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Balanced bipartite 4-cycle designs

Hung-Lin Fu*

Department of Applied Mathematics
National Chiao Tung University
Hsin Chu, Taiwan 30050

Abstract

Let $(\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$ denote the graph G with $V(G) = V_1 \cup V_2,\ V_1 \cap V_2 = \emptyset,\ |V_1| = v_1,\ |V_2| = v_2$, and the edges of G are obtained by joining (a) each pair of vertices in $V_i,\ i = 1, 2$, exactly λ_i times and (b) each pair of vertices from V_1 to V_2 exactly λ_3 times. In this paper, we determine all quintuples $(\lambda_1, \lambda_2, \lambda_3; v_1, v_2)$ such that $(\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$ can be decomposed into 4-cycles.

1 Introduction

A balanced bipartite block design BBBD $(v_1, v_2; k; \lambda_1, \lambda_2, \lambda_3)$ is a triple (V_1, V_2, B) where V_1 and V_2 are disjoint sets and B is a collection of subsets of $V_1 \cup V_2$ called blocks each of size k, such that

(a) each pair of elements from V_1 or from V_2 occur together in exactly λ_i blocks, $i = 1, 2$, and

(b) each pair of elememts one from V_1 and the other from V_2, occur together in exactly λ_3 blocks.

The pairs of elements in (a) are called the first and second associates respectively, and the pairs of elements in (b) are called the third associates. We say the BBBD is defined on $V_1 \cup V_2$.

The notion of BBBD was first introduced by Nair and Rao in 1942 [15]. Since then, quite a few special BBBD's have been constructed; see [1, 4, 11, 12, 14, 16]. Recently, the combined works of Fu, Rodger and Savarte [8], and Fu and Rodger [7] settled the existence of group divisible designs (GDD) with two associates for block size 3.

A group divisible design GDD $(n, m; k; \lambda_1, \lambda_2)$ with two associates is an ordered triple (X, G, B) where X is a set of elements or varieties, G is a partition of X into

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m sets of size n, each set being called a group, and B is a collection of subsets of X, called blocks, each of size k, such that

(a) each pair of elements from the same group occur together in exactly λ_1 blocks, and
(b) each pair of elements from different groups occur together in exactly λ_2 blocks.

Theorem 1.1 [7,8] *Let $n,\ m,\ \lambda_2 \geq 1$ and $\lambda_1 \geq 0$. Then there exists a GDD$(n,m;3;\lambda_1,\lambda_2)$ if and only if*

(a) 2 divides $\lambda_1(n-1)+\lambda_2(m-1)m$,

(b) 3 divides $\lambda_1 mn(n-1)+\lambda_2 m(m-1)n^2$,

(c) if $m=2$ then $\lambda_1 \geq \lambda_2 n/2(n-1)$, and

(d) if $n=2$ then $\lambda_1 \leq (m-1)\lambda_2$.

As a consequence of this result, when $m=2$, the existence of BBBD$(v,v;3;\lambda_1,\lambda_1,\lambda_3)$ is settled. We say that $G|H$ if H can be decomposed into isomorphic copies of graphs G. For example, it is well-known that $C_3|K_7$. In terms of graph decompositions, the above result determines the triples (v,λ_1,λ_3) such that the graph $(\lambda_1,\lambda_3)K_{v,v}$ can be decomposed into triangles (K_3 or C_3), i.e., $K_3|(\lambda_1,\lambda_3)K_{v,v}$. Here $(\lambda_1,\lambda_3)K_{v,v}$ denotes the graph on $V_1 \cup V_2$ where V_1 and V_2 are disjoint sets of size v, and each pair of vertices from V_1 or from V_2 are joined by λ_1 edges, and each pair of vertices, one from V_1 and the other from V_2, are joined by λ_3 edges. Therefore, the existence of a group divisible design GDD$(n,m;k;\lambda_1,\lambda_2)$ is equivalent to decomposing the modified balanced complete multipartite graph $H=(A_1,A_2,...,A_m)$ into $K_k^{'}s$ where $A_1,A_2,...,A_m$ are disjoint sets each of size n, and each pair of vertices from A_1, A_2, $\cdots$, or A_m are joined by λ_1 edges and each pair of vertices from two different sets are joined by λ_2 edges. For convenience, H is also denoted by $(\lambda_1,\lambda_2)K_{m(n)}$. By replacing K_k with C_4, we can determine the quadruples $(\lambda_1,\lambda_2;m,n)$ such that $C_4|(\lambda_1,\lambda_2)K_{m(n)}$. If $C_4|(\lambda_1,\lambda_2)K_{m(n)}$, then we say a 4-cycle GDD$(n,m;C_4;\lambda_1,\lambda_2)$ with two associate classes exists.

Theorem 1.2 [10] *There exists a 4-cycle group divisible design GDD$(n,m;C_4;\lambda_1,\lambda_2)$ if and only if*

(a) 2 divides $\lambda_1(n-1)+\lambda_2 n(m-1)$,

(b) 8 divides $\lambda_1 mn(n-1)+\lambda_2 n^2 m(m-1)$,

(c) if $n=2$ then $\lambda_1 \leq 2(m-1)\lambda_2$, and

(d) if $n=3$ then $\lambda_1 \leq 3(m-1)\lambda_2/2-\delta(m-1)/9$, where $\delta=0$ or 1 if and only if λ_2 is even or odd, respectively.

