AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **32** (2005), Pages 147–162

Decycling regular graphs

Narong Punnim*

Department of Mathematics, Srinakharinwirot University
Sukhumvit 23, Bangkok 10110
Thailand
narongp@swu.ac.th

Abstract

For a graph G and $S \subseteq V(G)$, if $G - S$ is acyclic, then S is said to be a *decycling set* of G. The cardinality of the smallest decycling set of G is called the *decycling number* of G and is denoted by $\phi(G)$. We prove in this paper that if G runs over the set of graphs with a fixed degree sequence $\mathbf{d}$, then the values $\phi(G)$ completely cover a line segment $[a, b]$ of positive integers. Let $\mathcal{R}(\mathbf{d})$ be the class of all graphs having degree sequence $\mathbf{d}$. For an arbitrary graphic degree sequence $\mathbf{d}$, two invariants

$$a := \min(\phi, \mathbf{d}) = \min\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}$$

and

$$b := \max(\phi, \mathbf{d}) = \max\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\},$$

arise naturally. For a regular graphic degree sequence $\mathbf{d} = r^n := (r, r, \ldots, r)$, where r is the vertex degree and n is the order of the graph, the exact value of $\min(\phi, r^n)$ and $\max(\phi, r^n)$ are found in all situations. As an application, we can find all cubic graphs of order $2n$ having the smallest decycling number.

1. Introduction

The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph G is difficult even for some simply defined graphs. For a graph G, this minimum number is known as the *decycling number* of G, and is denoted by $\phi(G)$. The class of those graphs G of which $\phi(G) = 0$ consists of all forests, and $\phi(G) = 1$ if and only if G has at least one cycle and a vertex is on all of its cycles. It is also easy to see that $\phi(K_n) = n - 2$ and $K_{p,\,q} = p - 1$ if $p \leq q$, where

* Work supported by The Thailand Research Fund, under the grant number BRG/09/2545.

K_n denotes the complete graph of order n and $K_{p,\,q}$ denotes the complete bipartite graph with partite sets of cardinality p and q. The exact values of decycling numbers for many classes of graphs were obtained and cited in [1]. In the same paper, they posed the following problems:

Problem 1. Which cubic graphs G with $|G| = 2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$?

Problem 2. Which cubic planar graphs G with $|G| = 2n$ satisfy $\phi(G) = \lceil \frac{n+1}{2} \rceil$?

Problem 1 has been answered in [2] by proving that for a random cubic graph G of order n, $\phi(G) = \lceil \frac{n}{2} + \frac{1}{2} \rceil$ holds asymptotically almost surely, but no answer for the second problem yet.

We prove in this paper that if G runs over the set of graphs with a fixed degree sequence $\mathbf{d}$, the values $\phi(G)$ completely cover a line segment $[a, b]$ of positive integers. Let $\mathcal{R}(\mathbf{d})$ be the class of all graphs having degree sequence $\mathbf{d}$. For an arbitrary graphic degree sequence $\mathbf{d}$, two invariants

$$a := \min(\phi, \mathbf{d}) = \min\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}$$

and

$$b := \max(\phi, \mathbf{d}) = \max\{\phi(G) : G \in \mathcal{R}(\mathbf{d})\},$$

arise naturally. For a regular graphic degree sequence $\mathbf{d} = r^n := (r, r, \ldots, r)$ where r is the vertex degree and n is the number of graph vertices, the exact values of $\min(\phi, r^n)$ and $\max(\phi, r^n)$ are found in all situations. Finally we shall answer Problem 1.

In this paper we only consider finite simple graphs. For the most part, our notation and terminology follows that of Bondy and Murty [3]. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Since we only deal with finite and simple graphs, we will use the following notation and terminology for a typical graph G. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. As usual, we use $|S|$ to denote the cardinality of a set S and therefore we define $n = |V|$ to be the *order* of G and $m = |E|$ the *size* of G. To simplify writing, we write $e = uv$ for the edge e that joins the vertex u to the vertex v. The *degree* of a vertex v of a graph G is defined as $d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree of a graph G is usually denoted by $\Delta(G)$. Let S and T be disjoint subsets of $V(G)$ of a graph G. We denote by $e(S, T)$ the number of edges in G that connect S to T. If $S \subseteq V(G)$, the graph $G|_S$ is the subgraph induced by S in G and denotes $e(S)$ the number of edges in the graph $G|_S$. A graph G is said to be *regular* if all of its vertices have the same degree. A 3-regular graph is called *cubic graph.*

Let G be a graph of order n and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is called a *degree sequence* of G, and we simply write $(d(v_1), d(v_2), \ldots, d(v_n))$ if the underline graph G is clear in the context. A graph H of order n is said to have the same degree sequence as G if there is a bijection f from $V(G)$ to $V(H)$ such that $d_G(v_i) = d_H(f(v_i))$ for all $i = 1, 2, \ldots, n$. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G. In this case, G is called a *realization* of $\mathbf{d}$.

An algorithm for determining whether or not a given sequence of non-negative integers is graphic was independently obtained by [5] and [4]. We state their results in the following theorem.

Theorem 1.1 *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a non-increasing sequence of non-negative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n) = \mathbf{d}'.$$

Then $\mathbf{d}$ *is graphic if and only if* $\mathbf{d}'$ *is graphic.*

□

Let G be a graph and $ab, cd \in E(G)$ be independent, where $ac, bd \notin E(G)$. Let

$$G^{\sigma(a,b;c,d)} = (G - \{ab, cd\}) \cup \{ac, bd\}.$$

The operation $\sigma(a, b; c, d)$ is called a *switching operation.* It is easy to see that the graph obtained from G by a switching has the same degree sequence as G. The following theorem has been shown by [5] and [4].

Theorem 1.2 *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a graphic degree sequence. If* G_1 *and* G_2 *are any two realizations of* $\mathbf{d}$*, then* G_2 *can be obtained from* G_1 *by a finite sequence of switchings.*

□

As a consequence of Theorem 1.2, we can define *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ whose vertices are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. Thus we obtain the following theorem.

Theorem 1.3 *The graph* $\mathcal{R}(\mathbf{d})$ *is connected.*

□

2. Interpolation theorem

Let $\mathcal{G}$ be the class of all simple graphs, a function $f : \mathcal{G} \to \mathbb{Z}$ is called a *graph parameter* if $f(G) = f(H)$, whenever $G \cong H$. If f is a graph parameter and $\mathcal{J} \subseteq \mathcal{G}$, f is called an *interpolation graph parameter with respect to* $\mathcal{J}$ if there exist integers a and b such that

$$\{f(G) : G \in \mathcal{J}\} = [a, b] = \{k \in \mathbb{Z} : a \leq k \leq b\}.$$

We have shown in [7, 8, 9] that the chromatic number χ, the clique number ω, and the matching number α_1 are interpolation graph parameters with respect to $\mathcal{J} = \mathcal{R}(\mathbf{d})$.

If f is an interpolation graph parameter with respect to $\mathcal{J}$, it is natural to write $\min(f, \mathcal{J}) = \min\{f(G) : G \in \mathcal{J}\}$ and $\max(f, \mathcal{J}) = \max\{f(G) : G \in \mathcal{J}\}$.

In the case where $\mathcal{J} = \mathcal{R}(\mathbf{d})$ we simply write $\min(f, \mathbf{d})$ and $\max(f, \mathbf{d})$ for $\min(f, \mathcal{R}(\mathbf{d}))$ and $\max(f, \mathcal{R}(\mathbf{d}))$ respectively.

