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Chords in graphs

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Abstract

We characterize all simple graphs such that each edge is a chord of some cycle. As a consequence, we characterize all simple 2-connected graphs such that, for any two adjacent vertices x and y, the local connectivity $k(x, y) \geq 3$. We also make a conjecture about chords for 3-connected graphs.

1 Introduction

All graphs considered in the paper are undirected and simple. Suppose that e is an edge of a graph G. For simplicity, we will use $E(G)\backslash e$ to denote $E(G)\backslash\{e\}$. An edge e is a *chord* of a cycle C if $E(C)$ can be partitioned into two sets C_1 and C_2 such that both $C_1 \cup e$ and $C_2 \cup e$ are cycles. We say that an edge is a *chord* of the graph if it is a chord of some cycle in the graph. A graph is chordal if each cycle of size at least four has a chord. The following is a characterization for chordal graphs due to Hajnal and Surányi, and Dirac, respectively (see [3, Theorem 8.11]).

Theorem 1.1 *A graph G is chordal if and only if either G is complete or G can be obtained from two chordal graphs G_1 and G_2 (having fewer vertices than that of G) by identifying two complete subgraphs of the same order in G_1 and G_2.*

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A graph G is minimally k-connected if G is k-connected, but for each edge e of G, the deletion $G\backslash e$ is not k-connected. Dirac [4] and Plummer [6] independently proved the following result.

Theorem 1.2 *Let G be a 2-connected simple graph. Then no cycle has a chord if and only if G is minimally 2-connected.*

If a graph G has minimum degree at least three, then it is shown that it has a cycle with a chord. In [1], Ali and Staton investigated minimum degree conditions to force the existence of a cycle with k chords. Theorem 1.1 determines all graphs such that each cycle of size at least four has a chord. Theorem 1.2 determines all 2-connected graphs such that no cycle has a chord. In this paper, we answer the following natural question.

Problem 1: Determine all simple graphs such that each edge is a chord of some cycle.

We will use $\mathcal{D}$ to denote the class of graphs such that each edge is a chord. By Whitney's theorem [8], a simple graph with at least four vertices is 3-connected if and only if every two vertices are connected by at least three internally disjoint paths. Thus each edge in a 3-connected graph is a chord of some cycle. Hence the class of graphs $\mathcal{D}$ contains all 3-connected graphs. Our main result shows that we can generate all such graphs based on 3-connected graphs.

First we describe two well-known useful graph operations (see, for example, [5]). Let G_1 and G_2 be two graphs with no common vertices or edges. Let $e_1 = u_1v_1$ be an edge of G_1 and $e_2 = u_2v_2$ be an edge of G_2. Identify u_1 and u_2 and relabel as u. Identify v_1 and v_2 and relabel as v (the edges e_1 and e_2 are identified too and relabelled as e). We obtain a graph $P(G_1, G_2)$, called the *parallel connection* of G_1 and G_2 with respect to the edges e_1 and e_2. The graph $P(G_1, G_2)\backslash e$ is called the *2-sum* of G_1 and G_2, denoted by $G_1 \oplus_2 G_2$. Let S be a subset of $V(G)$. We use $G[S]$ to denote the subgraph induced by S. Before stating our main result, we define a class of graphs $\mathcal{G}$ and a graph operation first.

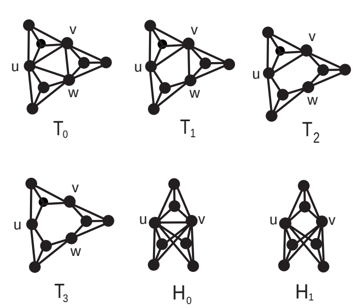


Figure 1.

Our class of graphs $\mathcal{G}$ is constructed from K_2, K_3, or any 3-connected graph G. First suppose that we start with a 3-connected graph G. Let S be a non-empty set of edges of G. Take a graph $N \cong K_4$. For each edge e in S, label an edge of N as e first, then perform either the parallel connection or 2-sum operation between G and N along the edge e. We obtain a graph in $\mathcal{G}$ by repeating the operation for each element of S, while replacing G with the newly-obtained graph. There are six more graphs in $\mathcal{G}$ which we will describe next. Start with a triangle. Let S be the set of all edges in the triangle in the above construction. One of the graphs constructed is T_0, shown in Figure 1. The other graphs T_1, T_2, and T_3 can be obtained from T_0 by deleting one, two, and three edges in the set $\{uv, uw, vw\}$. Finally, sticking three copies of K_4 along a common edge (K_2), we get the the graph H_0. Remove uv from H_0, we get the graph H_1 (see Figure 1.) Next we define a graph operation $\uplus$. We use K_4^- to denote the graph obtained by deleting an edge from K_4.

