the folded hypercube FQ_n with at most $n-1$ faulty edges lies on a fault-free cycle of every even length from 4 to 2^n and, furthermore, also every odd length from $n+1$ to 2^n-1 if n is even. These improve some known results and are all optimal.

1 Introduction

It is well-known that when the underlying topology of an interconnection network is modelled by a connected graph $G=(V,E)$, where V is the set of processors and E is the set of communication links in the network, the study of the structure of G is of quite great interest. A cycle structure, which is a fundamental topology for parallel and distributed processing, is suitable for local area networks and for the development of simple parallel algorithms with low communication cost. A cycle structure can also be used as a control/data flow structure for distributed computation in arbitrary networks.

A graph G is *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$. The concept of pancyclicity has been extended to vertex-pancyclicity and edge-pancyclicity. A graph G is *vertex* (resp. *edge*)*-pancyclic* if every vertex (resp. edge) lies on a cycle of every length from 3 to $|V(G)|$. It is clear that an edge-pancyclic graph is vertex-pancyclic and, hence, pancyclic. A graph G is *panconnected* if for any two different vertices u and v in G there exists a uv-path of length l with $d_G(u,v) \le l \le |V(G)|-1$, where $d_G(u,v)$ is the distance between u and v in G.

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Since a bipartite graph contains no cycles of odd length, a bipartite graph G is *edge-bipancyclic* if every edge of G lies on a cycle of every even length from 4 to $|V(G)|$, and *bipanconnected* if for any two different vertices u and v in G there exists a uv-path of length l with $d_G(u,v) \leq l \leq |V(G)| - 1$ and $2|(l - d_G(u,v))$. A graph G is *k-edge-fault-tolerant panconnected* (*edge-pancyclic*) if the resulting graph by deleting any k edges from G is panconnected (edge-pancyclic). A subgraph of G is *fault-free* if it contains no faulty edges in G.

The bipancyclicity of the hypercubes has been investigated by Saad and Schultz [5]. Recently, Li et al. [3] showed that the hypercube Q_n is $(n-2)$-edge-fault-tolerant edge-bipancyclic for $n \geq 2$.

In this paper, we consider the hypercube Q_n and folded hypercube FQ_n with faulty edges. We show that for any two different vertices u and v of Q_n with at most $n-2$ faulty edges there exists a fault-free uv-path of length l with $d_{Q_n}(u,v) + 2 \leq l \leq 2^n - 1$ and $2|(l - d_{Q_n}(u,v))$. We also show that every non-faulty edge of FQ_n with at most $n-1$ faulty edges lies on a fault-free cycle of every even length from 4 to 2^n and, moreover, also every odd length from $n+1$ to $2^n - 1$ if n is even.

As consequences of our results, we immediately obtain Li *et al*'s result that Q_n is bipanconnected and $(n-2)$-edge-fault-tolerant edge-bipancyclic [3], and Wang's result that FQ_n is $(n-1)$-edge-fault-tolerant Hamiltonian [7].

The proofs of our results are given in Section 2 and Section 3, respectively. Throughout this paper, we follow Xu [8] for graph-theoretical terminology and notation not defined here.

2 Edge-fault-tolerant bipanconnectivity of Q_n

The n-dimensional hypercube Q_n is a graph with 2^n vertices, each vertex with a distinct binary string $u_n \ldots u_2 u_1$ on the set $\{0,1\}$. Two vertices are linked by an edge if and only if their strings differ in exactly one bit. As a topology for an interconnection network of a multiprocessor system, the hypercube structure is a widely used and well-known interconnection model since it possesses many attractive properties [5, 8]. In particular, Q_n is vertex-transitive and edge-transitive.

By the definition, for any $k \in \{1, 2, \ldots, n\}$, Q_n can be expressed as $Q_n = L_k \odot R_k$, where L_k and R_k are the two $(n-1)$-subcubes of Q_n induced by the vertices where the k-th position is 0 and 1, respectively. We call edges between L_k and R_k to be *k-dimensional*, which form a perfect matching of Q_n. Clearly, for any edge e of Q_n, there is some $k \in \{1, 2, \ldots, n\}$ such that e is k-dimensional. For convenience, we will write L and R instead of L_n and R_n, respectively. Use u_L and u_R to denote two vertices in L and R, respectively, linked by the n-dimensional edge $u_L u_R$ in Q_n.

Lemma 2.1 (Saad and Schultz [5]) Let u and v be two vertices in Q_n and $d_{Q_n}(u,v) = d$. Then there exist n internally disjoint uv-paths in Q_n such that d of them are of length d, which lie in a d-dimensional subcube, otherwise of length $d+2$.

