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On conjectures on the defining set of (vertex) graph colourings

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Abstract

In a given graph $G = (V, E)$, a set of vertices S with an assignment of colours to them is a defining set of the vertex colouring of G, if there exists a unique extension of the colours of S to a $\chi(G)-$colouring of the vertices of G. A defining set with minimum cardinality is called a minimum defining set (of vertex colouring) and its cardinality, the defining number, is denoted by $d(G, \chi)$. In *Combinatorics, Graph Theory and Algorithms* (1999), 461–467, Mahdian et al. have studied $d(C_m \times K_3, \chi)$, $d(C_m \times K_4, \chi)$ and $d(C_m \times K_5, \chi)$. They have conjectured:

(a) $d(C_m \times K_5, \chi) = 2m + 1$ for m odd; and

(b) $d(C_m \times K_4, \chi) = m + 1$.

In this paper we disprove conjecture (a) for m (≥ 5) odd and prove conjecture (b).

1 Introduction

A c-colouring (proper c-colouring) of a graph G is an assignment of c different colours to the vertices of G, such that no two adjacent vertices receive the same colour. The vertex chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number c for which there exists a c-colouring of G. In a given graph $G = (V, E)$, a set of vertices S with an assignment of colours to them is called a defining set of the vertex colouring of G if there exists a unique extension of the colours of S to a $\chi(G)$-colouring of the vertices of G. A defining set with minimum cardinality is called a minimum defining set (of vertex colouring) and its cardinality, the defining number, is denoted by $d(G, \chi)$; (see [6, 7]). The concept of defining set has been studied, to some extent, for block design and also under another name, critical set, for latin squares; (see [1, 3, 9, 10]). In [6, 8] this concept is extended to colourings of graphs. Let G be a graph and let $L(v)$ denote a list of colours available for a vertex v of G. A list colouring from the given collection of lists is a proper colouring c such that the colour of vertex v, $c(v)$, is in $L(v)$ (see [2]). Defining sets of vertex colourings are closely related to the list colouring, (see [7]).

In [5] Mahdian et al. proved the following result.

Theorem A.

(1) $d(C_m \times K_3, \chi) = \lfloor \frac{m}{2} \rfloor + 1$.

(2) $m \leq d(C_m \times K_4, \chi) \leq m + 1$.

(3) $d(C_m \times K_5, \chi) = 2m$ *for* m *even and* $2m \leq d(C_m \times K_5, \chi) \leq 2m + 1$ *for* m *odd.*

They have conjectured:

(a) $d(C_m \times K_5, \chi) = 2m + 1$ for m odd; and

(b) $d(C_m \times K_4, \chi) = m + 1$.

The following will be useful.

Theorem B. [8] *For any graph* $G = (V(G), E(G))$ *such that* $\chi(G) \leq n$, *we have* $d(G \times K_n, \chi) \geq |V(G)|(n-1) - 2|E(G)|$.

Definition [4] *A graph* G *with* n *vertices is called a uniquely* $2-$*list colourable graph, if there exists* $S_1, S_2, \cdots S_n$, *a list of colours on its vertices, each of size* 2, *such that there is a unique colouring for* G *from this list of colours.*

Theorem C. [4] *A connected graph is uniquely* $2-$*list colourable if and only if at least one of its blocks is not a cycle, a complete graph, or a complete bipartite graph.*

Let $G = C_m \times K_n$. Each subgraph K_n of G is called a row and each subgraph C_m of G is called a column. It is well-known that $\chi(G) = n$ when $n \geq 3$.

2 $d(C_m \times K_5, \chi)$

In this section we disprove conjecture (a); in other words we show that $d(G = C_m \times K_5, \chi(G)) = 2m$ for odd integers $m \geq 5$. Note that all vertices in arrays of this section are labelled by their colours. The non-indexed labels in the arrays denote the defining set and the indexed labels are the vertices, whose colours are uniquely determined by the defining set, while the indices denote the order of colouring of these vertices.

Lemma 2.1. *$d(C_5 \times K_5, \chi) = 10$ and $d(C_7 \times K_5, \chi) = 14$.*

Proof. For $m = 5$ consider the array

$$\begin{bmatrix} 2 & 1 & 4_2 & 5_1 & 3_3 \\ 1_4 & 5_7 & 3 & 4 & 2_8 \\ 5 & 2_6 & 4_9 & 3_5 & 1 \\ 1_{11} & 4 & 5_{12} & 2 & 3_{10} \\ 4_{13} & 2_{14} & 1_{15} & 3 & 5 \end{bmatrix},$$

where the cardinality of the defining set (non-indexed labels) is 10.