Definition 1.3 *Let G_1 and G_2 be two simple graphs. Suppose that for $i = 1, 2$, the graph G_i has a 2-element vertex-cut $\{u_i, v_i\}$, such that $G_i - \{u_i, v_i\}$ has a component with vertex set S_i where the subgraph $G[S_i \cup \{u_i, v_i\}] - u_iv_i$ is isomorphic to K_4^-. We define a new graph $G_1 \uplus G_2$ as follows (see Figure 2). From $G_1 - S_1$ and $G_2 - S_2$, identify u_1 and u_2 and relabel as u, then identify v_1 and v_2 and relabel as v, and remove any multiple edges if there are any (when $u_iv_i \in E(G_i)$ for $i = 1, 2$).*

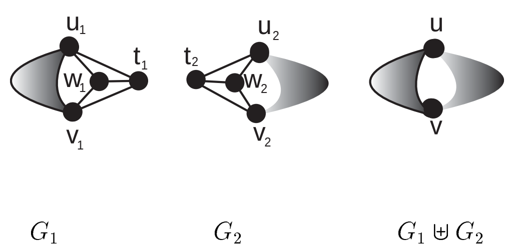


Figure 2.

The next result is the main result of this paper. It has a similar flavor to that of Theorem 1. A proof of the theorem will be given in the next section.

Theorem 1.4 *Let G be a simple graph. Then each edge of G is a chord of some cycle if and only if each block of any component of G is either 3-connected, or is a graph in $\mathcal{G}$, or can be constructed by a sequence of operations $\uplus$ starting from graphs in $\mathcal{G}$.*

For any two non-adjacent vertices $x, y \in V(G)$, the local connectivity $k(x, y)$ is defined to be the minimum number of vertices that separates x and y. For $xy \in E(G)$, $k(x, y) = k_{G-xy}(x, y) + 1$. A graph is 3-connected if and only if the local connectivity is at least three for any two vertices of G. Our main result characterizes all simple 2-connected graphs such that the local connectivity of any two adjacent vertices is at least three.

Corollary 1.5 *Let G be a simple 2-connected graph. Then for each edge $xy \in E(G)$, the local connectivity $k(x,y) \geq 3$ if and only if G is either 3-connected, or is a graph in $\mathcal{G}$, or can be constructed by a sequence of operations $\uplus$ starting from graphs in $\mathcal{G}$.*

Next we introduce some notation which will be needed in the proof. Let S be a vertex cut of G. We use $G - S$ to denote the subgraph obtained by deleting S from G. A *minimum component* of a graph G is a component of G with minimum size.

2 Proof of the main result

In this section, we will prove our main result. First we prove several lemmas.

Lemma 2.1 *Let G be a graph such that each edge is a chord. Then each block of any component of G is both 2-connected and 3-edge-connected.*

Proof. Let H be a block of a component of G. Then each edge of H is a chord of some cycle in H. Hence H is not a single edge and therefore is 2-connected. For each edge $e = uv$ in H, there are at least two internally disjoint paths connecting u and v in the graph $H - uv$. Suppose that H is not 3-edge-connected and $T = \{uv, u_1v_1\}$ is a 2-element edge-cut. Then $H - uv$ has a cut edge. Thus $H - uv$ cannot have two internally disjoint paths connecting u and v, a contradiction. □

Lemma 2.2 *Let G be a 2-connected graph such that each edge is a chord. Then each edge of $H = G \oplus_2 K_4$ (or $P(G, K_4)$) is a chord of some cycle of H.*