Lemma 2.2 If Q_3 has exactly one faulty edge, then there exists a fault-free path of length l joining any two different vertices u and v for any l with $d_{Q_n}(u,v)+2 \leq l \leq 7$ and $2|(l - d_{Q_n}(u,v))$.

Proof Without loss of generality, we may suppose that the faulty edge is 3-dimensional since Q_3 is edge transitive. Let $Q_3 = L \odot R$, where both L and R are fault-free. Let u and v be any two vertices in Q_3.

Case 1 Both u and v are in L or R. Without loss of generality, we may assume that $u = u_L$ and $v = v_L$ in L.

If $d_{Q_n}(u_L, v_L) = 1$ then, by Lemma 2.1, there are two u_Lv_L-paths in L, one of them is of length 1 and the other of length 3. For $l \geq 5$, let $P_L = (u_L, x_L, y_L, v_L)$ be a u_Lv_L-path of length 3 in L. Since there is only one faulty edge between L and R, at least one of two sets of edges $\{u_Lu_R, x_Lx_R\}$ and $\{y_Ly_R, v_Lv_R\}$ is fault-free. Without loss of generality, we may assume $\{u_Lu_R, x_Lx_R\}$ is fault-free. Then there is a u_Rx_R-path P_R of length l in R for $l = 1$ or 3. Thus $P = (u_L, u_R, P_R, x_R, x_L, y_L, v_L)$ is a fault-free u_Lv_L-path of length 5 or 7 in Q_3.

If $d_{Q_n}(u_L, v_L) = 2$, by Lemma 2.1, there are two internally disjoint u_Lv_L-paths of length 2 in L. Since there is only one faulty edge between L and R, there is a path $P_L = (u_L, x_L, v_L)$ such that at least one of two sets of edges $\{u_Lu_R, x_Lx_R\}$ and $\{x_Lx_R, v_Lv_R\}$ is fault-free. Without loss of generality, we may assume $\{x_Lx_R, v_Lv_R\}$ is fault-free. Then there is an x_Rv_R-path P_R of length l in R for $l = 1$ or 3. Thus $P = (u_L, x_L, x_R, P_R, v_R, v_L)$ is a fault-free u_Lv_L-path of length 4 or 6 in Q_3.

Case 2 If u and v are in different parts. Without loss of generality, we may assume that $u = u_L \in L$ and $v = v_R \in R$.

If $d_{Q_n}(u_L, v_R) = 1$, by Lemma 2.1, there are two internally disjoint u_Lv_R-paths of length 3 in Q_3. We may assume $P = (u_L, x_L, x_R, v_R)$ is fault-free. There are a u_Lx_L-path P_L in L and an x_Rv_R-path P_R in R, both are of length 3. Thus $(u_L, P_L, x_L, x_R, v_R)$ and $(u_L, P_L, x_L, x_R, P_R, v_R)$ are two fault-free u_Lv_R-paths of length 5 and 7, respectively, in Q_3.

If $d_{Q_n}(u_L, v_R) = 2$, by Lemma 2.1, there are two internally disjoint u_Lv_R-paths of length 2 in Q_3. Without loss of generality, we may assume $P = (u_L, v_L, v_R)$ is a fault-free path. Then there is a u_Lv_L-path P_L of length 3 in L and, thus, $(u_L, P_L, v_L, v_R$ is a fault-free u_Lv_R-path of length 4. Let $P_L = (u_L, x_L, y_L, v_L)$ be the u_Lv_L-path of length 3 in L. Since there is only one faulty edge between L and R, at least one of $\{u_Lu_R, x_Lx_R\}$ and $\{y_Ly_R, v_Lv_R\}$ is fault-free. If $\{u_Lu_R, x_Lx_R\}$ is fault-free, then $(u_L, u_R, x_R, x_L, y_L, y_R, v_R)$ is a fault-free u_Lv_R-path of length 6. If $\{y_Ly_R, v_Lv_R\}$ is fault-free, there is a y_Rv_R-path P_R of length 3 in R, thus $(u_L, x_L, y_L, y_R, P_R, v_R)$ is a fault-free u_Lv_R-path of length 6.

If $d_{Q_n}(u_L, v_R) = 3$ then, since u_L has two neighbors in L, there are two internally disjoint u_Lv_R-paths $P_1 = (u_L, x_L, x_R, v_R)$ and $P_2 = (u_L, y_L, y_R, v_R)$ of length 3 in Q_3. We may assume P_2 is fault-free. There are a u_Ly_L-path P_L of length 3 in L and a y_Rv_R-path P_R of length 3 in R. Thus $(u_L, P_L, y_L, y_R, v_R)$ and $(u_L, P_L, y_L, y_R, P_R, v_R)$ are fault-free u_Lv_R-paths of length 5 and 7 in Q_3, respectively.