For $m = 7$ consider the array

$$\begin{bmatrix} 2 & 1 & 4_3 & 3_1 & 5_2 \\ 1_4 & 5_6 & 3 & 4 & 2_5 \\ 5 & 2_8 & 4_9 & 3_7 & 1 \\ 1_{11} & 4 & 5_{12} & 2 & 3_{10} \\ 4_{15} & 1_{13} & 2_{14} & 3 & 5 \\ 5_{18} & 2 & 4 & 1_{16} & 3_{17} \\ 1_{20} & 3_{19} & 2_{21} & 5 & 4 \end{bmatrix},$$

where the cardinality of the defining set (non-indexed labels) is 14. □

Theorem 2.1. *Let $G = (C_m \times K_5)$ then $d(C_m \times K_5, \chi) = 2m$, for m (≥ 5) odd.*

Proof. For $m = 5$ and $m = 7$, the result follows from Lemma 2.1. If $m \geq 9$ is an odd number then $m = 4n + 5$ or $m = 4n + 7$ for some positive integer n.

For $m = 4n + 5$ consider the colouring $\begin{bmatrix} 2 & 1 & 5_3 & 4_1 & 3_2 \\ 1_5 & 4_4 & 3 & 5 & 2_6 \\ 3_9 & 2 & 5_8 & 4_7 & 1 \\ 1_{11} & 4_{10} & 2_{12} & 3 & 5 \end{bmatrix}$ for $C_4 \times K_5$.

Now we take the defining set of $C_4 \times K_5$ n times and then combine this with the defining set for $(C_5 \times K_5)$ as in array X overleaf.

Then, as the array X shows, the cardinality of the defining set of $(C_{4n+5} \times K_5)$ is $(4n + 5)5 - (12n + 15) = 8n + 10 = 2m$.

For $m = 4n + 7$ consider the colouring $\begin{bmatrix} 2 & 1 & 3_2 & 4_1 & 5_3 \\ 1_5 & 4_4 & 5 & 3 & 2_6 \\ 5_9 & 2 & 3_8 & 4_7 & 1 \\ 1_{11} & 3_{10} & 2_{12} & 5 & 4 \end{bmatrix}$ for $C_4 \times K_5$.

Now take the defining set of $C_4 \times K_5$ n times and then combine this with the defining set for $(C_7 \times K_5)$ as given in array Y overleaf.

$$X:\quad \begin{bmatrix}
2 & 1 & 5_3 & 4_1 & 3_2 \\
1_5 & 4_4 & 3 & 5 & 2_6 \\
3_9 & 2 & 5_8 & 4_7 & 1 \\
1_{11} & 4_{10} & 2_{12} & 3 & 5 \\
2 & 1 & 5_{15} & 4_{13} & 3_{14} \\
1_{17} & 4_{16} & 3 & 5 & 2_{18} \\
3_{21} & 2 & 5_{20} & 4_{19} & 1 \\
1_{23} & 4_{22} & 2_{24} & 3 & 5 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
2 & 1 & 5_{12(n-1)+3} & 4_{12(n-1)+1} & 3_{12(n-1)+2} \\
1_{12(n-1)+5} & 4_{12(n-1)+4} & 3 & 5 & 2_{12(n-1)+6} \\
3_{12(n-1)+9} & 2 & 5_{12(n-1)+8} & 4_{12(n-1)+7} & 1 \\
1_{12(n-1)+11} & 4_{12(n-1)+10} & 2_{12n} & 3 & 5 \\
2 & 1 & 4_{12n+2} & 5_{12n+1} & 3_{12n+3} \\
1_{12n+4} & 5_{12n+7} & 3 & 4 & 2_{12n+8} \\
5 & 2_{12n+6} & 4_{12n+9} & 3_{12n+5} & 1 \\
1_{12n+11} & 4 & 5_{12n+12} & 2 & 3_{12n+10} \\
4_{12n+13} & 2_{12n+14} & 1_{12n+15} & 3 & 5
\end{bmatrix}.$$

