AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **32** (2005), Pages 205–211

Gracefulness of a cycle with parallel P_k-chords

G. Sethuraman A. Elumalai*

Department of Mathematics
Anna University
Chennai–600 025
INDIA
sethu@annauniv.edu

Abstract

In this paper we prove that every n-cycle ($n \geq 6$) with parallel P_k-chords is graceful for $k = 3$ and for $k = 2r$, where $2 \leq r \leq 5$, and we discuss a related problem.

1 Introduction

A function f is called a *graceful labeling* of a graph G with m edges if f is an injection from the vertex set of G to the set $\{0, 1, 2, \ldots, m\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.

A line of work on graceful graphs has concentrated on graphs related to the cycles stemming from Rosa's result [5] that a cycle C_n is graceful iff $n = 0$ or $3(\text{mod}4)$. A *chord* of a cycle is an edge joining two non adjacent vertices of the cycle. Bodendiek, Schumacher and Wegner conjectured in [1] that every cycle with a chord is graceful. The validity of this conjecture has been proved by Delorme, Maheo et al. in [2]. A natural extension of the structure of a cycle with a chord is that of a cycle with a P_k-chord. A cycle with a P_k-chord ($k > 2$) is a graph obtained by joining a pair of non adjacent vertices of a cycle of order n ($n > 4$) by a path of order k.

Koh and Yap have shown that cycles with P_3-chords are graceful and conjectured that all cycles with P_k-chords are graceful. This was proved for $k \geq 4$ by Punnim and Pabhapote [4]. For an excellent survey on graceful labeling refer [3].

A graph G is called a *cycle with parallel P_k-chords* if G is obtained from the cycle C_n of order n: $u_0 u_1 \cdots u_{n-1}$ ($n \geq 6$) by adding a disjoint path P_k ($k \geq 3$) between each pair of vertices $(u_1, u_{n-1}), (u_2, u_{n-2}), \ldots, (u_i, u_{n-i}) \ldots, (u_\alpha, u_\beta)$ of C_n, where $\alpha = \lfloor \frac{n}{2} \rfloor - 1$ and $\beta = \lfloor \frac{n}{2} \rfloor + 2$ if n is odd or $\beta = \lfloor \frac{n}{2} \rfloor + 1$ if n is even.

* Currently working in the Department of Mathematics, Crescent Engineering College, Chennai–600 048.

In this note we prove that a cycle with parallel P_k-chords is graceful for $k = 3$ and for $k = 2r$, where $2 \leq r \leq 5$ and we discuss a related problem.

2 Gracefulness of a cycle with parallel P_k-chords

In this section we prove our main results that the cycle C_n with parallel P_k-chords is graceful for $k = 3$ and for $k = 2r$, where $2 \leq r \leq 5$.

Let G be a cycle C_n: $u_0u_1u_2\ldots u_{n-1}$ with parallel P_k-chords. We call the P_k-chords joining the pair (u_i, u_{n-i}) of C_n in G, the ith P_k-chord, for $1 \leq i \leq \lfloor\frac{n}{2}\rfloor - 1$. Observe that G has a hamiltonian path starting at $v_0 = u_0$ and ending up with u_γ of the cycle C_n of G, where $\gamma = \lfloor\frac{n}{2}\rfloor + 1$ if n is odd or $\gamma = \lfloor\frac{n}{2}\rfloor$ if n is even.

Let $v_0v_1\ldots v_{N-1}$, where $N = |V(G)|$ be a hamiltonian path in G starting with u_0 of C_n in G and ending up with u_γ of C_n in G, where $\gamma = \lfloor\frac{n}{2}\rfloor + 1$ if n is odd or $\gamma = \lfloor\frac{n}{2}\rfloor$ if n is even.

Theorem 1: *For $n \geq 6$, every n-cycle with parallel P_3-chords is graceful.*

Proof: Let G be an n-cycle with parallel P_3-chords. Observe that G has $N = \frac{3n-\alpha}{2}$ vertices and $M = 2n - \alpha$ edges, where $\alpha = 3$ if n is odd or $\alpha = 2$ if n is even.

Let $v_0v_1\ldots v_{N-1}$ be a hamiltonian path in G.

Now we give the labeling to the vertices $v_0, v_1, v_2, \ldots, v_{N-1}$ in four cases depending on the remainder of n mod 4, where n is the length of the cycle C_n in G.