Proof. We will give a proof for $H = G \oplus_2 K_4$ only. The case for $P(G, K_4)$ is obvious. Suppose that H is obtained by first identifying a common edge uv of G and $K \cong K_4$, then deleting the edge uv. Now let $e \neq uv$ be any edge of H. If $e \in E(G)$, then e is a chord of some cycle C of G. If C is also a cycle of H, then e is a chord of a cycle of H. If C is not a cycle of H, then $uv \in C$. Let $w \in V(K) - \{u, v\}$, then $C \backslash uv \cup \{uw, wv\}$ is a cycle of H. Clearly, e is still a chord of this cycle. Now suppose that $f \notin E(G)$. Then f is a chord of cycle C_1 in K. If $uv \notin C_1$, then C_1 is still a cycle of H. If $uv \in C_1$, then take a $uv-$path P in $G - uv$. This path exists as G is 2-connected. Now $C_2 = C_1 \cup P - uv$ is also a cycle having f as a chord. This completes the proof of the lemma. □

Lemma 2.3 *Let G_1 and G_2 be two 2-connected graphs and $G = G_1 \uplus G_2$. Suppose that each edge of G_i is a chord of G_i for $i = 1, 2$. Then each edge of G is a chord of some cycle.*

Proof. We use the same notation as in Definition 1.3. Let e be any edge of H. Without loss of generality, assume that $e \in E(G_1)$. Thus e is a chord of a cycle C of G_1. First suppose that $e \neq uv$. If neither w_1 nor t_1 is in C (see Figure 2), then C is also a cycle of G having e as a chord. . If either w_1 or t_1 is in C, then as $\{u_1, v_1\}$ is a 2-element vertex-cut, we deduce that both u_1 and v_1 are in C. The vertices u_1 and v_1 divides the cycle C into two u_1v_1-paths. As $e \neq uv$ is a chord of C, we conclude that one path, denoted by $P_1[u, v]$, must be in G. As G_2 is 2-connected, there is a path $P_2[u_2, v_2]$ in the graph $G_2 - \{w_2, t_2\}$ (otherwise, both u_2 and v_2 are cut-vertices of G_2, a contradiction). Now in G, $P_1[u_1, v_1] \cup P_2[u_2, v_2]$ is a cycle. Clearly this cycle has e as a chord. Now suppose that $e = uv$. Pick a u_iv_i-path $P[u_i, v_i]$ in $G_i - \{w_i, t_i\}$ for $i = 1, 2$. Then $P_1[u_1, v_1] \cup P_2[u_2, v_2]$ is a cycle having e as a chord. This completes the proof of the lemma. □

Next we prove our main theorem. Recall that we use $\mathcal{D}$ to denote the class of graphs such that each edge is a chord of some cycle.

Proof of Theorem 1.4. Clearly, each edge of G is a chord if and only if each edge of all blocks of any component of G is a chord. Thus we may assume that G is a block.

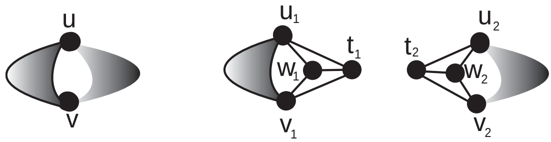
If G is 3-connected, then from Whitney's theorem, each edge is a chord. It is easy to check that each graph T_0, T_1, T_2, T_3, H_0, and H_1 is in $\mathcal{D}$. By Lemma 2.2, each graph of $\mathcal{G}$ is in $\mathcal{D}$. By Lemma 2.3, if a graph G is constructed by a sequence of operations $\uplus$ starting from graphs in $\mathcal{G}$, then each edge of G is a chord.

Now we prove the converse. Suppose that every edge of G is a chord and that G is connected. Clearly G has at least four vertices. Thus G is 2-connected as it is a block. We use induction on $n = |V(G)|$. If $n \leq 4$, it is straightforward to see that G must be K_4 and the theorem holds. Suppose that G is not 3-connected, and G is not a member of $\mathcal{G}$. By Lemma 2.1, G is 3-edge-connected. As G is not 3-connected, there is a 2-element vertex-cut $S = \{u, v\}$ of G. Then $G - S$ has at least two components.

Claim 1. Let T be the vertex set of a component of $G - S$. If $|T| = 2$, then $G[(S \cup T)] - uv \cong K_4^-$.