$$Y:\quad \begin{bmatrix}
2 & 1 & 3_2 & 4_1 & 5_3 \\
1_5 & 4_4 & 5 & 3 & 2_6 \\
5_9 & 2 & 3_8 & 4_7 & 1 \\
1_{11} & 3_{10} & 2_{12} & 5 & 4 \\
2 & 1 & 3_{14} & 4_{13} & 5_{15} \\
1_{17} & 4_{16} & 5 & 3 & 2_{18} \\
5_{21} & 2 & 3_{20} & 4_{19} & 1 \\
1_{23} & 3_{22} & 2_{24} & 5 & 4 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
2 & 1 & 3_{12(n-1)+2} & 4_{12(n-1)+1} & 5_{12(n-1)+3} \\
1_{12(n-1)+5} & 4_{12(n-1)+4} & 5 & 3 & 2_{12(n-1)+6} \\
5_{12(n-1)+9} & 2 & 3_{12(n-1)+8} & 4_{12(n-1)+7} & 1 \\
1_{12(n-1)+11} & 3_{12(n-1)+10} & 2_{12n} & 5 & 4 \\
2 & 1 & 4_{12n+3} & 3_{12n+1} & 5_{12n+2} \\
1_{12n+4} & 5_{12n+6} & 3 & 4 & 2_{12n+5} \\
5 & 2_{12n+8} & 4_{12n+9} & 3_{12n+7} & 1 \\
1_{12n+11} & 4 & 5_{12n+12} & 2 & 3_{12n+10} \\
4_{12n+15} & 1_{12n+13} & 2_{12n+14} & 3 & 5 \\
5_{12n+18} & 2 & 4 & 1_{12n+16} & 3_{12n+17} \\
1_{12n+20} & 3_{12n+19} & 2_{12n+21} & 5 & 4
\end{bmatrix}.$$

Thus, as the array Y shows, the cardinality of the defining set of $(C_{4n+7} \times K_5)$ is $(4n+7)5 - (12n+21) = 8n + 14 = 2m$. $\square$

As an immediate result we have

Corollary 2.2. *The conjecture (a) for* $m \geq 5$ *is disproved.*

3 $d(C_m \times K_4, \chi)$

In this section we prove conjecture (b); in other words we prove that $d(G = C_m \times K_4, \chi(G)) = m + 1$.

By (2) of Theorem A, we have $m \leq d(C_m \times K_4, \chi) \leq m + 1$. We show that $d(C_m \times K_4, \chi) \neq m$.

Lemma 3.1. *Let* $G = C_m \times K_4$, *then each row has at least one vertex in the defining set of* G.

Proof. On the contrary suppose that one of the rows has no vertex in the defining set. So if all of the vertices of the other rows have been coloured, then by Theorem C the vertices of this row cannot be uniquely coloured. $\square$

Theorem A implies that the defining number $d(C_m \times K_4, \chi) \geq m$.

Lemma 3.2. *Let* $G = C_m \times K_4$ *and* $d(G, \chi) = m$. *If* v *and* u *are two adjacent vertices of the same column of* G, *then they cannot be simultaneously in the defining set of* G.

Proof. Suppose that $m > 3$ (for $m = 3$ it is easily verified). On the contrary, assume that v, u are in a defining set and all vertices of the rows of G except the rows including vertex v and vertex u have already been coloured. So there remain two rows in which six vertices are colourless; these are shown with a $(\star\text{s})$ in the following diagram: $\begin{bmatrix} v & \star & \star & \star \\ u & \star & \star & \star \end{bmatrix}$.

Without loss of generality, let the colours of v and u be $c(v) = 1$ and $c(u) = 2$. So four vertices $(\star\text{s})$ have two choices for colouring and the others have three choices for colouring. By colouring the vertices of all of the rows except the rows including v and u, one of the following arrays of the set A or B must occur. Note that for $m = 4$ one of the arrays of the set B occurs and for $m > 4$ one of the arrays of the set B or the set A occurs.

$$A = \left\{ \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 1 & 2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 1 & 3 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 1 & 4 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 3 & 1 & 4 & 2 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 3 & 1 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 3 & 2 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 1 & 3 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 1 & 2 & 3 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 1 & 4 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 4 & 3 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 4 & 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 4 & 1 & 3 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 3 & 1 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 3 & 1 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 3 & 4 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 3 & 4 & 1 & 2 \end{bmatrix},$$

$$\left.\begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 3 & 2 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 3 & 2 & 4 & 1 \end{bmatrix}\right\}.$$

$$B = \left\{\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 1 & 2 & 4 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 1 & 3 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 1 & 4 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 3 & 4 & 2 & 1 \end{bmatrix},\right.$$

