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The second largest eigenvalue of trees

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Abstract

Let T be a tree of order $2n+1$ and edge independence number n. In this paper, a tight upper bound for the second largest eigenvalue of T is obtained. This result can play an important role in investigating the third largest eigenvalue of trees.

1 Introduction

Let G be a simple graph with vertex set $\{v_1, v_2, \cdots, v_n\}$. Its adjacency matrix is defined to be the $n \times n$ matrix $A(G) = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if v_i is adjacent to v_j; and $a_{ij} = 0$ otherwise. The characteristic polynomial of G is just $\det(\lambda I - A(G))$, which is denoted by $\phi(G, \lambda)$ or $\phi(G)$. Since $A(G)$ is real symmetric, all of its eigenvalues are real. We assume, without loss of generality, that they are ordered in non-increasing order, that is,

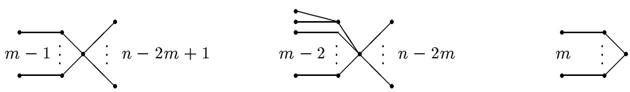
$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G),$$

and $\lambda_k(G)$ is called the kth largest eigenvalue of G.

Two distinct edges in G are independent if they are not adjacent in G. A set of pairwise independent edges of G is called a matching in G, while a matching of maximum cardinality is called a maximum matching in G and the number of edges in a maximum matching of G is called the edge independence number of G. Let $M(G)$ be a matching and v a vertex of G. If v is incident to an edge in $M(G)$, then v is called saturated by $M(G)$. If each vertex of G is saturated by $M(G)$, then $M(G)$ is called a perfect matching of G. If graphs G and H are isomorphic, we write $G \cong H$, and $G \ncong H$ otherwise.

For a tree T with order $2n$ and a perfect matching, $\lambda_1(T)$, $\lambda_2(T)$ and $\lambda_n(T)$ have been completely studied and their precise upper and lower bounds have been obtained (see [1–7]). For a tree T with order $2n+2$ and edge independence number n, the tight upper bound for $\lambda_2(T)$ were also obtained in [8]. Let T be a tree of order

$2n+1$ and edge independence number n. In this paper, a tight upper bound for the second largest eigenvalue of T is obtained. This result can play an important role in investigating the third largest eigenvalue of trees.



$T^{(1)}_{n,m}$ $\quad$ $T^{(2)}_{n,m}$ $\quad$ $T^{(3)}_{2m+1,m}$

Fig. 1.

Throughout this paper, let $o(G)$ and $i(G)$ denote the order and edge independence number of a graph G, respectively. For positive integers n and m, let $T^{(1)}_{n,m}(n \geq 2m)$, $T^{(2)}_{n,m}(m \geq 2, n \geq 2m+1)$ and $T^{(3)}_{2m+1,m}$ be the three trees shown in Fig. 1. For a positive integer n and a real number r such that $n^2 \geq r$, let $\delta_{n,r}, \beta_{n,r}$ and γ_n be defined as follows:

$$\delta_{n,r} = \sqrt{\frac{1}{2}(n+\sqrt{n^2-r})}, \;\; \beta_{n,r} = \sqrt{\frac{1}{2}(n-\sqrt{n^2-r})}, \;\; \gamma_n = \sqrt{n+1+\frac{1}{2n}}.$$

Lemma 1.1 [9] *Let $e=uv$ be an edge of a simple graph G and $C(e)$ the set of all cycles containing edge e. Then*

$$\phi(G,\lambda) = \phi(G-uv) - \phi(G-u-v) - 2\sum_{Z\in C(e)} \phi(G-V(Z)).$$

Lemma 1.2 [5] *Let T be a tree such that $o(G)=n$ and $i(G)=m$. Then*

$$\lambda_1(T) \leq \sqrt{\frac{1}{2}(n-m+1+\sqrt{(n-m+1)^2-4(n-2m+1)})},$$

and the equality holds if and only if $T \cong T^{(1)}_{n,m}$, where

$$\phi(T^{(1)}_{n,m},\lambda) = \lambda^{n-2m}(\lambda^2-1)^{m-2}[\lambda^4-(n-m+1)\lambda^2+(n-2m+1)].$$