The lemma is proved. ∎

Theorem 2.3 If Q_n $(n \geq 2)$ has at most $n-2$ faulty edges, then for any two different vertices u and v there exists a fault-free uv-path of length l with $d_{Q_n}(u,v)+2 \leq l \leq 2^n-1$ and $2|(l-d_{Q_n}(u,v))$. Moreover, there must exist a fault-free uv-path of length $d_{Q_n}(u,v)$ if $d_{Q_n}(u,v) \geq n-1$.

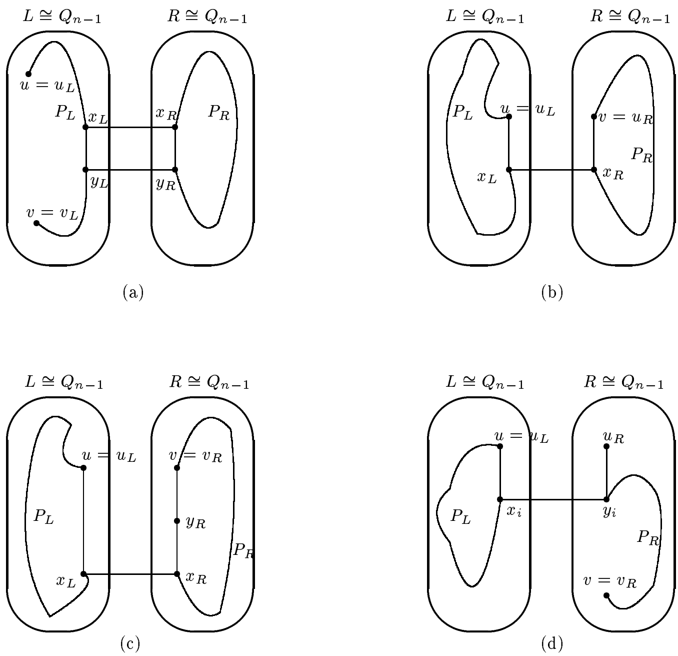

Figure 1: Illustrations for Theorem 2.3

Proof Obviously the latter conclusion is true. In fact, by Lemma 2.1, there are $d_{Q_n}(u,v)$ internally disjoint uv-paths of length $d_{Q_n}(u,v)$, at least one of which is fault-free since $d_{Q_n}(u,v) \geq n-1$ and Q_n has at most $n-2$ faulty edges.

We now prove the former conclusion by induction on $n \geq 2$. Obviously, the conclusion holds for $n=2$. By Lemma 2.2, the conclusion is true for $n=3$. Assume that the conclusion is true for any k with $3 \leq k < n$. Let F be a set of faulty edges in Q_n. Then $|F| \leq n-2$ by the hypothesis. For $1 \leq i \leq n$, let F_i denote the set of i-dimensional edges in F. Thus, $\sum_{i=1}^{n} |F_i| = |F|$, without loss of generality by the edge-transitivity of Q_n, we assume that $|F_1| \leq |F_2| \leq \ldots \leq |F_n|$. With the expression $Q_n = L \odot R$, we use F_L and F_R to denote the set $E(L) \cap F$ and $E(R) \cap F$ respectively. Hence, $F = F_L \cup F_n \cup F_R$ and $|F_L| + |F_R| \leq n-3$.

Let u and v be two arbitrary vertices in Q_n. To prove that there exists a fault-free uv-path of length l with $d_{Q_n}(u,v)+2 \leq l \leq 2^n-1$ and $2|(l-d_{Q_n}(u,v))$, we need to consider the location of the vertices u and v.

Case 1 Both u and v are in L or R. Without loss of generality, we may assume that $u = u_L$ and $v = v_L$ are in L.

If $d_{Q_n}(u_L, v_L) + 2 \leq l \leq 2^{n-1} - 1$ and $2|(l - d_{Q_n}(u,v))$ then, since $|F_L| \leq n-3$, there exists a u_Lv_L-path of length l in $L - F_L$ by the induction hypothesis. In particular, we use $P_L = (u_L, \ldots, x_L, y_L, \ldots, v_L)$ to denote the longest fault-free u_Lv_L-path in L and $\varepsilon(P_L)$ to denote its length. Then $2|(\varepsilon(P_L) - d_{Q_n}(u,v))$ and $\varepsilon(P_L) = 2^{n-1} - 2$ if $d_{Q_n}(u,v)$ is even or $2^{n-1} - 1$ if $d_{Q_n}(u,v)$ is odd.