Clearly, the case $m = 2$ gives the bipartite designs and the above result shows that $C_4 | \ (\lambda_1, \lambda_2)K_{v,v}$ if the triple $(\lambda_1, \lambda_2, v)$ satisfies the conditions in Theorem 1.2. Indeed, this motivates the study of balanced bipartite block design with each block a 4-cycle, C_4. Now, instead of two associate classes we have three associate classes; furthermore the two partite sets can be of different sizes. In this paper, we shall use the notation BBQD$(v_1, v_2; \lambda_1, \lambda_2, \lambda_3)$ to represent such a design, and prove the following theorem.

Recall that the existence of a BBQD$(v_1, v_2; \lambda_1, \lambda_2, \lambda_3)$ is equivalent to $C_4 \mid (\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$. Therefore, throughout this paper, we determine all the quintuples $(\lambda_1, \lambda_2, \lambda_3; v_1, v_2)$ such that $C_4 \mid (\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$.

Theorem 1.3 *Let $v_1, v_2, \lambda_1, \lambda_2$ and λ_3 be non-negative integers such that $2 \leq v_1 \leq v_2$, and $\lambda_3 \geq 1$. Then $C_4 \mid (\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$ if and only if*

(a) 2 divides $\lambda_1(v_1 - 1) + \lambda_3 v_2$ and $\lambda_2(v_2 - 1) + \lambda_3 v_1$,

(b) 8 divides $\lambda_1 v_1(v_1 - 1) + \lambda_2 v_2(v_2 - 1) + 2\lambda_3 v_1 v_2$,

(c) if $v_1, v_2 \leq 3$, then $\lambda_1\binom{v_1}{2} + \lambda_2\binom{v_2}{2} \leq \lambda_3 v_1 v_2$,

(d) if $v_1 = 2$ and $v_2 \geq 4$, then $\lambda_2\binom{v_2}{2} \geq \lambda_1$ and $\lambda_3 v_2 \geq \lambda_1$, and

(e) if $v_1 = 3$ and $v_2 \geq 4$, then $\lambda_3 v_2 \geq \lambda_1 + \delta_{\lambda_3}\lceil v_2/6\rceil$ where $\delta_x = 0$ if x is even and $\delta_x = 1$ if x is odd, and $\lambda_2 > 0$ provided that λ_1 is odd.

2 Preliminary results

First, we prove the necessity of Theorem 1.3.

Here, $(\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$ is the modified complete bipartite graph $G = (V_1, V_2)$ where $|V_1| = v_1, |V_2| = v_2, V_1 \cap V_2 = \emptyset$, each pair of vertices from V_1 are joined by λ_1 edges, each pair of vertices from V_2 are joined by λ_2 edges, and each pair of vertices from V_1 and V_2 respectively are joined by λ_3 edges. For convenience, we shall also use (V_1, V_2) to denote $(\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$.

Lemma 2.1 *Let $v_1, v_2, \lambda_1, \lambda_2$ and λ_3 be non-negative integers such that $2 \leq v_1 \leq v_2$, and $\lambda_3 \geq 1$. If $C_4 | \ (\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$, then*

(a) 2 divides $\lambda_1(v_1 - 1) + \lambda_3 v_2$ and $\lambda_2(v_2 - 1) + \lambda_3 v_1$,

(b) 8 divides $\lambda_1 v_1(v_1 - 1) + \lambda_2 v_2(v_2 - 1) + 2\lambda_3 v_1 v_2$,

(c) if $v_1, v_2 \leq 3$, then $\lambda_1\binom{v_1}{2} + \lambda_2\binom{v_2}{2} \leq \lambda_3 v_1 v_2$,

(d) if $v_1 = 2$ and $v_2 \geq 4$, then $\lambda_2\binom{v_2}{2} \geq \lambda_1$ and $\lambda_3 v_2 \geq \lambda_1$, and

(e) if $v_1 = 3$ and $v_2 \geq 4$, then $\lambda_3 v_2 \geq \lambda_1 + \delta_{\lambda_3}\lceil v_2/6\rceil$ where $\delta_x = 0$ if x is even and $\delta_x = 1$ if x is odd, and $\lambda_2 > 0$ provided that λ_1 is odd.

Proof. Let $G = (V_1, V_2) = (\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$, where $|V_1| = v_1$ and $|V_2| = v_2$. Let E_i be the set of edges of G that are incident with two vertices in V_i for $i = 1, 2$, and let E_3 be the set of edges incident with a vertex in V_1 and a vertex in V_2. Suppose that $C_4 \mid G$. Then the degree of each vertex in G is even, so (a) is necessary. Also, the number of edges in G is divisible by 4, so (b) is necessary. If $v_1, v_2 \leq 3$, then every 4-cycle has either two edges in $E_1 \cup E_2$ and two edges in E_3 or four edges in E_3. It follows that $|E_1| + |E_2| \leq |E_3|$. (Note that the fact $|E_3|$ is even follows from (a).)

If $v_1 = 2$, then there exist λ_1 4-cycles that have one edge in E_1, one edge in E_2 and two edges in E_3. It follows that $|E_2| \geq |E_1|$ and $|E_3| \geq 2|E_1|$. That is $\lambda_2\binom{v_2}{2} \geq \lambda_1$ and $2\lambda_3 v_2 \geq 2\lambda_1$, so (d) is necessary.

Finally suppose $v_1 = 3$. Then any 4-cycle has at most two edges in E_1. If a 4-cycle has one edge in E_1, then it has two edges in E_3 and one edge in E_2. If a 4-cycle has two edges in E_1, then it has two edges in E_3.