Proof. The claim follows from the fact that $d_G(x) \geq 3$ for each $x \in T$ since, by Lemma 2.1, G is 3-edge-connected. □

Claim 2. Let $S = \{u, v\}$ be a 2-element vertex-cut of G. Suppose that there are two subgraphs H_1 and H_2 of G such that $V(H_1) \cup V(H_2) = V(G)$ and $V((H_1) \cap V(H_2) = S$. If $min\{|V(H_1)|, |V(H_2)|\} \geq 5$, then there are two graphs $G_1 \in \mathcal{D}, G_2 \in \mathcal{D}$ such that $G = G_1 \uplus G_2$. Moreover, both G_1 and G_2 have fewer number of vertices than G.



G G_1 G_2

Figure 3.

Proof. If $uv \in E(G)$, let $G_1 = P(H_1, K_4)$ and $G_2 = P(H_2, K_4)$, where uv is the commonly identified edge. If $uv \notin E(G)$, add uv to G first then define G_1 and G_2 similarly with the only difference being deleting the edge uv in the construction of G_1 and G_2(see Figure 2). Then as $|V(H_1)|, |V(H_2)| \geq 5$, both G_1 and G_2 have fewer number of vertices than G has. Moreover, it is straightforward to check that $G_1 \in \mathcal{D}, G_2 \in \mathcal{D}$ by the construction of G_1 and G_2. □

Now we continue the proof of the theorem. If G has no 2-element vertex-cut, then G is 3-connected and the theorem holds. Suppose that G has a 2-element vertex-cut S. Then there are two subgraphs H_1 and H_2 of G such that $V((H_1) \cup V(H_2) = V(G)$ and $V(H_1) \cap V(H_2) = S$. Suppose that there exist such subgraphs H_1 and H_2 such that $min\{|V(H_1)|, |V(H_2)|\} \geq 5$. By Claim 2, there are two graphs $G_1 \in \mathcal{D}, G_2 \in \mathcal{D}$ such that $G = G_1 \uplus G_2$. Moreover, both G_1 and G_2 have fewer number of vertices than G has. By induction, the theorem holds.

Otherwise, for each 2-element vertex-cut S and each subgraphs H_1 and H_2 of G such that $V(H_1) \cup V(H_2) = V(G)$ and $V((H_1) \cap V(H_2) = S$, we have $\min\{|V(H_1)|, |V(H_2)|\} = 4$. Next we complete the proof of the theorem by showing that $G \in \mathcal{G}$. By Lemma 2.1, G is 3-edge connected. Thus $\delta(G) \geq 3$. Hence each component of $G - S$ has at least two vertices. By our assumption, it is straightforward to see that $G - S$ has at most three components. Moreover, if $G - S$ has exactly three components, then each such component has exactly two vertices. Using Claim 1, it is easy to see that G is either H_0 or H_1.

Therefore, we may assume that $G - W$ has exactly two components for each 2-element vertex-cut W of G. For each minimum component T of $G - W$, we deduce that $|V(T)| = 2$ by our assumption. Suppose that $W = \{p, q\}$. By Claim 1, $G[(W \cup T)] - uv \cong K_4^-$. Remove $V(T)$ from G, then add an edge pq. Repeat this process for each 2-vertex cut of G. Finally remove any multiple edges to get a graph H. During this process, all subgraphs of the form K_4^- has been reduced. If H has exactly one edge, then $G \cong P(K_4, K_4)$, or $K_4 \oplus_2 K_4$ and the theorem holds. Now suppose that H is a triangle with edges labelled as a, b, and c. Then G can be obtained from H and copies of K_4 by performing the operation of parallel connection or 2-sum along the edges a, b, and c sequentially. As each edge is a chord of some cycle, it is straightforward to check now that G is one of T_0, T_1, T_2 or T_3. Suppose that H has at least four vertices. Clearly H is 2-connected. Now we show that H must be

3-connected. Suppose that H has 2-element vertex-cut W. Then W is clearly also a 2-element vertex-cut of G. Thus $G - W$ has exactly two components and has one 2-element component T_1. By Claim 1, we conclude that $G[(S \cup T_1] - uv \cong K_4^-$ and therefore T_1 is removed in H and W cannot be a vertex-cut of H, a contradiction. This completes the proof of the theorem. □

3 A conjecture

A well-known conjecture of Thomassen (see, for example, [2]) is the following:

Conjecture 1 Each longest cycle in a 3-connected graph has a chord.

We make the following conjecture which is stronger than the above conjecture.