If $2^{n-1} \leq l \leq 2^n - 1$ and $2|(l - d_{Q_n}(u,v))$, let $l' = l - \varepsilon(P_L) - 1$, then l' is odd and $1 \leq l' \leq 2^{n-1} - 1$. Since $\varepsilon(P_L) \geq 2^{n-1} - 2$ and $\frac{2^{n-1}-2}{2} = 2^{n-2} - 1 > n - 2$ for $n \geq 4$, there is an edge x_Ly_L on the path P_L such that the set of edges $\{x_Lx_R, y_Ly_R, x_Ry_R\}$ is fault-free. By the induction hypothesis, there is a fault-free x_Ry_R-path P_R of odd length l' in R, where P_R consist of a single edge x_Ry_R if $l' = 1$. Thus, $(u_L, \ldots, x_L, x_R, P_R, y_R, y_L, \ldots, v_L)$ is a fault-free u_Lv_L-path of length l in Q_n (see Fig. 1 (a)).

Case 2 If u and v are in different parts. Without loss of generality, we may assume that $u = u_L \in L$ and $v = v_R \in R$.

Subcase 2.1 If $d_{Q_n}(u_L, v_R) = 1$, then $v_R = u_R$. In this case, l is odd with $3 \leq l \leq 2^n - 1$. Since there are $n - 1$ internally disjoint u_Lu_R-paths of length 3 by Lemma 2.1 and there are at most $n - 2$ faulty edges in Q_n, there must exist a u_Lu_R-path of length 3 in $Q_n - F$. Let (u_L, x_L, x_R, u_R) be a fault-free u_Lu_R-path. By the induction hypothesis, there exist a fault-free u_Lx_L-path P_L of odd length l' with $1 \leq l' \leq 2^{n-1} - 1$ in L and a fault-free x_Ru_R-path P_R of odd length $(l - l' - 1)$ in R, where P_L consist of a single edge u_Lx_L if $l' = 1$ and P_R consist of a single edge x_Ru_R if $l - l' - 1 = 1$. Thus, $(u_L, P_L, x_L, x_R, P_R, u_R)$ is a fault-free u_Lu_R-path of odd length l with $3 \leq l \leq 2^n - 1$ (see Fig. 1 (b)).

Subcase 2.2 $d_{Q_n}(u_L, v_R) = 2$. In this case, l is even with $4 \leq l \leq 2^n - 2$. By Lemma 2.1 there are n internally disjoint u_Lv_R-paths in Q_n such that two of them are of length 2, otherwise of length 4.

We can choose a fault-free u_Lv_R-path of length four. In fact, suppose that $P_i = (u_L, x_i, y_i, z_i, v_R)$ $(i = 1, 2, \ldots, n-2)$ are $n-2$ internally disjoint u_Lv_R-paths of length four in Q_n such that each of them contains at least one faulty edge. Then every such a path contains exactly one faulty edge and, hence, all faulty edges must be in these paths since $|F| \leq n-2$. Since $|F_L| + |F_R| \leq n-3$, there is some i with $1 \leq i \leq n-2$ such that the only one faulty edge in P_i is n-dimensional that is either x_iy_i or y_iz_i. Since $d_{Q_n}(x_i, z_i) = 2$, by Lemma 2.1 there exists a 2-dimensional subcube in which there are two internally disjoint x_iz_i-paths of length two, one of which is (x_i, y_i, z_i) and the other, say, (x_i, y_i', z_i), is not contained in P_j for each $j = 1, 2, \ldots, n-2$. Then $(u_L, x_i, y_i', z_i, v_R)$ is a fault-free u_Lv_R-path of length four, as required.

Without loss of generality, we assume $(u_L, x_L, x_R, y_R, v_R)$ is a fault-free u_Lv_R-path. Note $d_{Q_n}(u_L, x_L) = 1$ and $d_{Q_n}(x_R, v_R) = 2$, $|F_L| \leq n-3$ and $|F_R| \leq n-3$. By the induction hypothesis, there are a fault-free u_Lx_L-path P_L of odd length l' in L with $3 \leq l' \leq 2^{n-1} - 1$ and a fault-free x_Rv_R-path P_R of even length l'' in R, where $l'' = l - l' - 1$ with $4 \le l'' \le 2^{n-1} - 2$. If let $P_L = u_L x_L$ when $l' = 1$ and let $P_R = (x_R, y_R, v_R)$ when $l'' = 2$, then $(u_L, P_L, x_L, x_R, P_R, v_R)$ is a fault-free $u_L v_R$-path of even length l with $4 \le l \le 2^n - 2$ (see Fig. 1 (c)).