Since the edges in E_1 cannot be decomposed into 4-cycles, they have to be combined with either the edges in E_3 or E_2 to obtain 4-cycles. Also every 4-cycle has at most two adjacent edges of E_1. This implies that if λ_3 is even, then $|E_3| \geq |E_1|$ is necessary. But, if λ_3 is odd, then v_2 is even and for each vertex x in V_2, there is at least one edge of E_3 incident to x which cannot be used to form the type of 4-cycles mentioned above. It can only be used to obtain a 4-cycle which uses an edge in E_1, an edge in E_2 and two edges in E_3. This implies that $3\lambda_3 v_2 - v_2 \geq 3\lambda_1 - \frac{v_2}{2}$, i.e., $\lambda_3 v_2 \geq \lambda_1 + \lceil \frac{v_2}{6} \rceil$. By combining the case λ_3 is even, we have $\lambda_3 v_2 \geq \lambda_1 + \delta_{\lambda_3}\lceil \frac{v_2}{6} \rceil$. This concludes the proof of the first half of condition (e). Finally, if λ_1 is odd, then $|E_1| = 3\lambda_1$ is also odd. Thus, some of the edges in E_1 have to be a part of 4-cycles using the edges in E_2. Hence $\lambda_2 > 0$ as mentioned in condition (e). ■

Now, if $v_1 = v_2$ and $\lambda_1 = \lambda_2$, then Theorem 1.2 shows that the necessary conditions in Theorem 1.3 are also sufficient. Therefore, our main goal in this paper is to consider the case when $v_1 \neq v_2$ or $\lambda_1 \neq \lambda_2$, and prove the necessary condition obtained in Lemma 2.1 is also sufficient. That is, we have to construct a BBQD$(v_1, v_2; \lambda_1, \lambda_2, \lambda_3)$ (or equivalently prove that $C_4 | \ (\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$) for each admissible $(\lambda_1, \lambda_2, \lambda_3; v_1, v_2)$.

For clarity, we use Table 1 to depict the relationship between the pairs (v_1, v_2) and the triples $(\lambda_1, \lambda_2, \lambda_3)$ by using conditions (a) and (b), i.e., at this stage, we consider only conditions (a) and (b).

Lemma 2.2 [3] *A complete graph of order n and multiplicity λ, λK_n, can be decomposed into 4-cycles if and only if (a) 2 divides $\lambda(n-1)$, and (b) 8 divides $\lambda n(n-1)$.*

Whenever the pair (n, λ) does not satisfy the above two conditions, we can pack λK_n with 4-cycles. After taking away the 4-cycles, we have a leave L left. If L contains no 4-cycles, then the packing is maximal. A maximal packing with minimum leave (size) is called a maximum packing. The following result is well-known and plays an important role in our constructions.

$\lambda_3 \setminus \lambda_2$	$\lambda_1 = 0$: 1	2	3	4	$\lambda_1 = 1$: 1	2	3	4
1	(odd;0)	(even;0,4)	(odd;0)	(even;even)	(0:1)(1:0) (2:7)(7:2) (4:5)(5:4) (3:6)(6:3)	(0;1,5) (4;3,7)	(0:1)(1:0) (2:3)(3:2) (4:5)(5:4) (6:7)(7:6)	(0;odd)
2	(odd;5) (even;1)	(even;0,1,4,5) (odd;0,3,4,7)	(even;1) (odd;5)	(even;any) (odd;even)	(1;5) (5;1) (3;3) (7;7)	(1;0,3,4,7) (5;1,2,5,6)	(1;5) (3;7) (5;1) (7;3)	(1;even) (5;odd)
3	(odd;0)	(even;0,4)	(odd;0)	(even;even)	(0:1)(1:0) (2:7)(7:2) (4:5)(5:4) (3:6)(6:3)	(0;1,5) (4;3,7)	(0:1)(1:0) (2:3)(3:2) (4:5)(5:4) (6:7)(7:6)	(0;odd)
4	(any;1)	(any;0,1,4,5)	(any;1)	(any;any)	(1;1) (5;5) (3;7) (7;3)	(1;0,1,4,5) (5;2,3,6,7)	(1;1) (3;3) (5;5) (7;7)	(1;any)

$\lambda_3 \setminus \lambda_2$	$\lambda_1 = 2$: 1	2	3	4	$\lambda_1 = 3$: 1	2	3	4
1	(0;1,5) (4;3,7)	(0,4;0,4) (2,6;2,6)	(1,5;0) (3,7;4)	(0,4;even)	(0:1)(1:0) (2:3)(3:2) (4:5)(5:4) (6:7)(7:6)	(0;1,5) (3,7;4)	(0:1)(1:0) (2:7)(7:2) (3:6)(6:3) (4:5)(5:4)	(0;odd)
2	(0;1,5) (5;1,2,5,6)	(0,4;0,1,4,5) (2,6;2,3,6,7) (1,5;0,3,4,7) (3,7;1,2,5,6)	(0,3,4,7;1) (1,2,5,6;5)	(0,4;any) (1,5;even) (3,7;odd)	(1;5) (3;7) (5;1) (7;3)	(0,3,4,7;1) (1,2,5,6;5)	(1;5) (5;1) (3;3) (7;7)	(1;even) (5;odd)
3	(0;1,5) (4;3,7)	(0,4;0,4) (2,6;2,6)	(1,5;0) (3,7;4)	(0,4;even)	(0:1)(1:0) (2:3)(3:2) (4:5)(5:4) (6:7)(7:6)	(1,5;0) (3,7;4)	(1:0)(0:1) (2:7)(7:2) (3:6)(6:3) (4:5)(5:4)	(0;odd)
4	(1;0,1,4,5) (5;2,3,6,7)	(0,1,4,5; 0,1,4,5) (2,3,6,7; 2,3,6,7)	(0,1,4,5;1) (2,3,6,7;5)	(0,1,4,5;any)	(1;1) (3;3) (5;5) (7;7)	(0,1,4,5;1) (2,3,6,7;5)	(1;1) (5;5) (3;7) (7;3)	(1;any)

Table 1: Admissible pairs for $(v_1; v_2)$.