Case I: *When $n = 4r$, where $r \geq 2$ is any positive integer.*

Define
$$\begin{aligned}
\phi(v_0) &= 0\\
\phi(v_2) &= 1\\
\phi(v_{2i}) &= i+1, \text{ for } 2 \leq i \leq \frac{N-5}{2}\\
\phi(v_{N-3}) &= \frac{N+1}{2}\\
\phi(v_{N-1}) &= \frac{N+3}{2}\\
\phi(v_1) &= M\\
\phi(v_3) &= M-2\\
\phi(v_{2i+1}) &= \phi(v_{2i-1}) - \alpha, \text{where } \alpha = \begin{cases} 1 & \text{if } 2i+1 \text{ is not a multiple of } 3\\ 3 & \text{if } 2i+1 \text{ is a multiple of } 3\\ & \text{and } 2 \leq i \leq \frac{N-5}{2}. \end{cases}\\
\phi(v_{N-2}) &= \phi(v_{N-4}) - 2
\end{aligned}$$

Case II: *When* $n = 4r + 1$*, where* $r \geq 2$ *is any positive integer.*

$$\text{Define}\quad \begin{aligned} \phi(v_0) &= 0 \\ \phi(v_2) &= 1 \\ \phi(v_{2i}) &= i+1,\ \text{for } 2 \leq i \leq \frac{N-2}{2} \\ \phi(v_1) &= M \\ \phi(v_3) &= M-2 \\ \phi(v_{2i+1}) &= \phi(v_{2i-1}) - \alpha, \text{where } \alpha = \begin{cases} 1 & \text{if } 2i+1 \text{is not a multiple of } 3 \\ 3 & \text{if } 2i+1 \text{ is a multiple of } 3 \\ & \text{and } 2 \leq i \leq \frac{N-2}{2}. \end{cases} \end{aligned}$$

Case III: *When* $n = 4r + 2$*, where* r *is any positive integer.*

$$\text{Define}\quad \begin{aligned} \phi(v_0) &= 0 \\ \phi(v_2) &= 1 \\ \phi(v_{2i}) &= i+1,\ \text{for } \ 2 \leq i \leq \frac{N-2}{2} \\ \phi(v_1) &= M \\ \phi(v_3) &= M-2 \\ \phi(v_{2i+1}) &= \phi(v_{2i-1}) - \alpha, \text{where } \alpha = \begin{cases} 1 & \text{if } 2i+1 \text{ is not a multiple of } 3 \\ 3 & \text{if } 2i+1 \text{ is a multiple of } 3 \\ & \text{and } 2 \leq i \leq \frac{N-6}{2}. \end{cases} \\ \phi(v_{N-3}) &= \frac{N+2}{2} \\ \phi(v_{N-1}) &= \frac{N+6}{2}. \end{aligned}$$

Case IV: *When* $n = 4r + 3$*, where* r *is any positive integer.*

$$\text{Define}\quad \begin{aligned} \phi(v_0) &= 0 \\ \phi(v_2) &= 1 \\ \phi(v_{2i}) &= i+1,\ \text{for } 2 \leq i \leq \frac{N-7}{2} \\ \phi(v_{N-5}) &= \frac{N+3}{2} \\ \phi(v_{N-3}) &= \frac{N-1}{2} \\ \phi(v_{N-1}) &= \frac{N+1}{2} \\ \phi(v_1) &= M \\ \phi(v_3) &= M-2 \end{aligned}$$

$$\begin{aligned}
\phi(v_{2i+1}) &= \phi(v_{2i-1}) - \alpha, \text{where } \alpha = \begin{cases} 1 & \text{if } 2i+1 \text{is not a multiple of 3} \\ 3 & \text{if } 2i+1 \text{ is a multiple of 3} \\ & \text{and } 2 \leq i \leq \frac{N-5}{2}. \end{cases} \\
\phi(v_{N-2}) &= \phi(v_{N-4}) + 2
\end{aligned}$$

It is clear that ϕ is injective and the edge values are distinct and range from 1 to M. Hence the graph G is graceful. □

Theorem 2: *For $n \geq 6$ every n-cycle with parallel P_k-chords is graceful for $k = 2r$, where $2 \leq r \leq 5$.*

Proof: Let G be an n-cycle with parallel P_k-chords, where $n \geq 6$. Observe that G has $N = \frac{nk-\alpha(k-2)}{2}$ vertices and $M = \frac{n(k+1)-\alpha(k-1)}{2}$ edges, where $\alpha = 3$ if n is odd or $\alpha = 2$ if n is even. Let $v_0 v_1 \ldots v_{N-1}$ be a hamiltonian path in G.