Subcase 2.3 $d_{Q_n}(u_L, v_R) \ge 3$. Let $d_{Q_n}(u_L, v_R) = k$. Then $d_{Q_n}(u_R, v_R) = k - 1$, by Lemma 2.1, there are $n-1$ internally disjoint $u_R v_R$-paths $R_i = (u_R, y_i, \ldots, v_R)$ $(i = 1, 2, \ldots, n-1)$ in R such that $k-1$ of them are of length $k-1$, otherwise of length $k+1$. We use x_i $(i = 1, 2, \ldots, n-1)$ to denote the neighbors of y_i $(i = 1, 2, \ldots, n-1)$ in L. Clearly x_i $(i = 1, 2, \ldots, n-1)$ are the $n-1$ neighbors of u_L in L. Hence $P_i = (u_L, x_i, y_i, \ldots, v_R)$ $(i = 1, 2, \ldots, n-1)$ are $n-1$ internally disjoint $u_L v_R$-paths such that $k-1$ of them are of length k, otherwise of length $k+2$. Since $|F| \le n-2$, there is some i with $1 \le i \le n-1$ such that P_i is a fault-free $u_L v_R$-path of length k or $k+2$. By the induction hypothesis, there are a fault-free $u_L x_i$-path P_L of length l' in L with $1 \le l' \le 2^{n-1} - 1$ and $2|(l'-1)$ and a fault-free $y_i v_R$-path P_R of length l'' in R with $k \le l'' \le 2^{n-1} - 1$ and $2|(l'' - k)$, from which the required length $u_L v_R$-path can be constructed (see Fig. 1 (d)).

This completes the proof of the theorem. ∎

This result is optimal in the following sense. Let $n-1$ faulty edges all be incident to the same vertex x. Hence there is only one fault-free edge incident to x. There is no uv-path of length $2^n - 2$ or $2^n - 1$ for any pair of vertices u and v different from x.

From Theorem 2.3, the following result, due to Li *et al* [3], can be obtained immediately.

Corollary 2.4 For any $n \ge 2$, Q_n is bipanconnected and $(n-2)$-edge-fault-tolerant edge-bipancyclic.

3 Edge-fault-tolerant edge-pancyclicity of FQ_n

As a variant of the hypercube, the n-dimensional folded hypercube FQ_n, proposed first by El-Amawy and Latifi [1], can be obtained from the hypercube Q_n by adding an edge, called a *complementary edge*, between any pair of complementary vertices $u = u_n \ldots u_2 u_1$ and $\bar{u} = \bar{u}_n \ldots \bar{u}_2 \bar{u}_1$, where $\bar{u}_i = 1 - u_i$ for $i = 1, 2, \ldots, n$. We denote the set of complementary edges by E_c. To distinguish them from the edges in Q_n, we call edges in Q_n *regular edges* and denote the set of i-dimensional regular edges by E_i for $i = 1, 2, \ldots, n$.

It has been shown that FQ_n is $(n+1)$-regular $(n+1)$-connected, vertex- and edge-transitive. FQ_n is also superior to Q_n in some properties. For example, it has diameter $\lceil \frac{n}{2} \rceil$, about half the diameter of Q_n [1]. Thus, the folded hypercube FQ_n is an enhancement on the hypercube Q_n. As a result, the study of the folded hypercube has recently attracted much attention of researchers [1, 2, 4, 6, 7].

Lemma 3.1 For $n \ge 2$, every edge e of FQ_n lies on n cycles $C_1, C_2, \ldots, C_n$ of length $n+1$ such that $C_i \cap C_j = \{e\}$ for $i, j = 1, 2, \ldots, n$ and $i \ne j$.

Proof Let $e = u\bar{u}$ be a complementary edge. Since $d_{Q_n}(u, \bar{u}) = n$, by Lemma

2.1, there are n internally disjoint $u\bar{u}$-paths P_i $(i = 1, 2, \ldots, n)$ of length n in $FQ_n - E_c \cong Q_n$. Then $C_i = P_i + e$ $(i = 1, 2, \ldots, n)$ are required cycles. Since FQ_n is edge-transitive, the conclusion also holds for any regular edge. The lemma follows. ∎

Lemma 3.2 There is an automorphism σ of FQ_n such that $\sigma(E_i) = E_j$ for any $i, j \in \{1, 2, \ldots, n, c\}$.