Lemma 2.3 [3] *The minimum leave L of a maximum packing of λK_n $(n \geq 4)$ can be described as in the following table.*

$\lambda \setminus n$	0	1	2	3	4	5	6	7
1	F	ϕ	F	C_3	F	E_6	F	C_5
2	ϕ	ϕ	D	D	ϕ	ϕ	D	D
3	F	ϕ	F_2	C_5	F	D	F_2	C_3
4	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ

(mod 4)

Table 2. F is a 1-factor, E_6 is an even graph with 6 edge, $D = \{uv, uv\}$ is a set of double edges, and F_2 is a spanning odd graph with $\frac{n}{2} + 2$ edges.

We also need a lemma about the packing of $\lambda K_{m,n}$. But, first, the following result is worth mentioning.

Lemma 2.4 [17] $C_{2t} \mid K_{m,n}$ *if and only if m and n are even, m, $n \geq t$ and $2t$ divides mn.*

The following result is known.

Lemma 2.5 [3, 17] *Let λ be an even integer. Then $C_4 |\ \lambda K_{m,n}$ if and only if $m, n \geq 2$ and 4 divides λmn.*

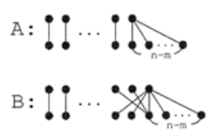
Now we are ready to describe the maximum packing of $\lambda K_{m,n}$ with 4-cycles. By Lemma 2.5, it suffices to consider $\lambda = 1, 2, 3$ for $m, n \geq 2$.

Lemma 2.6 [2] *The minimum leaves of the maximum packings of $\lambda K_{m,n}$ $(m \leq n)$ are as follows:*

	$\lambda = 1$				$\lambda = 2$				$\lambda = 3$			
$n \setminus m$	1	2	3	4	1	2	3	4	1	2	3	4
1	A	S_m	B	S_m	D	ϕ	D	ϕ	A	S_m	B	S_m
2	S_n	ϕ	S_n	ϕ	ϕ	ϕ	ϕ	ϕ	S_n	ϕ	S_n	ϕ
3	A	S_m	B	S_m	D	ϕ	D	ϕ	A	S_m	B	S_m
4	S_n	ϕ	S_n	ϕ	ϕ	ϕ	ϕ	ϕ	S_n	ϕ	S_n	ϕ

(mod 4)

Table 3 : S_k is a star with k edges, D is a double edge and A, B are as in the following figures.



The following lemma shows that we can combine two decompositions together.

Lemma 2.7 *Let $\lambda_i' \leq \lambda_i$ for $i = 1, 2, 3$. If $C_4 \mid (\lambda_1', \lambda_2', \lambda_3')K_{v_1,v_2}$, and $C_4 \mid (\lambda_1 - \lambda_1', \lambda_2 - \lambda_2', \lambda_3 - \lambda_3')K_{v_1,v_2}$, then $C_4 \mid (\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$.*

For the proof of the main theorem, we need several special decompositions given in the next three lemmas.

Lemma 2.8 [9] *Let H be a 2-regular subgraph of K_{2m+1} such that $\binom{2m+1}{2} - |E(H)|$ is a multiple of 4. Then $C_4 \mid K_{2m+1} \setminus H$.*

A graph is odd (even) if each vertex is of odd (even) degree.

Lemma 2.9 [5] *Let H be a spanning odd subgraph of K_{2m}, $m \geq 2$, such that $\Delta(H) \leq 3$. Then $K_{2m} - H$ can be decomposed into 4-cycles if and only if $4 \mid \binom{2m}{2} - |E(H)|$ except $m = 4$ and H is one of the following two graphs in Figure A.*

Figure A.

Lemma 2.10 [6] *Let H be a 2-regular subgraph of $2K_{2m}$ such that $2\binom{2m}{2} - |E(H)|$ is a multiple of 4. Then $C_4 \mid 2K_{2m} \setminus H$.*

In fact, we have a stronger results about Lemma 2.8, 2.9 and 2.10. In what follows, we let $L_0 = \emptyset$, $L_1 = C_5$, $L_2 = B(bowtie)$ and $L_3 = C_3$. Note here that a bowtie is the graph which is the union of two triangles with one common vertex.

Theorem 2.11 [6] *Let H be a 2-regular subgraph of K_{2m+1}, $m \geq 2$. Then $K_{2m+1} - H$ has a maximum packing with leave L_i if and only if $\binom{2m+1}{2} - |E(H)| \equiv i \pmod 4$, $i = 0, 1, 2, 3$.*

Theorem 2.12 [6] *Let F be an odd spanning forest of K_{2m}, $m \geq 2$. Then $K_{2m} - F$ has a maximum packing with leave L_i if and only if $\binom{2m+1}{2} - |E(F)| \equiv i \pmod 4$, $i = 0, 1, 2, 3$.*

Theorem 2.13 [6] *Let H be a 2-regular subgraph of K_v. Then $2K_v - H$ has a maximum packing with leave L_i if and only if $2\binom{v}{2} - |E(H)| \equiv i \pmod 4$, $i = 0, 1, 2, 3$, and L_2 can be a bowtie or a double edge here.*