Conjecture 2 Let G be a 3-connected graph and e be an edge of G. Then among all the cycles containing e, each longest such cycle has a chord.

While the above two conjectures seem to be very hard, the following weaker result can be proved.

Theorem 3.1 *Let G be a 2-connected simple graph with minimum degree at least three. Then each edge lies in a cycle which contains a chord.*

This result follows from a stronger result of Voss [7]. But the proof of that result is long. Here we give a simple proof of this weaker result.

Proof. Suppose that the theorem fails. As $\delta(G) \geq 3$, G has at least 4 vertices. Let e be any edge of G. Take a longest cycle C containing e. Then clearly $|C| \geq 3$. Suppose that $|C| = 3$. Let $f \in E(G)\backslash C$. As G is 2-connected, there is a cycle C_1 containing both e and f. As G is simple and $|C| = 3$, we deduce that $|C_1| = 3$ and $C \cap C_1 = \{e\}$. Thus $C \cup C_1 \backslash e$ is a 4-element cycle, a contradiction. Hence $|C| \geq 4$. Let $C = a_1 a_2 \ldots a_n a_1$, where $n \geq 4$ and $e = a_n a_1$. As G is 2-connected with minimum degree at least three, for each vertex a_i, there is an a_{k_i} for some $1 \leq k_i \leq n, k_i \neq i$ and a path $P[a_i, a_{k_i}]$ such that $V(P[a_i, a_{k_i}]) \cap V(C) = \{a_i, a_{k_i}\}$. Although there might be more than one path, we assume that path from a_{k_i} to the cycle is also $P[a_i, a_{k_i}]$. In other words, we assume that $a_{k_{k_i}} = a_i$. As C is a longest cycle of G containing e, we conclude that

(a) $k_i \neq i-1, i+1$ for all $2 \leq i \leq n-1$.

(b) For $1 \leq i \leq n-1$, $P[a_i, a_{k_i}]$ and $P[a_{i+1}, a_{k_{i+1}}]$ do not meet except possibly at the end vertex (i.e., the only possible vertex they could meet is when $a_{k_i} = a_{k_{i+1}}$.)

(c) Suppose that $k_i > i$ for some i where $2 \leq i \leq n$. Then $k_{i+1} \leq k_i$.

Proof. by (b), $V(P[a_i, a_{k_i}]) \cap V(P[a_{i+1}, a_{k_{i+1}}]) = \emptyset$. Suppose that $k_{i+1} > k_i$. We deduce that $a_n a_1 a_2 \ldots a_i P[a_i, a_{k_i}] a_{k_i - 1} \ldots a_{i+1} P[a_{i+1}, a_{k_{i+1}}] a_{k_{i+1}+1} \ldots a_n$ is a cycle containing e. Clearly this cycle has a chord $a_i a_{i+1}$, a contradiction.

Now we complete the proof of the theorem. Clearly, $k_2 > 2$ by (a). Hence $k_3 \leq k_2$. If $k_3 > 3$, then $k_4 \leq k_3$. Continue the process and suppose that t (where $k_2 \geq t \geq 2$) is the largest number such that $k_i > i$ for all $2 \leq i \leq t$. Using (c) and the fact that $P[a_2, a_{k_2}]$ is a path, it is straightforward to see that t exists and that $2 \leq t \leq k_2 - 1$. By (c), we deduce that $k_2 \geq k_3 \geq \ldots \geq k_t$. By (a), $t \leq k_t - 2$. Hence $t + 1 < k_t$. By the choice of t and (a), we have $1 \leq k_{t+1} \leq t - 1$. Denote k_{t+1} by j. As $2 \leq j+1 \leq t$, $k_{j+1} \geq k_t \geq t + 2$. By (b) again, $V(P[a_j, a_{k_j}]) \cap V(P[a_{j+1}, a_{k_{j+1}}]) = \emptyset$, noting that $k_j = t + 1$. Thus $a_n a_1 \ldots a_j P[a_j, a_{k_j}]a_t \ldots a_{j+1} P[a_{j+1}, a_{k_{j+1}}]a_{k_{j_1}+1} \ldots a_n$ is a cycle containing e. Clearly this cycle has a chord $a_j a_{j+1}$, a contradiction. This completes the proof of the theorem. □

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