Theorem 2.1 *If G is a graph and $\sigma(a,b;c,d)$ is a switching on G, then* $\phi(G^{\sigma(a,b;c,d)}) \leq \phi(G) + 1$.

Proof. Let S be a decycling set of G with $|S| = \phi(G)$. Let $\sigma(a,b;c,d) = \sigma$ be a switching on G. We claim that $(G - S)^\sigma$ contains at most one cycle. If $ab, cd \in E(G - S)$, then $(G - S)^\sigma$ is well defined. Since $G - S$ is a forest, there is at most one path in $G - S$ from a to c. If there is a path from a to c in $G - S$, then the path can be modified as the unique path from b to d. Thus the claim is true. If one or both of ab, cd are not edges in $G - S$, then the claim is also true. Therefore $\phi(G^\sigma) \leq \phi(G) + 1$.

□

Corollary 2.2 *If σ is a switching on G, then $|\phi(G) - \phi(G^\sigma)| \leq 1$.*

Proof. Since a switching is symmetric, we may assume that $\phi(G) \leq \phi(G^\sigma)$. By Theorem 2.1, $\phi(G^\sigma)$ is either $\phi(G) + 1$ or $\phi(G)$. In both cases we have $|\phi(G) - \phi(G^\sigma)| \leq 1$.

□

Theorem 2.3 *For a given graphic degree sequence* $\mathbf{d}$*, there exist integers a and b such that there is a graph G with degree sequence* $\mathbf{d}$ *and $\phi(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

Proof. The proof follows directly from Theorem 1.3 and Corollary 2.2.

□

Let $\mathbf{d}$ be a graphic degree sequence. We have already defined the graph $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$. The *range of decycling numbers* of $\mathcal{R}(\mathbf{d})$ can be defined as the interval of integers specified by Theorem 2.3, i.e.,

$$\phi(\mathbf{d}) = \{\phi(G) \ : \ G \in \mathcal{R}(\mathbf{d})\} = [a, b] = \{c \in \mathbb{Z} \ : \ a \leq c \leq b\}.$$

Naturally, we can call a the *minimum decycling number* for $\mathbf{d}$, thus $a := \min(\phi, \mathbf{d})$ and $b := \max(\phi, \mathbf{d})$, the *maximum decycling number* for $\mathbf{d}$. We write $\mathbf{d} = r^n$ for the sequence $(r, r, \ldots, r)$ of length n, where r is a non-negative integer and n is a positive integer. By the definition of graphic degree sequence, $\mathbf{d} = r^n$ is graphic if and only if $rn \equiv 0 (\text{mod } 2)$ and $n \geq r + 1$. Moreover, $\mathcal{R}(r^n)$ contains a disconnected graph if and only if $n \geq 2r + 2$. It is easy to see that $\min(\phi, 0^n) = \max(\phi, 0^n) = 0$ and $\min(\phi, 1^{2n}) = \max(\phi, 1^{2n}) = 0$. When $r = 2$, we have $\min(\phi, 2^n) = 1$ and $\max(\phi, 2^n) = \lfloor \frac{n}{3} \rfloor$. Because of this fact, from now on we will consider the cases when $r \geq 3$ and $n \geq r + 1$.

3. $\min(\phi, r^n)$

It was a remark in [1] that if G is a connected graph with maximum degree Δ, then $\phi(G) \geq \frac{|E(G)|-|V(G)|+1}{\Delta-1}$. Thus if G is a connected r-regular graph of order n, then $\phi(G) \geq \frac{nr-2n+2}{2(r-1)}$.

In order to obtain the exact values of $\min(\phi, r^n)$ we first state some useful facts arising from elementary arithmetic as follows:
1. Let n and r be integers, $n > r \geq 3$. Then $r-1 \leq \frac{nr-2n+2}{2(r-1)}$ if and only if $n \geq 2r$.
2. Let n and r be integers, $n \geq 2r$ and $nr \equiv 0 \pmod 2$. Then $\frac{nr-2n+2}{2(r-1)}$ is an integer if and only if n is even and $n = 2+2(r-1)q$ for some positive integers q, or n is odd and $n = r+1+2(r-1)q$ for some positive integers q.

Theorem 3.1

$$\min(\phi, r^n) = \begin{cases} r-1 & \text{if } r+1 \leq n \leq 2r-1, \\ \lceil \frac{nr-2n+2}{2(r-1)} \rceil & \text{if } n \geq 2r. \end{cases}$$

The proof of Theorem 3.1 follows from Lemmas 3.2, 3.3, 3.4, and 3.5.

□

If S is a decycling set of an r-regular graph G with $E(G-S) = \emptyset$, then for any $v \in S$, $S - \{v\}$ is also a decycling set of G. Thus for a minimum decycling set S of an r-regular graph G, there exists $v \in V(G-S)$ such that $d_{(G-S)}(v) = 1$. It follows that $\min(\phi, r^n) \geq r-1$. With this observation, we will see that the lower bound is precise when $n \leq 2r$. Moreover, the lower bound can be improved to $\lceil \frac{nr-2n+2}{2(r-1)} \rceil$, otherwise.

Lemma 3.2 $\min(\phi, r^n) = r-1$, if $r+1 \leq n \leq 2r-1$.

Proof. In order to achieve the exact the value of $\min(\phi, r^n)$, we now construct an r-regular graph G of order n such that $\phi(G) = r-1$. Put $n = r+j, 1 \leq j \leq r-1$. It should be noted that an r-regular graph of order $n = r+j$ exists if and only if j is odd or both r and j are even.

Let $X = \{s_1, s_2, \ldots, s_{r-2}\}$ and $Y = \{t_1, t_2, \ldots, t_{j+2}\}$, where $r \leq j+2$. Let G be a graph with $V(G) = X \cup Y$ and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{t_1t_2, t_2t_3, \ldots, t_{j+1}t_{j+2}, t_{j+2}t_1\}, E_2 = \{s_pt_q : 1 \leq p \leq r-2, 1 \leq q \leq j+2\}$, and $E_3 = E(H)$, where H is an $(r-j-2)$-regular graph on X. Note that an $(r-j-2)$-regular graph of order $r-2$ exists if and only if an r-regular graph of order $r+j$ exists. It is clear that G is an r-regular graph of order n with a decycling set $S = X \cup \{t_1\}$ of cardinality $r-1$. Hence S is minimum.

Suppose $r = j+1$. In this particular case, we see that an r-regular graph of order $n = r+j = 2r-1$ exists if and only if j is odd and r is even. Let $X = \{s_1, s_2, \ldots, s_{r-2}\}$ and $Y = \{t_1, t_2, \ldots, t_{r+1}\}$. Let G be a graph with $V(G) = X \cup Y$ and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{t_1t_2, t_2t_3, \ldots, t_rt_{r+1}, t_{r+1}t_1\}$,

$E_2 = \{s_pt_q : 1 \le p \le r-2, 1 \le q \le r\} - \{s_1t_2, s_2t_3, \ldots, s_{\frac{r-2}{2}}t_{\frac{r}{2}}\}$, and $E_3 = \{s_1t_{r+1}, t_2t_{r+1}, s_2t_{r+1}, t_3t_{r+1}, \ldots, t_{\frac{r}{2}}t_{r+1}\}$. Thus G is an r-regular graph of order $2r-1$ with a decycling set $S = X \cup \{t_{r+1}\}$ of cardinality $r-1$. Hence S is minimum. The proof is complete.