Lemma 2.14 *Let $V_1 = \{a_1, a_2, ..., a_{v_1}\}$ and $V_2 = \{b_1, b_2, ..., b_{v_2}\}$ be two disjoint sets. Let H_1, H_2 and H_3 be three graphs defined on V_1, $V_1 \cup V_2$ and V_2 respectively where H_2 is a bipartite graph with bipartite sets V_1 and V_2. Let (H_1, H_2, H_3) denote the graph formed by the union of H_1, H_2 and H_3. Then (C_3, C_6, C_3), $(C_3, 2C_4, C_5)$, $(C_5, C_6 \cup C_4, C_5)$, (C_6, C_6, ϕ), (D, C_4, D), $(C_6, 3C_4, C_6)$, $(5K_3, K_{3,6}, M_3)$, $(3K_3, K_{3,4}, K_{1,3})$, $(2K_3, K_{3,4}, M_2)$, $(3K_3, K_{3,6}, K_{1,5})$, $(2K_3, K_{3,8}, K_{1,3} \cup K_{1,3})$, and $(K_3, K_{3,6}, M_3)$, where M_i is a matching with i edges, can be decomposed into 4-cycles respectively, provided that the vertices of H_i, $i = 1, 2, 3$, are properly selected.*

Proof. Since the proofs are similar, we select five cases to prove. For the cases we don't prove here, we use Figure 1 to depict the graphs.

(i) $C_4 \mid (C_3, 2C_4, C_5)$.

Let $C_3 = (a_1, a_2, a_3)$, $C_5 = (b_1, b_2, b_3, b_4, b_5)$ and two 4-cycles be (a_2, b_1, a_3, b_3) and (a_1, b_2, a_3, b_5) respectively. Then the decomposition is obtained by using the 4-cycles: (a_3, b_3, b_4, b_5), (a_2, b_1, b_2, b_3), (a_1, b_5, b_1, a_3) and (b_2, a_1, a_2, a_3).

(ii) $C_4 \mid (C_6, 3C_4, C_6)$.

Let two 6-cycles be (a_1, a_2, a_6) and $(b_1, b_2, \ldots, b_6)$ respectively and three 4-cycles be (a_2, b_2, a_3, b_3), (a_4, b_4, a_5, b_5), and (a_6, b_6, a_1, b_1) respectively. Then the 4-cycles in the decomposition are: (a_1, a_2, b_2, b_1), (a_2, b_3, b_2, a_3), (a_3, b_3, b_4, a_4), (a_4, b_5, b_4, a_5), (a_5, b_5, b_6, a_6) and (a_6, b_1, b_6, a_1).

(iii) $C_4 \mid (5K_3, K_{3,6}, M_3)$.

Let $V(5K_3) = \{a_1, a_2, a_3\}$ and $V(M_3) = \{b_1, b_2, b_3, b_4, b_5, b_6\}$ where b_1b_2, b_3b_4 and b_5b_6 are edges of M_3. Let $V(K_{3,6}) = V(5K_3) \cup V(M_3)$. Then the 4-cycles in the decomposition are: (a_1, a_2, b_2, b_1), (a_2, a_3, b_4, b_3), (a_1, a_3, b_6, b_5), (b_1, a_2, a_1, a_3), (b_2, a_1, a_2, a_3), (b_3, a_1, a_2, a_3), (b_4, a_2, a_3, a_1), (b_5, a_2, a_1, a_3) and (b_6, a_2, a_3, a_1).

(iv) $C_4 \mid (3K_3, K_{3,4}, K_{1,3})$.

Let $V(3K_3) = \{a_1, a_2, a_3\}$, $V(K_{1,3}) = \{b_1, b_2, b_3, b_4\}$ where b_1b_2, b_1b_3 and b_1b_4 are edges, and $V(K_{3,4}) = \{a_1, a_2, a_3\} \cup \{b_1, b_2, b_3, b_4\}$ where a_ib_j is an edge for each $1 \leq i \leq 3$ and $1 \leq j \leq 4$. Then the 4-cycles in the decomposition are: (a_1, a_2, b_2, b_1), (a_2, a_3, b_3, b_1), (a_1, a_3, b_1, b_4), (b_2, a_1, a_2, a_3), (b_3, a_2, a_3, a_1) and (b_4, a_2, a_1, a_3).

(v) $C_4 \mid (3K_3, K_{3,6}, K_{1,5})$

Let $V(K_{1,5}) = \{b_1, b_2, b_3, b_4, b_5, b_6\}$ such that b_1b_i is an edge for $2 \leq i \leq 6$. Then the 4-cycle decomposition is $\{(a_1, a_2, b_2, b_1)$, (a_1, b_3, b_1, b_4), (a_3, b_5, b_1, b_6), (b_3, a_2, a_3, a_1), (b_2, a_1, a_2, a_3), (b_5, a_2, a_3, a_1), (b_3, a_2, b_4, a_3), $(b_6, a_2, a_3, a_1)\}$.

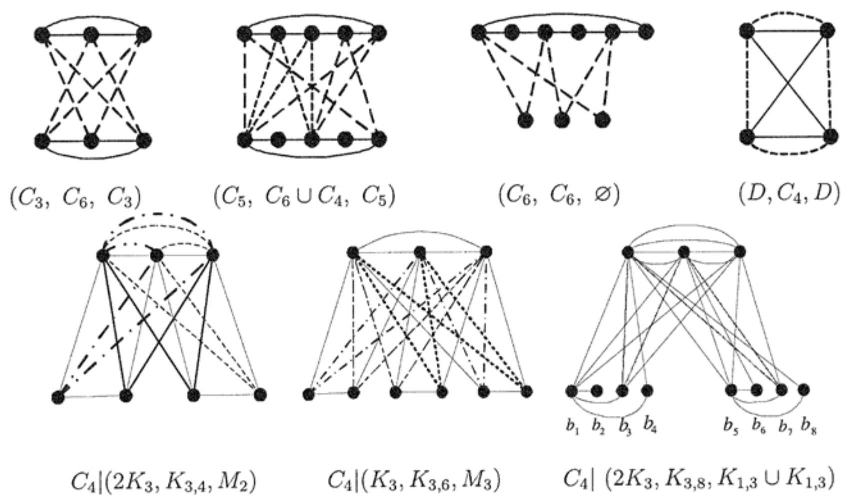


Figure 1

■

In what follows, we shall use $G \vee H$ to denoted the join of G and H, i.e. $V(G \vee H) = V(G) \cup V(H)$ (disjoint union) and $E(G \vee H) = E(G) \cup E(H) \cup E(K_{|V(G)|,|V(H)|})$. The following result is also essential to the proof of our main result.