Lemma 1.3 [6, 7] *Let T be a tree such that $o(T)=n$ and $i(T)=m$.*

(i) *Let $m \geq 2, n \geq \max\{2m+1, 6\}$ and $T \not\cong T^{(1)}_{n,m}$. Then $\lambda_1(T) \leq \lambda_1(T^{(2)}_{n,m})$, and the equality holds if and only if $T \cong T^{(2)}_{n,m}$, where $\lambda_1(T^{(2)}_{n,m})$ is the largest root of the equation*

$$(x^2-2)[x^4-(n-m)x^2+(n-2m-1)]-2=0.$$

(ii) *Let $n=2m+1$ and $T \notin \{T^{(1)}_{2m+1,m}, T^{(2)}_{2m+1,m}\}$. Then $\lambda_1(T) \leq \sqrt{m+1}$, and the equality holds if and only if $T \cong T^{(3)}_{2m+1,m}$.*

Remark The results in Lemma 1.3 (i) do not hold for $m=1$ and $(m,n)=(2,5)$, but they were not excluded in [6]. In fact, $m \geq 2$ is required by the definition of $T^{(2)}_{n,m}$ and $T^{(1)}_{n,m} \cong T^{(2)}_{n,m}$ for $(m,n)=(2,5)$.

Lemma 1.4 [10] *Let T be a tree with order n. Then for any positive integer k such that $1 \leq k \leq \frac{n}{2}$, there exists a vertex subset $V' \subseteq V(T)$ with $k-1$ vertices such that all components of $T - V'$ have order not exceeding $\frac{n}{k}$.*

Lemma 1.5 [9](Cauchy interlacing theorem) *Let G be a graph with order n, V' be a vertex subset with k vertices of G. Let $G - V'$ be the subgraph of G obtained by deleting all the vertices in V' together with their incident edges. Then*

$$\lambda_i(G) \geq \lambda_i(G - V') \geq \lambda_{i+k}(G), \quad i = 1, 2, \cdots, n-k.$$

2 On the second largest eigenvalue of trees

Lemma 2.1 *If $n \geq 3$, then $\lambda_1(T^{(2)}_{2n,n-1}) < \sqrt{n+1}$.*

Proof According to Lemma 1.3, $\lambda_1(T^{(2)}_{2n,n-1})$ is the largest root of the equation $f(\lambda) = 0$, where

$$f(\lambda) = (\lambda^2 - 2)[\lambda^4 - (n+1)\lambda^2 + 3] - 2.$$

$$= (\lambda^4 - 2\lambda^2 + 3))[\lambda^2 - (n+1)] + 3n - 5.$$

Since $f(\lambda) > 0$ for $\lambda \geq \sqrt{n+1}$, we have $\lambda_1(T^{(2)}_{2n,n-1}) < \sqrt{n+1}$.

This completes the proof. □

Lemma 2.2 *Let T be a tree, M a maximum matching of T and assume that T satisfies one of the following conditions:*

(i) *$o(T) = 2n, i(T) = n-1$ and $T \ncong T^{(1)}_{2n,n-1}$;*

(ii) *$o(T) < 2n$ and T has only two vertices not saturated by M;*

(iii) *$o(T) \leq 2n$ and T has at most one vertex not saturated by M.*

If $n \geq 3$, then $\lambda_1(T) < \sqrt{n+1}$.

Proof Firstly, assume that T satisfies condition (i). By Lemma 1.3 (i) and Lemma 2.1, we have

$$\lambda_1(T) \leq \lambda_1(T^{(2)}_{2n,n-1}) < \sqrt{n+1}.$$

Secondly, assume that T satisfies condition (ii). It is obvious that $o(T)$ is an even number. If $o(T) = 2s$, then $s \leq n-1$ and $i(T) = s-1$. By Lemma 1.2, we have

$$\lambda_1(T) \leq \delta_{s+2,12} \leq \delta_{n+1,12} < \sqrt{n+1}.$$

Finally, assume that T satisfies condition (iii). If M is a perfect matching, then $o(T)$ is an even number. If $o(T) = 2s$, then $s \leq n$ and $i(T) = s$. From Lemma 1.2, we have

$$\lambda_1(T) \leq \delta_{s+1,4} \leq \delta_{n+1,4} < \sqrt{n+1}.$$

If T has only one vertex not saturated by M, then $o(T)$ is an odd number. If $o(T) = 2s+1$, then $s \leq n-1$ and $i(T) = s$. By Lemma 1.2, we have

$$\lambda_1(T) \leq \delta_{s+2,8} \leq \delta_{n+1,8} < \sqrt{n+1}.$$

This completes the proof. □

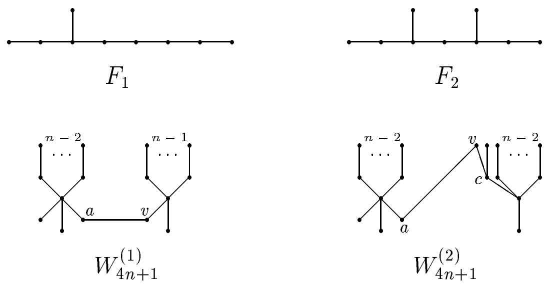


Fig. 2.