□

Lemma 3.3 $\min(\phi, r^{2r}) = r-1$, *for all* $r \ge 3$, *and* $\min(\phi, r^{2r+1}) = r$, *for all even integers* $r \ge 4$.

Proof. Let X and Y be disjoint sets, $|X| = |Y| = r$. The complete bipartite graph G on the partite sets X and Y is an r-regular graph of order $2r$. For each $x \in X$, put $S = X - \{x\}$ and $T = Y \cup \{x\}$ we see that S is a decycling set of G, $E(G|_S) = \emptyset$ and $G|_T$ is a tree. Since $|S| = r-1$, S is minimum. Thus $\min(\phi, r^{2r}) = r-1$.

For an even integer $r \ge 4$, let $X = \{s_1, s_2, \ldots, s_r\}$ and $Y = \{t_1, t_2, \ldots, t_r\}$ be disjoint sets. An r-regular graph G whose $V(G) = X \cup Y \cup \{v\}$, where v is a new vertex not in $X \cup Y$, and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{s_pt_q : 1 \le p \le r, 1 \le q \le r, p \ne q\}$, $E_2 = \{t_1t_2, t_3t_4, \ldots, t_{r-1}t_r\}$, and $E_3 = \{vs_p : 1 \le p \le r\}$. Since $\lceil \frac{(2r+1)r-2(2r+1)+2}{2(r-1)} \rceil = r, \min(\phi, r^{2r+1}) = r$.

□

Lemma 3.4 *If* $n \ge 2r$ *and* $\frac{nr-2n+2}{2(r-1)}$ *is an integer, then* $\min(\phi, r^n) = \frac{nr-2n+2}{2(r-1)}$.

Proof. *Case 1.* Suppose n is even and $n \ge 2r$. Since $\frac{nr-2n+2}{2(r-1)} = \frac{n}{2} - \frac{n-2}{2(r-1)}$, we write $n = 2(r-1)q+2, q \ge 1$. By induction on q, it is true when $q = 1$, by Lemma 3.3. Moreover, the graph G constructed in Lemma 3.3 has the property that $V(G) = S \cup T, E(G|_S) = \emptyset$, $|S| = r-1$, and $G|_T$ is a tree on $r+1$ vertices.

Suppose there exists an r-regular graph G_0 on $n = 2(r-1)q+2$ vertices with $V(G_0) = S_0 \cup T_0, S_0 \cap T_0 = \emptyset, S_0 = \{s_1, s_2, \ldots, s_a\}$ and $T_0 = \{t_1, t_2, \ldots, t_b\}$ where $a = (r-2)q+1$ and $b = rq+1$. We suppose further that $E(G_0|_{S_0}) = \emptyset$, and $G_0|_{T_0}$ is a tree with $d(t_b) = 1$. Let $S_1 = \{u_1, u_2, \ldots, u_{r-2}\}$ and $T_1 = \{v_1, v_2, \ldots, v_r\}$. Now let G be a graph with $V(G) = S \cup T$, where $S = S_0 \cup S_1, T = T_0 \cup T_1$, and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = E(G_0) - \{s_at_b\}, E_2 = \{u_pv_q : 1 \le p \le r-2, 1 \le q \le r\}, E_3 = \{t_bv_1, v_1v_2, \ldots, v_{r-1}v_r, v_rs_a\}$. It is clear that G is an r-regular graph on $2(r-1)(q+1)+2$ vertices, $E(G|_S) = \emptyset$, and $\phi(G) = (r-2)(q+1)+1$, where $S = S_0 \cup S_1$. Moreover $G|_T$ is a tree and $d_{G|_T}(t_b) = 1$.

Case 2. Suppose n is odd and r is even. Write $n = 2(r-1)q+r+1, q \ge 1$, and consider when $q = 1$. Let $S = \{s_1, s_2, \ldots, s_a\}$ and $T = \{t_1, t_2, \ldots, t_b\}$, where $a = \frac{3}{2}r-2$ and $b = \frac{3}{2}r+1$. Let H be an $(r-2)$-regular bipartite graph with partite sets S and $T' = \{t_1, t_2, \ldots, t_a\}$. Since $|S - \{s_a\}| = \frac{3}{2}r - 3 = 3(\frac{r}{2}-1)$, $S - \{s_a\}$ can be partitioned into $\frac{r}{2}-1$ sets each of which contains 3 elements. Let $P = \{S_1, S_2, \ldots, S_{\frac{r}{2}-1}\}$ be such a partition. Let K be a bipartite graph with partite sets $S-\{s_a\}$ and $\{t_{b-2}, t_{b-1}, t_b\}$ such that $E(K)$ is the union of all edges in 2-regular bipartite graphs with partite sets $\{t_{b-2}, t_{b-1}, t_b\}$ and S_i, for all $i = 1, 2, \ldots, \frac{r}{2}-1$. Finally let G be a graph with $V(G) = S \cup T$ and $E(G) = E(H) \cup E(K) \cup E_1$, where $E_1 = \{s_a t_1, s_a t_b, t_1 t_2, t_2 t_3, \ldots, t_{a-1} t_a\}$. Therefore G is an r-regular graph on $3r-1$ vertices with $\phi(G) = |S| = \frac{3}{2}r - 2$. Moreover $G|_T$ is a path with $d_{G|_T}(t_b) = 1$ and $E(G|_S) = \emptyset$.

Suppose there exists an r-regular graph G_0 on $2(r-1)q + r + 1, q \geq 1$, vertices with $V(G_0) = S_0 \cup T_0, E(G_0|_{S_0}) = \emptyset$, $|S_0| = a = \frac{3}{2}r - 2 + (q-1)(r-2)$, and $G_0|_{T_0}$ is a path $t_1 t_2 \ldots t_b$ of length $b = \frac{3}{2}r + 1 + (q-1)r$. Let $S_1 = \{u_1, u_2, \ldots, u_{r-2}\}$ and $T_1 = \{v_1, v_2, \ldots, v_r\}$. We may assume that $s_a t_b \in G_0$. Now let G be a graph with $V(G) = S_0 \cup S_1 \cup T_0 \cup T_1$ and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = E(G_0) - \{s_a t_b\}, E_2 = \{u_p v_q : 1 \leq p \leq r-2, 1 \leq q \leq r\}$, and $E_3 = \{s_a v_r, t_b v_1, v_1 v_2, v_2 v_3, \ldots, v_{r-1} v_r\}$. It is clear that G is an r-regular graph on $n = 2(r-1)(q+1) + r + 1$ vertices, $E(G|_S) = \emptyset$, and$|S| = \phi(G) = \frac{3}{2}r - 2 + q(r-2)$, where $S = S_0 \cup S_1$. Also note that $G - S = G|_T = t_1 t_2 \ldots t_b v_1 v_2 \ldots v_r$ is a path and $d_{G|_T}(t_1) = d_{G|_T}(v_r) = 1$.

□

Let $f(n,r) = \lceil \frac{nr-2n+2}{2(r-1)} \rceil$, $r \geq 3$ and $n \geq 2r$. If $\frac{nr-2n+2}{2(r-1)}$ is an integer, then it is easy to show that

$$f(n+2i, r) = \begin{cases} f(n,r) + i & \text{if } 1 \leq i \leq r-2, \\ f(n + 2(r-1), r) & \text{if } i = r-2. \end{cases}$$

Lemma 3.5 *If* $n \geq 2r$, *then* $\min(\phi, r^{n+2}) \leq \min(\phi, r^n) + 1$.