Lemma 2.15 *Let H be a 2-regular graph which has t edges and let F be a matching with s edges. Then $C_4 \mid tK_2 \vee H$ and $C_4 \mid sK_2 \vee 2F$.*

Lemma 2.16 *Let O_2 be an empty graph with two vertices. Then $C_4 \mid O_2 \vee C_{4t}$ for each $t \geq 1$.*

Next, we consider the decomposition of a complete bipartite graph.

Lemma 2.17 *Let m and n be odd integers such that m, $n \geq 3$. Let $\lambda \equiv 2(mod\ 4)$. Then $\lambda K_{m,n}$ can be packed with the leave a 6-cycle or a double edge.*

Proof. By Lemma 2.5, the maximum packing of $2K_{s,t}$ has empty leave provided that either one of s and t is even. Therefore, if $m,\ n \geq 3$, then $\lambda K_{m,n}$ can be packed with 4-cycles such that its leave is the same as the leave of packing $2K_{3,3}$, which is a 6-cycle or a double edge. ■

Now let $V(K_n) = Z_n$. Then the difference of two vertices x and y with $x \leq y$ is defined to be $\min\{|x-y|, n-|x-y|\}$. Let $G(A)$ denote the graph induced by the edges xy, where $x, y \in Z_n$, such that the difference of x and y is in A. If A is a set of several positive integers, say, $A = \{1, 2, 3, 4\}$, then we shall use $G(1, 2, 3, 4)$ to denote the graph $G(A)$. In the case that A is a multi-set, then $G(A)$ will be a multigraph.

If n is odd then for each difference $j \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, $G(j)$ induces a 2-factor. For distinct differences $i, i+1, j, j+1$, $G(i, i+1, j, j+1)$ induces an 8-factor which has a 4-cycle decomposition:$\{(k, k+i, k+i+j+1, k+i+1)|k \in \mathbb{Z}_n\}$. Now we consider the graph $G(1, 2, 3, 4)$ defined on Z_n where n is odd and $n \geq 9$. The following lemmas will be useful later.

Lemma 2.18 *Let x be a positive integer such that $v < x \leq 2v$. Then, $G = G(1, 2)$ can be written as the union of a spanning even subgraph H of G with $|E(H)| = x$, some 4-cycles and a cycle of length 3, 4, 5 or 6 if $2v - x$ is 3, 0, 1, or 2 modulo 4, respectively.*

Proof. The proof follows by constructing edge-disjoint cycles which are of length 4 except possibly the last one and the total length is $2v - x$. Let $2v - x = 4t + i$.

(i) $i = 3$ The collection of cycles is $B \cup (4t+1, 4t, 4t+2)$ where $B = \{(0, 1, 3, 2) + 4j | j = 0, 1, 2, ..., t-1\}$.

(ii) $i = 0$ The collection of cycles is B.

(iii) $i = 1$ The collection of cycles is $(B \setminus (4t-4, 4t-3, 4t-1, 4t-2)) \cup (4t-4, 4t-3, 4t-1, 4t, 4t-2)$.

(iv) $i = 2$ The collection of cycles is $(B \setminus (4t-4, 4t-3, 4t-1, 4t-2)) \cup (4t-4, 4t-3, 4t-1, 4t+1, 4t, 4t-2)$. ■

Lemma 2.19 $G(1, 2, 3)$ *has a maximum packing with leave a cycle of length 4, 5, 6 or 3 depending on $3v \equiv 0, 1, 2$ or 3 (mod 4).*

Proof. We split the proof into four cases.

(1) $3v \equiv 0 \pmod 4$ ($v \equiv 0 \pmod 4$)
The decomposition of $G(1, 2, 3)$ is $B_0 = \{(0, 1, 2, 3)+4i, (2, 4, 3, 5)+4i, (1, 4, 6, 3) + 4i \pmod v \mid i = 0, 1, 2, ..., \frac{v-4}{4}\}$.

(2) $3v \equiv 1 \pmod 4$ ($v \equiv 3 \pmod 4$)
The maximum packing of $G(1, 2, 3)$ is $B_1 = \{(0, 1, 2, 3) + 4i, (2, 4, 3, 5) + 4i, (1, 4, 6, 3) + 4i \ (mod\ v) \mid i = 0, 1, 2, ..., \frac{v-7}{4}\} \cup \{(0, v-1, 1, v-2)\}$ and its leave is $(0, 2, v-1, v-2, v-3)$.

(3) $3v \equiv 2 \pmod 4$ ($v \equiv 2 \pmod 4$)
The maximum packing of $G(1,2,3)$ is $B_2 = \{(0,1,2,3)+4i, (2,4,3,5)+4i, (1,4,6,3)+4i \ (mod\ v) \mid i = 0,1,2,..., \frac{v-6}{4}\} \cup \{(0, v-1, 1, v-2)\} \setminus \{(v-5, v-2, 0, v-3)\}$ and its leave is $(2, 0, v-3, v-5, v-2, v-1)$.