Proof If $i, j \in \{1, 2, \ldots, n\}$ and $i < j$, let σ be a mapping from $V(FQ_n)$ to itself defined by $\sigma(x_n \ldots x_j \ldots x_i \ldots x_1) = (x_n \ldots x_i \ldots x_j \ldots x_1)$ for any $x_n \ldots x_2 x_1 \in V(FQ_n)$. Clearly, $\sigma \in \mathrm{Aut}\,(FQ_n)$ and $\sigma(E_i) = E_j$, $\sigma(E_j) = E_i$.

If $i = c, j = n$, let σ be a mapping from $V(FQ_n)$ to itself defined by

$$\begin{cases} \sigma(0u) = 0u \\ \sigma(1u) = 1\bar{u} \end{cases} \quad \text{for any } u \in V(Q_{n-1}).$$

Clearly, $\sigma \in \mathrm{Aut}\,(FQ_n)$ and $\sigma(E_c) = E_n, \sigma(E_n) = E_c$.

If $i = c, j \neq n$, let $\sigma_n \in \mathrm{Aut}\,(FQ_n)$ such that $\sigma_n(E_c) = E_n$, $\sigma'_n \in \mathrm{Aut}\,(FQ_n)$ such that $\sigma'_n(E_n) = E_j$. Then $\sigma'_n\sigma_n \in \mathrm{Aut}\,(FQ_n)$ and $\sigma'_n\sigma_n(E_c) = E_j$. The lemma is proved. ∎

Like Q_n, we can express FQ_n as $L \otimes R$, where $L \cong Q_{n-1}$ and $R \cong Q_{n-1}$, the complementary edge $u\bar{u}$ is between L and R for any $u \in V(FQ_n)$. Let F be a set of faulty edges in FQ_n with $|F| \leq n - 1$, $F_c = F \cap E_c$ and $F_i = F \cap E_i$ for $1 \leq i \leq n$. Thus, $\sum_{i=1}^{n} |F_i| + |F_c| = |F|$. By Lemma 3.2, without loss of generality, we assume that $|F_c| \geq |F_n| \geq \ldots \geq |F_1|$. Moreover, we use F_L and F_R to denote the set $E(L) \cap F$ and $E(R) \cap F$ respectively. Hence, $F = F_L \cup F_n \cup F_c \cup F_R$ and $|F_L| + |F_R| \leq n - 3$.

Lemma 3.3 If FQ_n has at most $n - 1$ faulty edges for $n \geq 3$, then every non-faulty edge lies on a fault-free cycle of even length l with $4 \leq l \leq 2^n$.

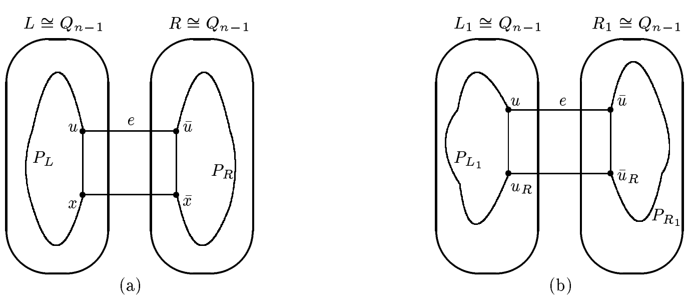

Figure 2: Illustrations for Lemma 3.3

Proof Let F be a set of faulty edges of FQ_n with $|F| \leq n-1$ and $|F_c| \geq |F_n| \geq \ldots \geq |F_1|$. Let e be any non-faulty edge of FQ_n and l any even integer with $4 \leq l \leq 2^n$. If e is an regular edge then, since $FQ_n - E_c \cong Q_n$ and $|F - F_c| \leq n-2$, the lemma follows immediately by Corollary 2.4. We now suppose that e is a complementary edge.

To complete the proof of the lemma, we need to construct a cycle of length l containing e. Let $e = u\bar{u}$, where $u \in L$ and $\bar{u} \in R$. Let N_L be the set of neighbors of u in L. Then $(u, x, \bar{x}, \bar{u}, u)$ is a cycle of length 4 containing e for each $x \in N_L$.

Case 1 If there is some $x \in N_L$ such that the cycle $(u, x, \bar{x}, \bar{u}, u)$ is fault-free then, by Theorem 2.3, there are a fault-free ux-path P_L of odd length l' in L with $1 \leq l' \leq 2^{n-1} - 1$ and a fault-free $\bar{x}\bar{u}$-path P_R of odd length l'' in R with $1 \leq l'' \leq 2^{n-1} - 1$. Thus, $(u, P_L, x, \bar{x}, P_R, \bar{u}, u)$ is a fault-free cycle of length $l = l' + l'' + 2$ with $4 \leq l \leq 2^n$ containing e(see Fig. 2 (a)).