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 3 & 4 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 3 & 2 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 1 & 3 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 1 & 2 & 4 & 3 \end{bmatrix},$$

$$\left.\begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 1 & 4 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 4 & 2 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 4 & 2 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & \star & \star & \star \\ 2 & \star & \star & \star \\ 4 & 3 & 1 & 2 \end{bmatrix}\right\}.$$

The other cases are similar, by the colours of v and u. It is easily verifiable that the non-coloured vertices in all arrays are not uniquely colourable. This shows that the two adjacent vertices v and u in the same column cannot be simultaneously in the defining set of G while $d(G, \chi) = m$. □

By the previous lemmas we have:

Corollary 3.1. *Let $G = C_m \times K_4$ and $d(G, \chi) = m$; then the vertices in the defining set are independent.*

Let $G = C_m \times K_n$. We show that, if $d(G, \chi) = m$, then the non-coloured vertices of every three consecutive rows of G cannot be uniquely coloured, and then all of the non-coloured vertices of G cannot be uniquely coloured.

Proposition 3.2. *Let $G = C_m \times K_4$ and $d(G, \chi) = m$; then every three consecutive rows of G admit two different colourings.*

Proof. There must be one of the following arrays for three consecutive rows in G when we assign (one, two or three) different colours to the three independent vertices of these rows which are in the defining set.

$$\begin{bmatrix} 1 & \star & \star & \star \\ \star & 2 & \star & \star \\ 3 & \star & \star & \star \end{bmatrix} \begin{bmatrix} 1 & \star & \star & \star \\ \star & 2 & \star & \star \\ 2 & \star & \star & \star \end{bmatrix} \begin{bmatrix} 1 & \star & \star & \star \\ \star & 2 & \star & \star \\ 1 & \star & \star & \star \end{bmatrix} \begin{bmatrix} 1 & \star & \star & \star \\ \star & 1 & \star & \star \\ 2 & \star & \star & \star \end{bmatrix} \begin{bmatrix} 1 & \star & \star & \star \\ \star & 1 & \star & \star \\ 1 & \star & \star & \star \end{bmatrix}$$

$$\begin{bmatrix} 1 & \star & \star & \star \\ \star & 2 & \star & \star \\ \star & \star & 3 & \star \end{bmatrix} \begin{bmatrix} 1 & \star & \star & \star \\ \star & 2 & \star & \star \\ \star & \star & 2 & \star \end{bmatrix} \begin{bmatrix} 1 & \star & \star & \star \\ \star & 2 & \star & \star \\ \star & \star & 1 & \star \end{bmatrix} \begin{bmatrix} 1 & \star & \star & \star \\ \star & 1 & \star & \star \\ \star & \star & 2 & \star \end{bmatrix} \begin{bmatrix} 1 & \star & \star & \star \\ \star & 1 & \star & \star \\ \star & \star & 1 & \star \end{bmatrix}.$$

Note that in every array except the third and the fifth, the first row and the third row can be adjacent.

For each array, there exist at least two different extensions as follows.