(4) $3v \equiv 3 \pmod 4$ ($v \equiv 1 \pmod 4$)
The maximum packing of $G(1,2,3)$ is $B_3 = \{(0,1,2,3)+4i, (2,4,3,5)+4i, (1,4,6,3)+4i \ (mod\ v) \mid i = 0,1,2,..., \frac{v-3}{4}\}$ and its leave is $(2, v-1, v-2)$. ∎

Lemma 2.20 *Let x be a positive integer such that $v \le x \le 2v$. Then $G = G(1,2,3)$ can be written as the union of a spanning even subgraph H of G with $2 \le \delta(H) \le \Delta(H) \le 4$ and $|E(H)| = x$, some 4-cycles and leave L_i where $3v - x \equiv i (mod\ 4)$ and $L_0 = C_4$, $L_1 = C_5$, $L_2 = B$, and $L_3 = C_3$.*

Proof. Assume that $3v - x = 4s + i$ for some i with $0 \le i \le 3$. Since $2 \le \delta(H) \le \Delta(H) \le 4$ and $x > v$, the graph $H' = G(1,2,3) - H$ also has maximum degree 4 and minimum degree 2. So, we construct H' directly depending on $3v - x \equiv i \pmod 4$, $i =$ 3, 0, 1, or 2. Let H_0 be the largest possible such graph. Then H' can be obtained by taking some edge-disjoint 4-cycles away from H_0.

(1) $v \equiv 0 \pmod 4$ (H_0 has about $2v$ edges.)

(i) $i = 0$ $\quad H_0 = B_0 = \{(0,1,2,3)+4i, (2,4,3,5)+4i \pmod v \mid i = 0,1,2,..., \frac{v-4}{4}\}$.

(ii) $i = 1$ $\quad H_0 = (B_0 \setminus \{(v-4, v-3, v-2, v-1), (v-2, 0, v-1, 1)\}) \cup \{(0, v-2, v-3, v-4, v-1)\}$.

(iii) $i = 2$ $\quad H_0 = (B_0 \setminus \{(v-4, v-3, v-2, v-1), (v-2, 0, v-1, 1)\}) \cup \{(v-4, v-3, v-2; v-2, v-1, 0)\}$.

(iv) $i = 3$ $\quad H_0 = (B_0 \setminus \{(v-4, v-3, v-2, v-1)\}) \cup \{(v-2, v-3, v-4)\}$.

(2) $v \equiv 2 \pmod 4$

(i) $i = 0$ $\quad H_0 = B_2 \cup \{(0, v-1, 1, v-2)\}$ where $B_2 = \{(0,1,2,3)+4i, (2,4,3,5)+4i \pmod v \mid i = 0,1,2,..., \frac{v-6}{4}\}$.

(ii) $i = 1$ $\quad H_0 = (B_2 \setminus \{(v-2, v-3, v-1, v-4)\}) \cup \{(1, v-1, v-4, v-2, 0)\}$.

(iii) $i = 2$ $\quad H_0 = (B_2 \setminus \{(v-2, v-3, v-1, v-4)\}) \cup \{(v-4, v-3, v-2; v-2, v-1, 0)\}$.

(iv) $i = 3$ $\quad H_0 = B_2 \cup \{(v-2, v-1, 0)\}$.

(3) $v \equiv 1 \pmod 4$

(i) $i = 0$ $\quad H_0 = B_1 = \{(0,1,2,3)+4i, (2,4,3,5)+4i \pmod v \mid i = 0,1,2,..., \frac{v-5}{4}\}$.

(ii) $i = 1$ $\quad H_0 = (B_1 \setminus \{(v-2, v-1, v-3, 0)\}) \cup \{(v-2, 1, v-3, 0, v-1)\}$.

(iii) $i = 2$ $\quad H_0 = (B_1 \setminus \{(v-2, v-1, v-3, 0)\}) \cup \{(v-3, 0, v-1; v-1, 1, v-2)\}$.

(iv) $i = 3$ $\quad H_0 = (B_1 \setminus \{(v-2, v-1, v-3, 0)\}) \cup \{(v-2, v-1, 0)\}$.

(4) $v \equiv 3 \pmod 4$

(i) $i = 0$ $\quad H_0 = B_3 \cup \{(v-1, 0, v-2, 1)\}$ where $B_3 = \{(0,1,2,3) + 4i, (2,4,3,5) + 4i \pmod v) \mid i = 0, 1, 2, ..., \frac{v-7}{4}\}$.

(ii) $i = 1$ $\quad H_0 = B_3 \cup \{(0, v-1, 1, v-2, v-3)\}$.

(iii) $i = 2$ $\quad H_0 = B_3 \cup \{(1, v-2, 1, v-1; v-1, 0, v-3)\}$.

(iv) $i = 3$ $\quad H_0 = B_3 \cup \{(0, v-2, v-1)\}$. ■

Combining the above two lemmas, we have the following result.

Lemma 2.21 *Let $3 \le x \le 2v$ be a non-negative integer. Then $G = G(1,2,3)$ can be written as the union of a spanning even graph H of G with $2 \le \delta(H) \le \Delta(H) \le 4$ and $|E(H)| = x$, some 4-cycles, and leave L_i where $3x - v \equiv i \pmod 4$ and $L_0 = C_4$, $L_1 = C_5$, $L_2 = C_6$ or B, and $L_3 = C_3$.*

Proof. By Lemma 2.20, it remains to consider the case when $x \le v$. The proof is divided into four cases, based on the congruence classes of v modulo 4. Since the proof of each of the four cases is similar, we present only one of them, $v \equiv 0 \pmod 4$.