Theorem 2.3 *Let T be a tree such that $o(T)=4n+1, i(T)=2n$ and $n\geq 2$. Let $F_1, F_2, W_{4n+1}^{(1)}$ and $W_{4n+1}^{(2)}$ be the four trees shown in Fig. 2.*

(i) *$\lambda_2(T)\leq\lambda_2(W_{4n+1}^{(1)})$, and the equality holds if and only if $T\cong W_{4n+1}^{(1)}$, where $\lambda_2(W_{4n+1}^{(1)})$ is the second largest root of the following equation*

$$[\lambda^4-(n+2)\lambda^2+2][\lambda^4-(n+2)\lambda^2+1]-\lambda^2=0.$$

(ii) *Let $T\not\cong W_{4n+1}^{(1)}$.*

If $n=2$, then $\lambda_2(T)\leq\frac{\sqrt{5}+1}{2}$, and the equality holds if and only if $T\in\{F_1,F_2\}$.

If $n\geq 3$, then $\lambda_2(T)\leq\lambda_2(W_{4n+1}^{(2)})$, and the equality holds if and only if $T\cong W_{4n+1}^{(2)}$, where $\lambda_2(W_{4n+1}^{(2)})$ is the second largest root of the following equation

$$(\lambda^2-1)^2[\lambda^4-(n+2)\lambda^2+2][\lambda^4-(n+1)\lambda^2+1]=\lambda^2[\lambda^4-(n+2)\lambda^2+3](\lambda^4-n\lambda^2+1).$$

Proof For $n=2$, the required result follows by the table of trees on 9 vertices (see [11]).

Suppose now that $n\geq 3$ and $T\notin\{W_{4n+1}^{(1)}, W_{4n+1}^{(2)}\}$. We next prove the following two claims.

Claim 1 $\lambda_2(T)\leq\sqrt{n+1}$.

Take $k=2$ in Lemma 1.4. Then there is a vertex $v\in V(T)$ such that each component of $T-v$, say $T_i (i=1,2,\cdots,l)$, is of order at most $2n$. By Lemma 1.5, we have

$$\lambda_2(T)\leq\lambda_1(T-v)=\max\{\lambda_1(T_1),\lambda_1(T_2),\cdots,\lambda_1(T_l)\}. \qquad (1)$$

Let M be a maximum matching of $T-v$, then there exist at most two vertices of $T-v$ not saturated by M. We distinguish the following two cases.

Case 1 Each component T_i of $T-v$ has at most one vertex not saturated by M.

Since each $T_i (i=1,2,\cdots,l)$ satisfies the condition (iii) of Lemma 2.2, by (1) and Lemma 2.2, Claim 1 holds.

Case 2 There exists a component of $T-v$ such that it has two vertices not saturated by M.

Without loss of generality, suppose that T_1 has only two vertices not saturated by M, then all the other components of $T-v$ have a perfect matching. Obviously the order of T_1 is an even number. If $o(T_1)=2s_1$, then $s_1 \le n$ and $i(T_1)=s_1-1$.

Case 2.1 Assume $T_1 \ncong T^{(1)}_{2n,n-1}$.

In this case, $s_1 \le n-1$, or $s_1=n$ and $T_1 \ncong T^{(1)}_{2n,n-1}$. So T_1 satisfies the condition (ii) or (i) of Lemma 2.2, while each $T_i (i \ge 2)$ satisfies the condition (iii) of Lemma 2.2. Hence by Lemma 2.2 and (1), Claim 1 holds.

Case 2.2 Assume $T_1 \cong T^{(1)}_{2n,n-1}$.

Let a be the unique vertex being adjacent to v in T_1, then a is one of two vertices of T_1 not saturated by M. It is obvious that $T-a$ has only two components, say $\tilde{T_1}$ and $\tilde{T_2}$. Without loss of generality, suppose that $\tilde{T_1}$ does not contain the vertex v. Then $\tilde{T_1} = T_1 - a \cong T^{(1)}_{2n-1,n-1}$, $o(\tilde{T_2}) = 2n+1$ and $i(\tilde{T_2}) = n$. If $\tilde{T_2} \cong T^{(1)}_{2n+1,n}$, then $T \cong W^{(1)}_{4n+1}$; if $\tilde{T_2} \cong T^{(2)}_{2n+1,n}$, then $T \cong W^{(2)}_{4n+1}$. But this contradicts $T \notin \{W^{(1)}_{4n+1}, W^{(2)}_{4n+1}\}$. Hence $\tilde{T_2} \notin \{T^{(1)}_{2n+1,n}, T^{(2)}_{2n+1,n}\}$. By Lemma 1.2 and Lemma 1.3(iii), we have