Proof. For $n \geq 2r$ we have constructed an r-regular graph G on n vertices having minimum decycling set when $n = 2r$ and $n = 2r+1$ in Lemma 3.3. We have also constructed an r-regular graph G on n vertices when $\frac{nr-2n+2}{2(r-1)}$ is an integer and $\phi(G) = \frac{nr-2n+2}{2(r-1)}$ in Lemma 3.4. Moreover, such constructions give us decycling sets S with $E(G|_S) = \emptyset$. Let G_0 be an r-regular graph on n vertices with $\phi(G_0) = \min(\phi, r^n)$. We may further assume that $V(G_0) = S_0 \cup T_0$ where $E(G_0|_{S_0}) = \emptyset$, and $G_0|_{T_0}$ is a forest. Since G_0 is r-regular and $|S_0| \geq r-1$, by Hall's Theorem (See, e.g. [6] page 227), there exists a set of $r-1$ independent edges M_0 joining S_0 to T_0. Let G be a graph with $V(G) = V(G_0) \cup \{x, y\}$ and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = E(G_0) - M_0, E_2 = \{xt_1, xt_2, \ldots, xt_{r-1}, s_1 y, s_2 y, \ldots, s_{r-1} y\}$, and $E_3 = \{xy\}$, where $M_0 = \{s_1 t_1, s_2 t_2, \ldots, s_{r-1} t_{r-1}\}$, with $s_i \in S_0$ and $t_i \in T_0$. Thus G is an r-regular graph on $n+2$ vertices with a decycling set $S_0 \cup \{x\}$. Therefore $\phi(G) \leq \phi(G_0) + 1$.
□

It is interesting to observe that the proof of Theorem 3.1 is now complete.

4. $\max(\phi, r^n)$

The problem of determining the decycling number of a graph is equivalent to finding the greatest order of an induced forest and the sum of the two numbers equals the order of the graph. Let $F \subseteq V(G)$ of a graph G. F is called an *induced forest* of G, if $G|_F$ contains no cycle. For a graph G, we define, $I(G)$ as:

$$I(G) := \max\{|F| : F \text{ is an induced forest in } G\}.$$

Observe that for a minimum decycling set S of a graph G, if $v \in S$, then there exists a connected component C of $G - S$ such that v is adjacent to at least two vertices of C. Thus $\Delta(G|_S) \leq \Delta(G) - 2$. With this observation, we find that if G is an r-regular graph and S is a minimum decycling set of G, the graph $G|_S$ may not be an $(r-2)$-regular graph. This causes a difficulty in finding $\max(\phi, r^n)$ if we consider only the class of regular graphs. It is reasonable to enlarge the class of regular graphs into the following class of graphs. Let Δ be a nonnegative integer and n be a positive integer such that $n \geq \Delta + 1$. Let $\mathcal{G}(\Delta, n)$ be the class of all graphs of order n and of maximum degree Δ. *The* (Δ, n)*-graph* is a graph having $\mathcal{G}(\Delta, n)$ as its vertex set and two such graphs being adjacent if one can be obtained from the other by either adding or deleting an edge.

Lemma 4.1 *The* (Δ, n)*-graph is connected.*

Proof. For any graph $G \in \mathcal{G}(\Delta, n)$, if $F = K_{1,\Delta} \cup (n - \Delta - 1)K_1$ and $G \neq F$, G can be obtained from F by a finite sequence of adding edges. The proof is complete. □

Lemma 4.2 *If* G_1 *and* G_2 *are adjacent in the* (Δ, n)*-graph, then* $|\phi(G_1) - \phi(G_2)| \leq 1$.

Proof. Without loss of generality we may assume that G_2 is a graph obtained from G_1 by adding an edge e. Thus $\phi(G_1) \leq \phi(G_2)$. On the other hand, if S is a minimum decycling set of G_1, then $G_2 - S$ contains at most one cycle. Thus $\phi(G_2) \leq |S| + 1$. Therefore $\phi(G_1) \leq \phi(G_2) \leq \phi(G_1) + 1$. The proof is complete. □

As a consequence of Theorems 4.1 and 4.2, we have the following corollary.

Corollary 4.3 *For any class of graphs* $\mathcal{G}(\Delta, n)$*, there exist integers* a *and* b *such that there is a graph* $G \in \mathcal{G}(\Delta, n)$ *with* $\phi(G) = c$ *if and only if* c *is an integer satisfying* $a \leq c \leq b$. □

Note that the result of Corollary 4.3 is an interpolation theorem of ϕ with respect to $\mathcal{G}(\Delta, n)$, and it is easy to see that $\min(\phi, \mathcal{G}(\Delta, n)) = 0$, the graph F in the proof of Lemma 4.1 is such a graph. In order to investigate the exact values of $\max(\phi, \mathcal{G}(\Delta, n))$, we first give its upper bound.

Lemma 4.4 *If G is a graph of order n with maximum degree $\Delta(G) = \Delta \geq 1$, then $\phi(G) \leq \frac{n(\Delta-1)}{\Delta+1}$.*

Proof. The theorem is trivial if $\Delta \leq 2$. If $\Delta = 3$ and S is a minimum decycling set of G, then for each $v \in S, d_{G|_S}(v) \leq 1$. This means that both $G|_S$ and $G - S$ are decycling sets of G and hence $|S| \leq \frac{n}{2} = \frac{n(\Delta-1)}{\Delta+1}$. Suppose $\Delta \geq 4$ and S is a minimum decycling set of G. Thus $\Delta(G|_S) \leq \Delta(G) - 2$. Since $I(G) \geq I(G|_S)$, it follows that $n - \phi(G) \geq |S| - \phi(S) \geq |S| - \frac{|S|(\Delta-3)}{\Delta-1}$. Therefore $\phi(G) \leq \frac{n(\Delta-1)}{\Delta+1}$.

□

Theorem 4.5 *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphic degree sequence and $d_1 + 1 \leq n \leq 2d_1 + 1$. Then*
(1) $\max(\phi, \mathbf{d}) = n - 2$ *if and only if* $\mathcal{R}(\mathbf{d}) = \{K_n\}$ *and*
(2) *if $K_n \notin \mathcal{R}(\mathbf{d})$, then* $\max(\phi, \mathbf{d}) = n - 3$ *if and only if there exists a union of stars as a realization of $\overline{\mathbf{d}}$, where $\overline{\mathbf{d}} = (n - d_n, n - d_{n-1}, \ldots, n - d_1)$.*

Proof. (1) By Lemma 4.4, we have $\max(\phi, \mathbf{d}) \leq n - 2$ and $\max(\phi, \mathbf{d}) = n - 2$ if and only if any induced subgraph of 3 vertices of $G \in \mathcal{R}(\mathbf{d})$ forms a triangle. Thus $\max(\phi, \mathbf{d}) = n - 2$ if and only if $\mathcal{R}(\mathbf{d}) = \{K_n\}$
(2) By (1), we have $\max(\phi, \mathbf{d}) \leq n - 3$. If $\mathcal{R}(\overline{\mathbf{d}})$ does not contain a union of stars as its realization, then for every realization $\overline{G}$ of $\overline{\mathbf{d}}$, G must contain a C_3 or P_4 as an induced subgraph. Thus $I(G) \geq 4$ which is equivalent to $\max(\phi, \mathbf{d}) \leq n - 4$. Conversely, if $\overline{G} \in \mathcal{R}(\overline{\mathbf{d}})$ and $\overline{G}$ is a union of stars, then any induced subgraph of 4 vertices of G must contain a triangle or C_4. Therefore $\max(\phi, \mathbf{d}) = n - 3$.