Case 2 For each $x \in N_L$ the cycle $(u, x, \bar{x}, \bar{u}, u)$ is faulty. Since $|F_c| \geq |F_n| \geq \ldots \geq |F_1|$ and $|N_L| = n - 1$, $|F_n| = 0$ and $F = F_c = \{x\bar{x} : x \in N_L\}$. Hence the cycle $C = (u, u_R, \bar{u}_R, \bar{u}, u)$ is a fault-free cycle of length 4. Considering the expression $FQ_n = L_1 \otimes R_1$, we can reduce this case to Case 1 (see Fig. 2 (b)).

This completes the proof of the lemma. ∎

It is well known that FQ_n is a bipartite graph if and only if n is odd. There are odd cycles in FQ_n if n is even and the shortest odd cycle is of length $n + 1$.

Lemma 3.4 If FQ_n has at most $n - 1$ faulty edges for any even integer $n \geq 2$, then every non-faulty edge e lies on a fault-free cycle of odd length l with $n + 1 \leq l \leq 2^n - 1$.

Proof It is easy to check that the conclusion holds for $n = 2$. We assume $n \geq 4$ below. Let F be an arbitrary set of faulty edges in FQ_n with $|F| \leq n - 1$. Let e be any non-faulty edge of FQ_n and $e \in E_i$, $i \in \{1, 2, \ldots, n, c\}$. Let l be any odd integer with $n + 1 \leq l \leq 2^n - 1$. To prove the lemma, we need to construct a cycle of length l containing e.

Case 1 If $|F_i| \geq 1$, by Lemma 3.2 there is an automorphism of FQ_n mapping E_i to E_c. Then the edge e is complementary and $|F_c| \geq 1$. Without loss of generality, assume $e = u\bar{u}$, where $u \in L$ and $\bar{u} \in R$. Then $d_{Q_n}(u, \bar{u}) = n$. Since $FQ_n - E_c \cong Q_n$ and $|F - F_c| \leq n - 2$, by Theorem 2.3, there is a fault-free $u\bar{u}$-path P of length $l - 1$ in FQ_n with $n \leq l - 1 \leq 2^n - 2$. Thus, $(u, P, \bar{u}, u)$ is a fault-free cycle of length l containing e with $n + 1 \leq l \leq 2^n - 1$.

Case 2 If $|F_i| = 0$, we can choose $j, k \in \{1, 2, \ldots, n, c\}$ and $j \neq i, k \neq i$ such that $|F_j| + |F_k| \geq 2$. By Lemma 3.2 there is an automorphism of FQ_n mapping E_j to E_c and mapping E_k to E_n. Then the edge $e \in L$ or $e \in R$ and $|F_L| + |F_R| \leq n - 3$. Without loss of generality, we assume $e = u_L v_L \in L$. Since $|F| \leq n - 1$, by Lemma 3.1, there is a fault-free cycle of length $n + 1$ containing e. For $n + 3 \leq l \leq 2^{n-1} + 1$, there are three subcases to be considered.

Subcase 2.1 Either $\{u_L\bar{u}_L, v_L v_R\}$ or $\{v_L\bar{v}_L, u_L u_R\}$ is fault-free. Without loss of generality, we assume $\{u_L\bar{u}_L, v_L v_R\}$ is fault-free. By Theorem 2.3, there is a fault-free $v_R\bar{u}_L$-path P_R of even length l_2 in R with $n - 2 \leq l_2 \leq 2^{n-1} - 2$. Thus, $(u_L, v_L, v_R, P_R, \bar{u}_L, u_L)$ is a fault-free cycle of length $l = l_2 + 3$ containing e with $n + 1 \leq l \leq 2^{n-1} + 1$ (see Fig. 3 (a)).