$$\lambda_1(\tilde{T_1}) = \lambda_1(T^{(1)}_{2n-1,n-1}) = \delta(n+1,8) < \sqrt{n+1}.$$

$$\lambda_1(\tilde{T_2}) \le \lambda_1(T^{(3)}_{2n+1,n}) = \sqrt{n+1}.$$

Hence by Lemma 1.5, we have

$$\lambda_2(T) \le \lambda_1(T-u) = \max\{\lambda_1(\tilde{T_1}), \lambda_1(\tilde{T_2})\} \le \sqrt{n+1}.$$

By the above discussion of Cases 1 and 2, we complete the proof of Claim 1.

Claim 2 $\lambda_2(W^{(1)}_{4n+1}) > \lambda_2(W^{(2)}_{4n+1})$.

According to Lemma 1.1 and $\phi(T^{(1)}_{n,m}, \lambda)$, we have

$$\begin{aligned}
\phi(W^{(1)}_{4n+1}) &= \phi(W^{(1)}_{4n+1} - av) - \phi(W^{(1)}_{4n+1} - a - v) \\
&= \phi(T^{(1)}_{2n,n-1} \bigcup T^{(1)}_{2n+1,n}) - \phi(T^{(1)}_{2n-1,n-1} \bigcup T^{(1)}_{2n,n}) \\
&= \lambda(\lambda^2-1)^{2n-4} f(\lambda), \\
\phi(W^{(2)}_{4n+1}) &= \phi(W^{(2)}_{4n+1} - vc) - \phi(W^{(2)}_{4n+1} - v - c) \\
&= \phi(T^{(1)}_{2n+1,n} \bigcup T^{(1)}_{2n,n}) - \phi(P_1 \bigcup T^{(1)}_{2n,n-1} \bigcup T^{(1)}_{2n-2,n-1}) \\
&= \lambda(\lambda^2-1)^{2n-6} g(\lambda),
\end{aligned}$$

where

$$f(\lambda) = [\lambda^4 - (n+2)\lambda^2 + 2][\lambda^4 - (n+2)\lambda^2 + 1] - \lambda^2.$$

$$g(\lambda) = (\lambda^2-1)^2[\lambda^4 - (n+2)\lambda^2 + 2][\lambda^4 - (n+1)\lambda^2 + 1]$$

$$-\lambda^2[\lambda^4-(n+2)\lambda^2+3][\lambda^4-n\lambda^2+1].$$

Since

$$f(0)=2>0,\ \ f(\beta_{n+2,8})=-\beta_{n+2,8}^2<0,$$

$$f(\sqrt{n+1+\frac{1}{n}})=\frac{n^2(n-1)^2+1}{n^3}>0,$$

$$f(\delta_{n+2,12})=-\delta_{n+2,12}^2+2<0,$$

$$f(\sqrt{n+3})=n^2+8n+17>0,$$

we have

$$\sqrt{n+1+\frac{1}{n}}<\lambda_2(W_{4n+1}^{(1)})<\delta_{n+2,12}. \tag{2}$$

Since

$$g(0)=2>0,\ \ g(\beta_{n,4})=-\beta_{n,4}^2[1-\beta_{n,4}^2]^2[1-2\beta_{n,4}^2]<0,$$

$$g(\beta_{n,8})=\beta_{n,8}^2\left\{2[1-\beta_{n,8}^2][1-\beta_{n,8}^4]+1-2\beta_{n,8}^2\right\}>0,$$

$$g(1)=-(n-2)^2<0,\ \ g(\sqrt{n+1})=2(n^2-2n-2)>0,$$

$$g(\gamma_n)=-\frac{16n^7(2n^2-11n+23)+8n^4(42n^2+49n+12)+4n^2(11n-1)-1}{64n^6}<0,$$

$$g(\sqrt{n+2})=n(2n^2-13)+4(n^2-6)>0,$$

we have

$$\sqrt{n+1}<\lambda_2(W_{4n+1}^{(2)})<\gamma_n. \tag{3}$$

By Equation (2) and (3), we have

$$\delta_{n+2,12}>\lambda_2(W_{4n+1}^{(1)})>\sqrt{n+1+\frac{1}{n}}>\gamma_n>\lambda_2(W_{4n+1}^{(2)}). \tag{4}$$

Combining Claims 1 and 2, the proof follows. □

Remark 2.1 (i) There are only two trees, P_5 and $T_{5,2}^{(1)}$, with order $4n+1$ and edge independence number $2n$ for $n=1$, and $\lambda_2(P_5)>\lambda_2(T_{5,2}^{(1)})$.