□

Theorem 4.6 *Let $n = (\Delta + 1)q + t, 0 \leq t \leq \Delta$. Then*
(1) $\max(\phi, \mathcal{G}(\Delta, n)) = n - 2q$, *if $t = 0$,*
(2) $\max(\phi, \mathcal{G}(\Delta, n)) = n - 2q - 1$, *if $t = 1$, and*
(3) $\max(\phi, \mathcal{G}(\Delta, n)) = n - 2q - 2$, *if $2 \leq t \leq \Delta$.*

Proof. (1) By Lemma 4.4, $\max(\phi, \mathcal{G}(\Delta, (\Delta + 1)q)) \leq n - 2q$. It is easy to see that the graph $qK_{\Delta+1} \in \mathcal{G}(\Delta, (\Delta + 1)q)$ and $\phi(qK_{\Delta+1}) = n - 2q$. Therefore $\max(\phi, \mathcal{G}(\Delta, n)) = n - 2q$, if $t = 0$.
(2) By Lemma 4.4, $\max(\phi, \mathcal{G}(\Delta, (\Delta+1)q+1)) \leq n-2q-1$. It is easy to see that the graph $qK_{\Delta+1} \cup K_1 \in \mathcal{G}(\Delta, (\Delta + 1)q + 1)$ and $\phi(qK_{\Delta+1} \cup K_1) = n - 2q - 1$. Therefore $\max(\phi, \mathcal{G}(\Delta, n)) = n - 2q - 1$, if $t = 1$.
(3) Let $n = (\Delta+1)q+t, 2 \leq t \leq \Delta$. Since $qK_{\Delta+1} \cup K_t \in \mathcal{G}(\Delta, n)$ and $\phi(qK_{\Delta+1} \cup K_t) = n - 2q - 2$, $\max(\phi, \mathcal{G}(\Delta, n)) \geq n - 2q - 2$. We first consider the case $q = 1$. Since $n = (\Delta+1)q+t, 2 \leq t \leq \Delta$, the minimum degree of $\overline{G}$ is t for all graphs $G \in \mathcal{G}(\Delta, n)$. Thus $\overline{G}$ is not a union of stars and therefore $\max(\phi, \mathcal{G}(\Delta, n)) = n - 4$ for $q = 1$.

Suppose $q \geq 2$ and q is the smallest integer such that $\max(\phi, \mathcal{G}(\Delta, (\Delta+1)q+1)) = n - 2q - 1$. Let v be a vertex of G of degree Δ and let H be the graph obtained from G by deleting the vertex v and its neighbors. It is clear that H has order $(\Delta + 1)(q - 1) + t$ and by minimality of q, there exists a minimum decycling set

S of H of order at most $n-2(q-1)-2=n-2q$. Equivalently, $|H-S|\geq 2q$. Since $(H-S)\cup\{v\}$ is an induced forest of G and $|(H-S)\cup\{v\}|\geq 2q+1$, we have $|(H-S)\cup\{v\}|=2q+1$. Put $F=(V(H)-S)\cup\{v\}$ and $D=G-F$, thus D is a minimum decycling set of G with $|D|=n-2q-1$. Since $v\in V(G|_F)$ and $d_{G|_F}(v)=0$, $G|_F$ contains at most $2q$ vertices of degree at least 1. Let T the subgraph of $G|_F$ consisting of all nontrivial components. By being maximality of $G|_F$, $V(T)\neq\emptyset$. Thus $1\leq|V(T)|\leq 2q$. Since $|D|=(\Delta-1)q+t-1$, $e(D,T)\geq 2(\Delta-1)q+2(t-1)$. But $\frac{2(\Delta-1)q+2(t-1)}{2q}>\Delta-1$, there exists a vertex $f\in T$ such that $d_G(f)>\Delta$. This is a contradiction. The proof is complete.

□

Theorem 4.7 *For $r\geq 3$, and $r+1\leq n\leq 2r+1$,*
(1) $\max(\phi,r^n)=n-2$, *if and only if* $n=r+1$,
(2) $\max(\phi,r^n)=n-3$, *if and only if* $n=r+2$,
(3) $\max(\phi,r^n)=n-4$, *for all even integers* n, $r+3\leq n$,
(4) $\max(\phi,r^n)=n-4$, *for all odd integers* n, $r+3\leq n$ *and* $n\geq f(j)$,
(5) $\max(\phi,r^n)=n-5$, *for all odd integers* n, $r+3\leq n$ *and* $n<f(j)$,
where $f(j)=\frac{5}{2}(j-1)$ *if* $j\equiv 3(\mathrm{mod}\ 4)$, *and*

$f(j)=1+\frac{5}{2}(j-1)$ *if* $j\equiv 1(\mathrm{mod}\ 4)$.

Proof. (1) and (2) follow directly from Theorem 4.5.
(3) From (1) and (2), we have $\max(\phi,r^n)\leq n-4$ for all integers $n=r+j$, $3\leq j\leq r+1$. Suppose n is even and let G be an r-regular graph of order n such that $\overline{G}$ is bipartite. Thus the induced subgraph of any 5 vertices of G must contain a cycle in G. Thus $\phi(G)=n-4$. Therefore $\max(\phi,r^n)=n-4$,
(4) In [7], we have shown that if j is odd and $j\geq 3$, then there exists a $(j-1)$-regular triangle-free graph of odd order n if and only if $n\geq f(j)$, where $f(j)=\frac{5}{2}(j-1)$ if $j\equiv 3(\mathrm{mod}\ 4)$, and $f(j)=1+\frac{5}{2}(j-1)$ if $j\equiv 1(\mathrm{mod}\ 4)$. Note that if H is a triangle-free graph of order at least 5 and for any subset K of H with $|K|=5$, then $E(H|_K)\leq 5$. Thus if $n\geq f(j)$ and H is a $(j-1)$-regular triangle-free graph on n vertices, then any induced subgraph of 5 vertices of H contains at most 5 edges in H. Therefore $\phi(\overline{H})\geq n-4$. Hence $\max(\phi,r^n)=n-4$.
(5) If $n<f(j)$ and $j\geq 5$, then any $(j-1)$-regular graph H on n vertices must contain a triangle. Let $T=\{u,v,w\}$ be a set of 3 vertices of H which induces a triangle. Since $n\leq\frac{5}{2}(j-1)$ and H is a $(j-1)$-regular graph, there exist $x,y\in T$ such that either $N_H(x)\cap N_H(y)\neq\emptyset$ or there exist $a\in N_H(x)$ and $b\in N_H(y)$ such that $ab\in E(H)$. In either case there are at least 5 vertices of $V(\overline{H})$ induced a forest in $\overline{H}$. Thus $I(\overline{H})\geq 5$ and it is equivalent to $\phi(\overline{H})\leq n-5$.