Now we give the labeling to the vertices $v_0, v_1, \ldots, v_{N-1}$ in two cases depending on whether n is odd or even, where n is the length of the cycle C_n in G

Case I: *When n is even*

$$\begin{aligned}
\text{Define} \quad \phi(v_0) &= 0 \\
\phi(v_{ki+j}) &= \frac{(k+2)i+j}{2}, \text{for } 2 \leq j \leq k, j \textit{ even} \quad \text{and } 0 \leq i \leq \frac{n}{2} - 3 \\
\phi(v_{N-(k-j)}) &= \phi(v_{N-(k-j+2)}) + 1, \text{ for } 0 \leq j \leq k-4 \text{ and } j \textit{ even}, \\
\phi(v_{N-2}) &= \phi(v_{N-4}) + 5 \\
\phi(v_{2i-1}) &= M - (i-1), \text{ for } \; 1 \leq i \leq \frac{N-(k-2)}{2} \\
\text{when } k = 4 & \\
\phi(v_{N-1}) &= \phi(v_{N-3}) - 4 \\
\text{when } k = 6 & \\
\phi(v_{N-3}) &= \phi(v_{N-5}) - 6 \\
\phi(v_{N-1}) &= \phi(v_{N-3}) + 2 \\
\text{when } k = 8 & \\
\phi(v_{N-5}) &= \phi(v_{N-7}) - 2 \\
\phi(v_{N-3}) &= \phi(v_{N-5}) - 4 \\
\phi(v_{N-1}) &= \phi(v_{N-3}) + 2
\end{aligned}$$

when $k = 10$

$$\begin{aligned}\phi(v_{N-7}) &= \phi(v_{N-9}) - 2\\ \phi(v_{N-5}) &= \phi(v_{N-7}) - 1\\ \phi(v_{N-3}) &= \phi(v_{N-5}) - 5\\ \phi(v_{N-1}) &= \phi(v_{N-3}) + 7\end{aligned}$$

Case II: *When n is odd*

$$\begin{aligned}\text{Define}\quad \phi(v_0) &= 0\\ \phi(v_{ki+j}) &= \frac{(k+2)i+j}{2},\ \text{for } 2 \le j \le k, j \ \textit{even} \quad \text{and } 0 \le i \le \frac{n-5}{2}\\ \phi(v_{2i-1}) &= M-(i-1),\ \text{ for } 1 \le i \le \frac{N-(k-\alpha)}{2}\\ &\quad \text{where } \alpha = 3 \text{ for } k = 4,6,8 \text{ and for } k = 10, \alpha = 5\\ \phi(v_{N-1}) &= \phi(v_{N-3}) + \alpha,\ \text{ where } \alpha = 3 \text{ for } k = 4,6,10 \ \&\, \text{for } k = 8,\ \alpha = 2\\ \text{when } k = 6 &\\ \phi(v_{N-2}) &= \phi(v_{N-4}) - 3\\ \text{when } k = 8 &\\ \phi(v_{N-4}) &= \phi(v_{N-6}) - 5\\ \phi(v_{N-2}) &= \phi(v_{N-4}) + 4\\ \text{when } k = 10 &\\ \phi(v_{N-4}) &= \phi(v_{N-6}) - 4\\ \phi(v_{N-2}) &= \phi(v_{N-4}) + 2.\end{aligned}$$

It is clear that ϕ is injective and the edge values are distinct and range from 1 to M. Hence the graph G is graceful. □

Discussion: In Theorem 1, we have shown that the cycle C_n with parallel P_3-chords is graceful. It appears that the cycle C_n with parallel P_k-chords may not be graceful for odd $k \ge 5$. However, we strongly feel that the cycle C_n with parallel P_k-chords are graceful for all even k, so we pose the following conjecture.

Conjecture: The cycle C_n with parallel P_k-chords is graceful for all even k.

Acknowledgement: The authors would like to thank the referee for his/her valuable comments and suggestions.

References

[1] R. Bodendiek, H. Schumacher and H. Wegner, *Uber graziose Graphen, Math.–Phys. Semesterberichte* **24** (1977), 103–106.

[2] C. Delorme, M. Maheo, H. Thuillier, K.M. Koh and H.K.Teo, Cycles with a chord are graceful, *J. Graph Theory* **4** (1980), 409–415.