(ii) By (4), we have $\lambda_2(W_{4n+1}^{(1)})>\gamma_n>\delta_{n+1,12}>\lambda_2(W_{4(n-1)+1}^{(1)})$. This indicates that $\lambda_2(W_{4n+1}^{(1)})$ is strictly increasing in $n (n\geq 2)$.

Lemma 2.4 *Let T be a tree such that $o(T)\leq 2n-1$, M be a maximum matching of T and there are at most two vertices of T not saturated by M. If $n\geq 2$, then*

$$\lambda_1(T)\leq\delta_{n+1,8}=\sqrt{\frac{1}{2}(n+1+\sqrt{(n+1)^2-8})}.$$

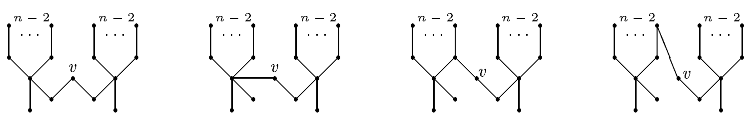
Proof Clearly $i(T) \le n-1$. Since there are at most two vertices of T not saturated by M, we have $i(T) \ge \frac{o(T)-2}{2}$. So we obtain

$$o(T) - i(T) + 1 \le o(T) - \frac{o(T)-2}{2} + 1 = \frac{1}{2}o(T) + 2$$
$$\le \frac{2n-1}{2} + 2 = n + 1 + \frac{1}{2}.$$

This implies that $c = o(T) - i(T) + 1 \le n + 1$. So by Lemma 1.2, we have

$$\lambda_1(T) \le \sqrt{\frac{1}{2}(c + \sqrt{c^2 - 4c + 4i(T)})}$$
$$\le \sqrt{\frac{1}{2}(n+1+\sqrt{(n+1)^2 - 4(n+1) + 4(n-1)})} = \delta_{n+1,8}.$$

This completes the proof. □



$W^{(1)}_{4n-1}$ $W^{(2)}_{4n-1}$ $W^{(3)}_{4n-1}$ $W^{(4)}_{4n-1}$

Fig. 3.

Theorem 2.5 *Let T be a tree such that $o(T) = 4n-1$, $i(T) = 2n-1$ and $n \ge 2$. Then*

$$\lambda_2(T) \le \sqrt{\frac{1}{2}(n+1+\sqrt{(n+1)^2-8})}\,, \tag{5}$$

and the equality holds if T is one of the trees shown in Fig. 3.

Proof Take $k = 2$ in Lemma 1.4. Then there exists one vertex $v \in V(T)$ such that each component of $T - v$, say $T_i (i = 1, 2, \cdots, l)$, is of order at most $2n-1$. By Lemma 1.5, we have

$$\lambda_2(T) \le \lambda_1(T - v) = \max\{\lambda_1(T_1), \lambda_1(T_2), \cdots, \lambda_1(T_l)\}. \tag{6}$$

Let M be a maximum matching of $T - v$, then $T - v$ has at most two vertices not saturated by M. So each $T_i (i = 1, 2, \cdots, l)$ has at most two vertices not saturated by M. Hence from Lemma 2.4 and (6), (5) follows.

On the other hand, by Lemma 1.2, we easily find

$$\lambda_1(W^{(i)}_{4n-1} - v) = \lambda_2(W^{(i)}_{4n-1} - v) = \lambda_1(T^{(1)}_{2n-1,n-1}) = \delta_{n+1,8}.$$

By Lemma 1.5, we have

$$\lambda_1(W^{(i)}_{4n-1} - v) \ge \lambda_2(W^{(i)}_{4n-1}) \ge \lambda_2(W^{(i)}_{4n-1} - v).$$

Hence we have

$$\lambda_2(W_{4n-1}^{(i)}) = \delta_{n+1,8} = \sqrt{\frac{1}{2}(n+1+\sqrt{(n+1)^2-8})}\ (1 \le i \le 4).$$

This completes the proof. □

Remark 2.2 (i) P_3 is the unique tree on order $4n-1$ and edge independence $2n-1$ for $n=1$.

(ii) We conjecture that $W_{4n-1}^{(i)} (i = 1, 2, 3, 4)$ are all trees such that the equality of (5) holds.

Acknowledgements

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