Finally, let $X=\{x_1,x_2,\ldots,x_t\}$ and $Y=\{y_1,y_2,\ldots,y_t\}, t=\frac{n-1}{2}$. Since $n\leq\frac{5}{2}(j-1), n$ is odd, and $j\geq 5$, there exists a $(j-1)$-regular bipartite graph K with the partite sets X and Y and $\{x_iy_i : i=1,2,\ldots,t\}\subseteq E(K)$. Now choose H to be a $(j-1)$-regular graph with $V(H)=V(K)\cup\{v\}$ and $E(H)=E_1\cup E_2$, where $E_1=E(K)-\{x_iy_i : i=1,2,\ldots,\frac{j-1}{2}\}$ and $E_2=\{vx_i,vy_i : i=1,2,\ldots,\frac{j-1}{2}\}$. It is clear that H is a $(j-1)$-regular graph on n vertices such that $\phi(\overline{H}) = n-5$.

□

Theorem 4.8 *For $n \geq 2r+2$ and $r \geq 3$, write $n = (r+1)q + t, q \geq 2$ and $0 \leq t \leq r$. Then*
(1) $\max(\phi, r^n) = n - 2q$ *if $t = 0$,*
(2) $\max(\phi, r^n) = n - 2q - 1$ *if $t = 1$,*
(3) $\max(\phi, r^n) = n - 2q - 2$ *if $2 \leq t \leq r-1$,*
(4) $\max(\phi, r^n) = n - 2q - 3$ *if $t = r$.*

Proof. The proof of (1), (2) and (3) follows directly from Theorems 4.6 and 4.7. (4) Note that $\max(\phi, \mathcal{G}(2, 3q+2)) = q$ and a graph $G \in \mathcal{G}(2, 3q+2)$ with $\phi(G) = q$ if and only if $G = (q-1)C_3 \cup C_5$ or $(q-2)C_3 \cup 2C_4$. It should be also noted that an r-regular graph of order $(r+1)q + r$ exists if and only if r is even. By Theorem 4.6, we have $\max(\phi, r^{(r+1)q+r}) \leq (r-1)q + (r-2)$. We first consider the case $r = 4$. Suppose there is a 4-regular graph G of order $5q+4$ and $\phi(G) = 3q+2$. Let S be a minimum decycling set of G such that $|S| = 3q+2$. Put $F = V(G) - S$. Since for each vertex $v \in S$ we have $e(\{v\}, F) \geq 2$, $e(S, F) \geq 2(3q+2)$. If $e(S, F) = 2(3q+2)$ holds for every minimum decycling set S of G, then S is a 2-regular graph of order $3q+2$. In this case $G|_S$ is either $(q-1)C_3 \cup C_5$ or $(q-2)C_3 \cup 2C_4$.

Let F_1 be an induced forest of S of order $2q+2$. Then F_1 is also a maximum induced forest of G and $G - F_1$ is a 2-regular graph of order $3q+2$. It is clear that F_1 can be obtained from S by removing one vertex on each cycle. Since F_1 contains an induced path of at least 3 vertices and there exists a vertex $v \in F$ adjacent to exactly two vertices in the path, we can choose a maximum induced forest F_2 of S of order $2q+2$ in such a way that $e(\{v\}, F_2) = 1$. Thus $F_2 \cup \{v\}$ is an induced forest of G of order $2q+3$, this is a contradiction. If $e(S, F) > 2(3q+2)$, then $S \in \mathcal{G}(2, 3q+2)$ and S is not regular. Therefore $I(G) \geq I(S) \geq 2q+3$. A graph $(q-1)K_5 \cup H$, where H is a 4-regular graph of order 9 satisfying the condition in Theorem 4.7(5), has the property that $\phi((q-1)K_5 \cup H) = 3(q-1) + 4 = 3q+1$. Thus $\max(\phi, 4^{5q+4}) = 3q+1$.

Now suppose $r \geq 6$ and let G be an r-regular graph of order $n = (r+1)q + r$ and $\phi(G) = (r-1)q + (r-2)$. Let S be a minimum decycling set of G such that $|S| = (r-1)q + (r-2)$. Put $F_1 = V(G) - S$. If $e(S, F_1) = 2((r-1)q + (r-2))$, then S is an $(r-2)$-regular graph of order $(r-1)q + (r-2)$. By induction on r, $I(S) \geq 2q+3$. Since $I(G) \geq I(S) \geq 2q+3$, we get a contradiction. Thus $e(S, F_1) > 2(n - 2q - 2)$. Since r is even and $e(\{v\}, F_1) \geq 2$ for all $v \in S$, there exists $v \in S$ such that $e(\{v\}, F_1) \geq 4$ or there exist two vertices $u, v \in S$ such that $e(\{u\}, F_1) \geq 3$ and $e(\{v\}, F_1) \geq 3$. Thus $e(S, F_1) \geq 2(r-1)q + 2(r-2) + 2$. On the other hand by counting edges from S to F_1, we find that $e(S, F_1) = 2(r-1)q + 2(r-2) + 2$ and $F_1 = (q+1)K_2$. Put $G_1 = G - F_1$ and $G_i = G_{i-1} - F_i$, for $2 \leq i \leq \frac{r-2}{2}$ and F_i is a maximum induced forest of G_{i-1}. If $2q+2 = I(G) = I(G_1) = I(G_2) = \ldots = I(G_{\frac{r-4}{2}})$, then $G_{\frac{r-2}{2}}$ has order $3q+2$, maximum degree 2 and not regular. Thus $I(G_{\frac{r-2}{2}}) \geq 2q+3$. This is a contradiction.

Therefore $\min(I, r^{(r+1)q+r}) \geq 2q+3$. A graph $(q-1)K_{r+1} \cup H$, where H is an r-regular graph of order $2r+1$ satisfying the condition in Theorem 4.7(5), has the property that $\phi((q-1)K_{r+1} \cup H) = (r-1)(q-1) + (2r+1-5) = (r-1)q + r - 3 = n - 2q - 3$. Therefore $\max(\phi, r^{(r+1)q+r}) = n - 2q - 3$.

□

5. Decycling number of cubic graphs

Let $\mathcal{R}(3^{2n})$ be the class of cubic graphs of order $2n$. As a consequence of the previous sections we have the following result concerning the class of cubic graphs.

Theorem 5.1 *For any integer $n \geq 2$, we have*

$$\min(\phi, 3^{2n}) = \lceil \frac{n+1}{2} \rceil, \text{ and}$$

$$\max(\phi, 3^{2n}) = \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

□

Thus, by Theorem 5.1, the range of decycling number can be obtained, namely

$$\phi(3^{2n}) = \{\phi(G) \ : \ G \in \mathcal{R}(3^{2n})\} = [\min(\phi, 3^{2n}), \max(\phi, 3^{2n})].$$

For each $c \in \phi(\mathbf{d})$, let $\mathcal{R}(\mathbf{d};\, c)$ denote the subgraph of the graph $\mathcal{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with decycling number c. We consider the problem of determining the structure of induced subgraph $\mathcal{R}(\mathbf{d};\, c)$. In general, what is the structure of $\mathcal{R}(\mathbf{d};\, c)$? In particular, are these graphs connected? If $\mathcal{R}(\mathbf{d};\, c)$ is connected, it must be possible to generate all realizations of $\mathbf{d}$ with decycling number c by beginning with one such realization and applying a suitable sequence of switchings producing only graphs with decycling number c. In this section, we prove that the induced subgraph $\mathcal{R}(3^{2n};\, \lceil \frac{n+1}{2} \rceil)$ is connected.