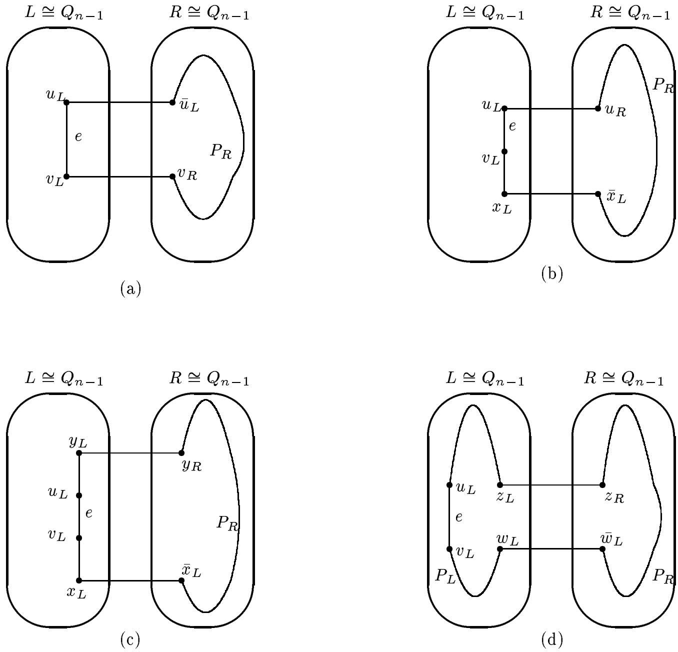

Figure 3: Illustrations for Lemma 3.4

Subcase 2.2 At least one of $\{u_L\bar{u}_L, v_Lv_R\}$ and at least one of $\{v_L\bar{v}_L, u_Lu_R\}$ are faulty, but there is at least one non-faulty edge in $\{u_Lu_R, u_L\bar{u}_L, v_Lv_R, v_L\bar{v}_L\}$. We may assume u_Lu_R is non-faulty. There is a neighbor x_L of v_L in L such that $\{v_Lx_L, x_L\bar{x}_L\}$ is fault-free since v_L has $n-2$ neighbors in L apart from u_L, which are incident with at most $n-3$ edges in F. Clearly, $d_{Q_n}(u_L, x_L) = 2$ and $d_{Q_n}(\bar{x}_L, u_R) = n-3$. By Theorem 2.3, there is a fault-free $\bar{x}_Lu_R$-path P_R of odd length l_2 in R with $n-1 \leq l_2 \leq 2^{n-1}-1$. Thus $(u_L, v_L, x_L, \bar{x}_L, P_R, u_R, u_L)$ is a fault-free cycle of length $l = l_2 + 4$ containing e with $n+3 \leq l \leq 2^{n-1}+3$ (see Fig. 3 (b)).

Subcase 2.3 All edges in $\{u_L\bar{u}_L, v_Lv_R, v_L\bar{v}_L, u_Lu_R\}$ are faulty. Then $n \geq 5$. There are a neighbor x_L of v_L and a neighbor y_L of u_L in L such that $\{v_Lx_L, x_L\bar{x}_L, u_Ly_L, y_Ly_R\}$ is fault-free. Note $d_{Q_n}(x_L, y_L) = 3$ and $d_{Q_n}(\bar{x}_L, y_R) = n-4$. By Theorem 2.3, there is a fault-free $\bar{x}_Ly_L$-path P_R of even length l_2 in R with $n-2 \leq l_2 \leq 2^{n-1}-2$. Thus, $(y_L, u_L, v_L, x_L, \bar{x}_L, P_R, y_R, y_L)$ is a fault-free cycle of length $l = l_2+5$ containing e with $n+3 \leq l \leq 2^{n-1}+3$ (see Fig. 3 (c)).

For $2^{n-1}+1 \leq l \leq 2^n-1$. By Corollary 2.4, there is a fault-free even cycle C_L of length l_1 containing e in L with $4 < l-n+1 \leq l_1 \leq 2^{n-1}$. Since $l_1 - 1 \geq l-n \geq 2^{n-1}+1-n > n-1$, there is an edge z_Lw_L on C_L apart from e with either $\{z_Lz_R, w_L\bar{w}_L\}$ or $\{z_L\bar{z}_L, w_Lw_R\}$ is fault-free. Assume $\{z_Lz_R, w_L\bar{w}_L\}$ is fault-free and let $P_L = C_L - z_Lw_L$. Then $d_{Q_n}(z_R, \bar{w}_L) = n-2$. By Theorem 2.3, there is a fault-free $z_R\bar{w}_L$-path P_R of even length l_2 in R with $n-2 \le l_2 \le 2^{n-1}-2$. Thus, $(z_L, P_L, w_L, \bar{w}_L, P_R, z_R, z_L)$ is a fault-free cycle of length $l = l_1 + l_2 + 1$ containing e with $2^{n-1}+3 \le l \le 2^n - 1$ (see Fig. 3 (d)).

This completes the proof of the lemma. ∎

Combining Lemma 3.3 and Lemma 3.4, we obtain the following theorem immediately.

Theorem 3.5 For $n \ge 3$, if n is odd, then FQ_n is $(n-1)$-edge-fault-tolerant edge-bipancyclic; if n is even and FQ_n has at most $n-1$ faulty edges, then every non-faulty edge of FQ_n lies on a fault-free cycle of every even length from 4 to 2^n and every odd length from $n+1$ to $2^n - 1$.

Corollary 3.6 (Wang [7]) FQ_n is $(n-1)$-edge-fault-tolerant Hamiltonian.

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