[3] J.A. Gallian, A dynamic survey of graph labeling, *Electronic J. Combinatorics* (2003), #DS6.

[4] N. Punnim and N. Pabhapote, On graceful graphs: cycles with a P_k-chord, $k \geq 4$, *Ars Combin.* **23A** (1987), 225–228.

[5] A. Rosa, On certain valuations of the vertices of a graph. *Theory of Graphs* (Internat. Symposium, Rome, 1966), Gordon and Breach, New York; Dunod, Paris (1967), 349–355.

Illustrations

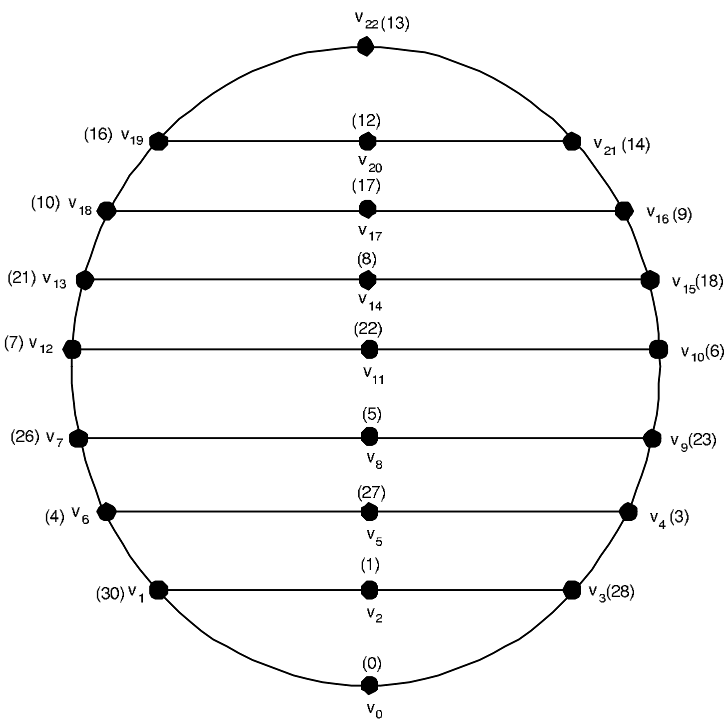

Fig 1(a) Graceful labelled C_{16} with parallel P_3 - chords

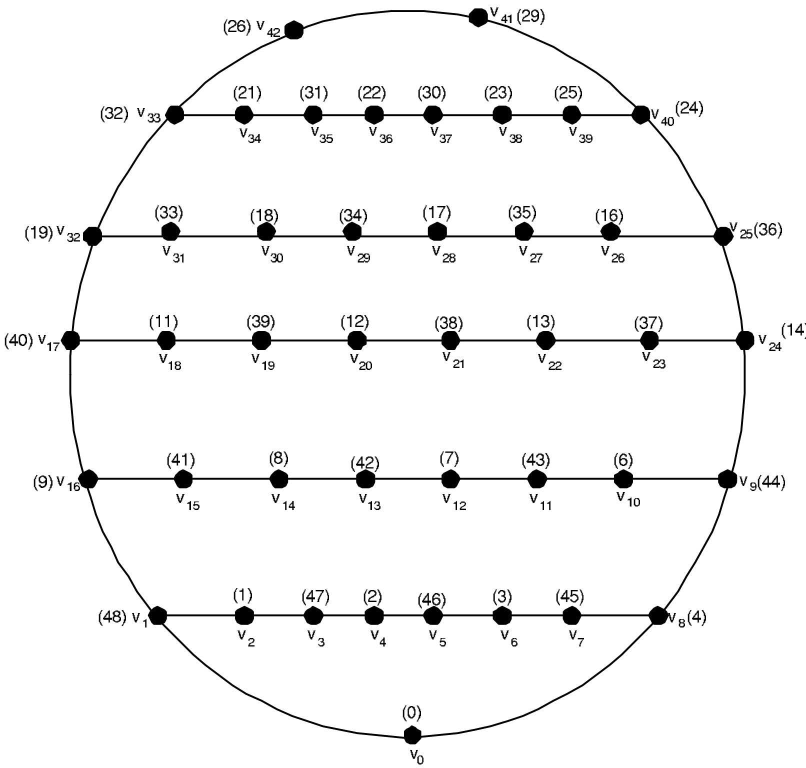

Fig 1(b) Graceful labelled C_{13} with parallel P_8 - chords

(Received 24 Sep 2003; revised 26 Feb 2004)