Let $G \in \mathcal{R}(3^{2n};\, \lceil \frac{n+1}{2} \rceil)$ and S be a minimum decycling set of G. The structure of $\mathcal{R}(3^{2n};\, \lceil \frac{n+1}{2} \rceil)$, for $n = 2, 3$, can be easily verified. From now on we will consider when $2n \geq 8$. Put $F = V(G) - S$. Thus, by counting edges in the graph G, we get $e(S) + e(S, F) + e(F) = 3n$, $2|S| \leq e(S, F) \leq 3|S|$ and $e(S, F) = 3|S| - 2e(S)$. Since $|E(G|_F)| \leq |F| - 1$, $\frac{3n-1}{2} - 1 \geq e(F) = 3n - 3\lceil \frac{n+1}{2} \rceil + e(S)$. The following Lemma is easily obtained.

Lemma 5.2 *Let $G \in \mathcal{R}(3^{2n};\, \lceil \frac{n+1}{2} \rceil)$ and S be a minimum decycling set of G. Put $F = V(G) - S$. Then*
(1) *$e(S) = 0$, if n is odd and $e(S) \leq 1$, if n is even,*
(2) *if n is odd, then $G|_F$ is a tree,*
(3) *if n is even and $e(S) = 1$, then $G|_F$ is a tree,*

(4) *if n is even and $e(S) = 0$, then $G|_F$ has 2 connected components.*

□

Let $\mathcal{F}(n_1, n_2, n_3)$ be the class of trees T having n_i vertices of degree $i, i = 1, 2, 3$ and $\Delta(T) \leq 3$. Thus for any $T \in \mathcal{F}(n_1, n_2, n_3)$ we have $|V(T)| = n_1 + n_2 + n_3$, $n_1 = n_3 + 2$, and $n_1 \geq 2$.

Lemma 5.3 *Let n be an odd integer with $n \geq 5$ and $N = \frac{3n-1}{2}$. Then there exists $G \in \mathcal{R}(3^{2n};\ \frac{n+1}{2})$ with a minimum decycling set S such that $G - S \in \mathcal{F}(n_1, n_2, n_3)$ if and only if there exist nonnegative integers $n_i, i = 1, 2, 3$, $n_1 + n_2 + n_3 = N$, $2n_1 + n_2 = \frac{3(n+1)}{2}$, and $n_1 \geq 2$.*

Proof. Suppose there is a cubic graph G of order $2n$ with $\phi(G) = \frac{n+1}{2}$. Then by Lemma 5.1, there exists a minimum decycling set S such that $|S| = \frac{n+1}{2}$, $e(S) = 0$, and $G - S$ is a tree of order N. Put $F = V(G) - S$. Since $\Delta(G|_F) \leq 3$, we define $n_i, i = 1, 2, 3$ as the number of vertices of $G|_F$ of degree i. It is clear that $n_1 + n_2 + n_3 = N$, $2n_1 + n_2 = \frac{n+1}{2}$ and $n_1 \geq 2$.

Conversely, suppose there exist nonnegative integers $n_i, i = 1, 2, 3$, $n_1 + n_2 + n_3 = N$, $2n_1 + n_2 = \frac{n+1}{2}$, and $n_1 \geq 2$. We first consider $n_1 = 2$. Thus $n_2 = \frac{3n-5}{2}$, $n_3 = 0$. Let T is a path of order N with $V(T) = F_1 \cup F_2$, where $F_1 = \{f_1, f_2\}$ and $F_2 = \{f_3, f_4, \ldots, f_N\}$ are the sets of vertices of T of degree 1 and of degree 2 of T, respectively. Let G be a graph with $V(G) = V(T) \cup S$, where $S = \{s_1, s_2, \ldots, s_{\frac{n+1}{2}}\}$, and $E(G) = \{s_1f_1, s_1f_2, s_1f_3, s_2f_1, s_2f_2, s_2f_4\} \cup E_1$, where

$$E_1 = \bigcup_{i=0}^{\frac{n-5}{2}} \{s_{3+i}f_{3i+5}, s_{3+i}f_{3i+6}, s_{3+i}f_{3i+7}\}.$$

It is clear that $G \in \mathcal{R}(3^{2n};\ \frac{n+1}{2})$ and $G - S \in \mathcal{F}(2, n_2, 0)$. Suppose $n_1 \geq 3$. Since $n_1' = n_1 - 1$, $n_2' = n_2 + 2$ and $n_3' = n_3 - 1$ satisfy the conditions of the theorem, by induction on n_1, there exists $G' \in \mathcal{R}(3^{2n};\ \frac{n+1}{2})$ and a minimum decycling set S such that $G' - S \in \mathcal{F}(n_1', n_2', n_3')$. Let $F_i' = \{f \in G' - S : d_{G'-S}(f) = i\}, i = 1, 2, 3$ be the corresponding vertices in $G' - S$ of degree i. Since $n_2' \geq 2$, there exists $v, w \in F_2'$ and $v \neq w$. There exist $s_1, s_2 \in S$ such that $vs_1, ws_2 \in E(G')$. If $s_1 \neq s_2$, then the graph $G = G'^{\sigma(u,v;w,s_2)}$ contains a maximum induced forest T with n_1 vertices of degree 1. If $s_1 = s_2$ and $us_1 \notin E(G')$, there exists $s \in S - \{s_1\}$ such that $us \in E(G')$. Thus $G'^{\sigma(u,s;v,s_1)}$ is a graph such that v and w have different neighbors in S. Finally, if $s_1 = s_2$ and $us_1 \in E(G')$, there exists $s \in S - \{s_1\}$ such that $us \in E(G')$ and there exists $x \in (G' - S) - \{u, v, w\}$ such that $xs \in E(G')$. Thus $G'^{\sigma(v,s_1;x,s)}$ is a graph such that v and w have different neighbors in S. Thus the proof is complete.

□

Lemma 5.4 *Let n be an even integer with $n \geq 4$ and $N = \frac{3n}{2} - 1$. Then there exists $G \in \mathcal{R}(3^{2n};\ \frac{n}{2} + 1)$ with a minimum decycling set S such that $G - S \in \mathcal{F}(n_1, n_2, n_3)$ and $e(S) = 1$ if and only if there exist nonnegative integers $n_i, i = 1, 2, 3$, $n_1 + n_2 + n_3 = N$, $2n_1 + n_2 = \frac{3n}{2} + 1$, and $n_1 \geq 2$.*

Proof. The proof follows from Lemma 5.2 and similar argument in Lemma 5.3. □

Let n be an even integer with $n \geq 4$ and $N = \frac{3n}{2} - 1$. For a set S of cardinality $\frac{n}{2}+1$, a graph $G \in \mathcal{R}(3^{2n};\ \frac{n}{2}+1)$ with the minimum decycling set S, G is called a cubic graph of type 0 or 1 if $e(S) = 0$ or $e(S) = 1$, respectively. Let $G \in \mathcal{R}(3^{2n};\ \frac{n}{2}+1)$ be of type 0. Then there exists a decycling set S of G such that $|S| = \frac{n}{2}+1$, $E(G|_S) = \emptyset$, and $F = G - S$ is a forest containing two connected components. Thus there exist $f_1, f_2 \in V(F)$ such that $d_F(f_1) \leq 1$, $d_F(f_2) \leq 1$, and f_1, f_2 are not in the same components of F. There must also exist $s_1, s_2 \in S$ such that $s_1 \neq s_2$ and $s_1f_1, s_2f_2 \in E(G)$. Thus $G^\sigma(s_1, f_1; s_2, f_2) \in \mathcal{R}(3^{2n};\ \frac{n}{2}+1)$ having S a minimum decycling set and $G^\sigma(s_1, f_1; s_2, f_2)$ is of type 1. Thus the graphs of two types can be transformed to each other by a suitable switching.