$$\begin{bmatrix} 1 & \star & \star & \star \\ \star & 2 & \star & \star \\ 3 & \star & \star & \star \end{bmatrix} \rightarrowtail \begin{bmatrix} 1 & 4 & 2 & 3 \\ 4 & 2 & 3 & 1 \\ 3 & 1 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 3 & 2 \\ 4 & 2 & 1 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix}. \quad \begin{bmatrix} 1 & \star & \star & \star \\ \star & 2 & \star & \star \\ 2 & \star & \star & \star \end{bmatrix} \rightarrowtail \begin{bmatrix} 1 & 4 & 3 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 3 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 & 4 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & \star & \star & \star \\ \star & 2 & \star & \star \\ 1 & \star & \star & \star \end{bmatrix} \rightarrowtail \begin{bmatrix} 1 & 4 & 3 & 2 \\ 4 & 2 & 1 & 3 \\ 1 & 3 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 & 4 \\ 4 & 2 & 1 & 3 \\ 1 & 4 & 3 & 2 \end{bmatrix}. \quad \begin{bmatrix} 1 & \star & \star & \star \\ \star & 1 & \star & \star \\ 2 & \star & \star & \star \end{bmatrix} \rightarrowtail \begin{bmatrix} 1 & 4 & 3 & 2 \\ 4 & 1 & 2 & 3 \\ 2 & 3 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 & 4 \\ 4 & 1 & 3 & 2 \\ 2 & 4 & 1 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & \star & \star & \star \\ \star & 1 & \star & \star \\ 1 & \star & \star & \star \end{bmatrix} \rightarrowtail \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \\ 1 & 4 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 1 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix}. \quad \begin{bmatrix} 1 & \star & \star & \star \\ \star & 2 & \star & \star \\ \star & \star & 3 & \star \end{bmatrix} \rightarrowtail \begin{bmatrix} 1 & 3 & 2 & 4 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 1 & 3 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & \star & \star & \star \\ \star & 2 & \star & \star \\ \star & \star & 2 & \star \end{bmatrix} \rightarrowtail \begin{bmatrix} 1 & 4 & 3 & 2 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix}. \quad \begin{bmatrix} 1 & \star & \star & \star \\ \star & 2 & \star & \star \\ \star & \star & 1 & \star \end{bmatrix} \rightarrowtail \begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 3 & 1 \\ 2 & 4 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 3 & 2 \\ 3 & 2 & 4 & 1 \\ 2 & 3 & 1 & 4 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & \star & \star & \star \\ \star & 1 & \star & \star \\ \star & \star & 2 & \star \end{bmatrix} \rightarrowtail \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}. \quad \begin{bmatrix} 1 & \star & \star & \star \\ \star & 1 & \star & \star \\ \star & \star & 1 & \star \end{bmatrix} \rightarrowtail \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 & 4 \\ 4 & 1 & 3 & 2 \\ 2 & 4 & 1 & 3 \end{bmatrix}.$$

These arrays show that every three consecutive rows of G admit two different colourings. By using induction, this is true for all rows. □

Theorem 3.3. *If $G = (C_m \times K_4)$ then $d(G, \chi) = m + 1$.*

Proof. By Lemma 3.1 there is at least one vertex in the defining set of G for each row, so $d(G, \chi) \geq m$. By Lemma 3.2 the m vertices of the m rows are independent. By Proposition 3.2, m independent vertices cannot form a defining set. So we have at least $m + 1$ vertices in the defining set of G. Therefore, by Theorem A, $d(G, \chi) = m + 1$. □

As an immediate result we have,

Corollary 3.4. *The conjecture (b) is proved.*

References

[1] J. Cooper, D. Donovan and J. Seberry, Latin squares and critical sets of minimal size, *Australas. J. Combin.* 4 (1991), 113–120.

[2] A. Daneshgar and R. Naserasr, On the small uniquely vertex colourable graphs and Xu's conjecture, *Discrete Math.* 223 (2000), 93–108.

[3] A. D. Keedwell, Critical sets and critical partial Latin squares, *Combinatorics, graph theory, algorithms and applications* (Beijing, 1993), 111–123, World Sci. Publishing, River Edge, NJ, 1994.

[4] M. Mahdian and E. S. Mahmoodian, A characterization of uniquely 2-list colourable graphs, *Ars Combinatoria* 51 (1999), 295–305.

[5] M. Mahdian, E. S. Mahmoodian, R. Naserasr and F. Harary, On defining sets of vertex colourings of the cartesian product of a cycle with a complete graph, *Combinatorics, Graph Theory and Algorithms* (1999), 461–467.

[6] E. S. Mahmoodian, Some problem in graph colouring, in: S. Javadpour, M. Radjabalipour (Eds), *Proc. 26th Annual Iranian Math. Conf.*, Kerman, Iran, March 1995, Iranian Math. Soc., University of Kerman, pp. 215–218.

[7] E. S. Mahmoodian and E. Mendelsohn, On defining numbers of vertex colouring of regular graphs, *Discrete Math.* 197/198 (1999), 543–554.

[8] E. S. Mahmoodian, R. Naserasr and M. Zaker, Defining sets in vertex colourings of graphs and Latin rectangles. *Discrete Math.* 167/168 (1997), 451–460.

[9] A. P. Street, Defining sets for block designs; an update, in: C. J. Colbourn and E. S. Mahmoodian (Eds), *Combinatorics Advances, Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, (1995), pp. 307–320.

[10] A. P. Street, Defining sets for t-designs and critical sets for Latin squares, *New Zealand J. Math.* 21 (1992), 133–144.

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