Lemma 5.5 *Let n be an even integer with $n \geq 4$ and $N = \frac{3n}{2} - 1$. Then there exists a cubic graph G of order $2n$ with minimum decycling set S such that $|S| = \frac{n}{2}+1$ and $e(S) = 0$ if and only if there exists a switching σ such that $G^\sigma \in \mathcal{R}(3^{2n};\ \frac{n}{2}+1)$ with a minimum decycling set S and $e(S) = 1$.*

□

Let n, N, n_1, n_2, n_3 be integers satisfying conditions in Lemma 5.3. Let $\mathcal{F}(n_1, n_2, n_3)$ be the class of trees T having n_i vertices of degree $i, i = 1, 2, 3$ and $\Delta(T) \leq 3$. We first consider in the case when $n_1 = 2$. Thus $n_3 = 0$ and $\mathcal{F}(2, n_2, 0)$ contains the path of N vertices. Let P_N be the path $f_1f_2 \cdots f_N$ and let $S_t = \{s_1, s_2, \ldots, s_t\}$, where $t = \frac{n+1}{2}$. It is clear that there are cubic graphs obtained by joining $3t$ edges from S to vertices in P_N and such graphs are not unique. Now we will construct a cubic graph of order $2n$. When $N = 4$, it is easy to see that there is a unique cubic graph G_4 obtained in this way. That is $V(G_4) = S_2 \cup V(P_4)$ and $E(G_4) = \{s_1f_1, s_1f_2, s_1f_4, s_2f_1, s_2f_3, s_2f_4\}$.

A cubic graph G_7 of order 10 can be constructed by taking $V(G_7) = S_3 \cup V(P_7)$ and $E(G_7) = (E(G_4) - \{s_2f_4\}) \cup \{s_2f_7, s_3f_5, s_3f_6, s_3f_7\}$. Thus G_7 is a cubic graph of order 10 with $\phi(G_7) = 3$. Similarly the graph G_{10} can be obtained from G_7 by extending the path P_7 to P_{10}, removing the edge s_3f_7 and inserting edges $s_3f_{10}, s_4f_8, s_4f_9, s_4f_{10}$. In general, if $t \geq 4$, then $N = 3t - 2$. We can construct the cubic graph G_N obtained from G_{N-3} by extending the path P_{N-3} to P_N, removing the edge $s_{t-1}f_{N-3}$ and inserting edges $s_{t-1}f_N, s_tf_{N-2}, s_tf_{N-1}, s_tf_N$.

Lemma 5.6 *Let G be a cubic graph of order $2n$, n is an odd integer, with $\phi(G) = \frac{n+1}{2}$. If G has a path P_N as a maximum induced forest, where $N = \frac{3n-1}{2}$, then G_N can be obtained from G by a finite sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that for all $i = 1, 2, \ldots, k$, $G^{\sigma_1\sigma_2\ldots\sigma_i}$ is a cubic graph with P_N as its induced forest.*

Proof. It is easy if $N = 4$. Let G be a cubic graph of order $2n$ with P_N as its induced forest. Put $P_N = f_1f_2 \cdots f_N$ and $S_t = \{s_1, s_2, \ldots, s_t\}$ where $t = \frac{n+1}{2}$. If $s_tf_N \notin E(G)$, there are exactly 2 vertices in S which are adjacent to f_N and there are exactly 3 vertices in $V(P_N)$ which are adjacent to s_t. Thus there exist $s_i \in S$ and $f_j \in V(P_N)$ such that $s_if_N, s_tf_j \in E(G)$ and $s_if_j \notin E(G)$. The graph $G^1 = G^{\sigma(s_t, f_j; f_N, s_i)}$ has a common edge s_tf_N with G_N. If $s_tf_{N-1} \notin E(G^1)$, there exists $s \in S_t$ such that $sf_{N-1} \in G^1$. Put $s = s_{t-1}$. Since P_N is a path and $t \geq 3$, $|N(s_{t-1}) \cap N(s_t)| \leq 1$. Thus there is a switching which transforms G^1 to G^2 such that G^2 has $s_t f_N, s_t f_{N-1}$ as common edges with G_N. By continuing in this way, we can transform the graph G by a finite number of switchings $\sigma_1, \sigma_2, \ldots, \sigma_k$ to G_N such that for all $i = 1, 2, \ldots, k$, $G^{\sigma_1 \sigma_2 \ldots \sigma_i}$ is a cubic graph with P_N as its induced forest.

□

Lemma 5.7 *Let G be a cubic graph of order $2n$, n is an odd integer, with $\phi(G) = \frac{n+1}{2}$. If G does not have P_N as its maximum induced forest. Then a cubic graph G_N can be obtained from G by a finite sequence of switchings $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that for all $i = 1, 2, \ldots, k$, $G^{\sigma_1 \sigma_2 \ldots \sigma_i} \in \mathcal{R}(3^{2n};\ \frac{n+1}{2})$ and $G^{\sigma_1 \sigma_2 \ldots \sigma_k} = G_N$.*

Proof. In the proof of Lemma 5.3 and Lemma 5.6 a sequence of suitable switchings can be obtained in order to transform G into G_N.

□

Similar arguments can be made to obtain the same result for cubic graphs of order $2n$ and n is even.

Combining the results in this section, we have the following theorem.

Theorem 5.8 *The induced subgraph $\mathcal{R}(3^{2n};\ \lceil \frac{n+1}{2} \rceil)$ is connected.*

□

Acknowledgment

A part of this work was written when the author visited the Department of Mathematics, Illinois State University on winter 2003. The hospitality is gratefully acknowledge. He would also like to thank Prof. R. B. Eggleton for his guidance and encouragement. Finally, the author wishes to thank the referee for his useful suggestions and comments.

References

[1] S. Bau and L. W. Beineke, The decycling number of graphs, *Australasian J. Combinatorics,* **25**(2002), 285–298.

[2] S. Bau, N. C. Wormald, S-M. Zhou, Decycling numbers of random regular graphs, *Random Struct. Algorithms,* **21**(2002) 397–413.

[3] J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications", 1^{st} Edition, The MacMillan Press, 1976.

[4] S. Hakimi, On the realizability of a set of integers as the degree of the vertices of a graph: *SIAM J. Appl. Math.,* **10**(1962), 496–506.

[5] M. Havel, A remark on the existence of finite graphs (in Hungarian), *Casopis Pest. Mat.,* **80**(1955), 477–480.

[6] K.R. Parthasarathy, "Basic Graph Theory", Tata McGraw-Hill Publishing Com. Ltd., New Delhi, 1994.

[7] N. Punnim, On $F(j)$-graphs and their applications. J. Combin. Math. Combin. Comput. **38,**(2001), 65–72.

[8] N. Punnim, Degree sequences and chromatic number of graphs, *Graphs Combin.,* **18**(3), (2002), 597–603.

[9] N. Punnim, The clique number of regular graphs, *Graphs Combin.,* **18**(4), (2002), 781–785.

[10] N. Punnim, The matching number of regular graphs, *Thai Journal of Mathematics,* **2** (2004), 133–140.

(Received 24 Aug 2003; revised 10 Feb 2004)