(i) $i = 1$ ($x \equiv 3 \pmod 4$)

So, we need a 2-regular subgraph of $G(1,2,3)$ which has $4t + 3$ edges, and this can be done by taking t edge-disjoint 4-cycles from $\{(0,1,2,3) + 4i | i = 0, 1, 2 \dots, \frac{v-8}{4}\}$ and transferring $(v-1, v-2, v-3, v-4) \cup (0, 2, v-1, v-3)$ into $(v-1, v-2, v-3) \cup (0, 2, v-1, v-4, v-3)$. Now, the t 4-cycles and the 3-cycle $(v-1, v-2, v-3)$ make up the graph H and the packing of $G - H$ has a leave $(0, 2, v-1, v-4, v-3)$.

(ii) $i = 2$ ($x \equiv 2 \pmod 4$ and $x = 4t + 2$.)

This can be done by finding $t-1$ 4-cycles and a 6-cycle. Now, since $(0,1,2,3) \cup (v-1, v-2, v-3, v-4) \cup (0, v-1, 1, v-2)$ can be replaced with two 6-cycles $(0, 3, 2, 1, v-1, v-2)$ and $(0, 1, v-2, v-3, v-4, v-1)$, H is obtained by taking the union of $(0, 3, 2, 1, v-1, v-2)$ and $\{(0,1,2,3) + 4i | i = 1, 2, \dots, t-1\}$.

(iii) $i = 3$ ($x \equiv 1 \pmod 4$)

Taking the 5-cycle in (i) instead of 3-cycle gives the graph H.

(iv) $i = 0$ ($x \equiv 0 \pmod 4$)

This is obtained by using $\{(0,1,2,3) + 4i \mid i = 0, 1, 2, \dots, \frac{v-4}{4}\}$ and choose the desired number of 4-cycles. ■

Lemma 2.22 *For v odd, the graph $G(\frac{v-3}{2}, \frac{v-1}{2})$ has a maximum packing with 4-cycles and leave $C_6 = (\frac{v-5}{2}, v-1, \frac{v-1}{2}, v-2, \frac{v-3}{2}, v-3)$.*

Proof. The packing of $G(\frac{v-3}{2}, \frac{v-1}{2})$ is $\{(0, \frac{v-1}{2}, 1, \frac{v+3}{2}) + i \pmod v) \mid i = 0, 1, 2, ..., \frac{v-5}{2}\}$. ■

Lemma 2.23 *For v odd, the graph $G(1,2,3,\frac{v-3}{2},\frac{v-1}{2})$ has a maximum packing with 4-cycles and leave either C_5 or C_3 if $5v \equiv 1$ or $3 \pmod 4$, respectively.*

Proof. The result follows by combining the leaves of the maximum packings of $G(1,2,3)$ obtained in Lemma 2.21 and $G(\frac{v-3}{2},\frac{v-1}{2})$ obtained above. The details are omitted here. ■

Lemma 2.24 *Let v be an even integer, then the differences i and two differences $\frac{v}{2}$ induce a graph $G(i,\frac{v}{2},\frac{v}{2})$ in $2K_v, i \in \{1,2,\ldots,\frac{v-2}{2}\}$, which has a 4-cycle decomposition. Moreover, if $v \equiv 0 \pmod 4$, then we can make the decomposition resolvable by taking $i=1$ i.e., two resolution classes of 4-cycles.*

Proof. The packing is $\{(0,i,\frac{i}{2}+i,\frac{v}{2})+j\ (mod\ v) \mid j=0,1,2,\ldots,\frac{v-2}{2}\}$. ■

Lemma 2.25 *In $2K_v(v$ even), $2G(i,j)$ has a 4-cycle decomposition for all $1 \le i \ne j \le \frac{v-2}{2}$.*

Proof. The decomposition is $\{(0,i,i+j,j)+k(\mathrm{mod}\ v) \mid k=0,1,2,\ldots,\frac{v-2}{2}\}$. ■

Lemma 2.26 *Let x be an integer such that $2 \le x \le 2v$ where v is an even integer. Then $2G(1,2)$ can be written as the union of an even subgraph H of $2K_v$ with $2 \le \delta(H) \le \Delta(H) \le 4$ and $|E(H)|=x$, some 4-cycles and leave $\emptyset$, C_5, D or C_3 if $4v-x$ is congruent to 0, 1, 2 or 3 modulo 4.*

Proof. We split the proof into two cases.

(1) $v \equiv 0 \pmod 4$

Let $x=4s+t$ for $0 \le t \le 3$. Clearly, if $t=0$, then we can choose s 4-cycles in $2G(1,2)$ starting with the first resolution class of 4-cycles $\{(0,1,3,2)+4i|i=0,1,2,...,\frac{v-4}{4}\}$ and then $\{(2,3,5,4)+4i|i=0,1,2,...,\frac{v-4}{4}\}$. Since $s \le \frac{v}{2}$, the proof follows. Now, consider $t \ne 0$. By combining $(0,1,3,2)$ with $(1,2,4,3)$ (instead of $(2,3,5,4)$) we have $(1,2,3) \cup (0,1,3,4,2)$, this handles the case when $t=1$ or 3. For $t=2$, we combine $(0,1,3,2)$ and $(2,3,5,4)$ to obtain double edge $(2,3)$ and $(0,1,3,5,4,2)$ which handles the case when $x \ge 6$.

(2) $v \equiv 2 \pmod 4$

Similar to the proof of (1), we consider $x=4s+t$ for $0 \le t \le 3$. If $t=0$, then we choose 4-cycles from $\{(0,1,3,2)+4i|i=0,1,2,...,\frac{v-6}{4}\}$ first and then from $\{(v-1,v-2,0,1)+4i \pmod v)|i=0,1,2,...,\frac{v-2}{4}\}$. So, if $t \ne 0$, use the same technique to combine two 4-cycles as in (1). This concludes the proof. ■

Lemma 2.27 *Let H be a spanning even subgraph of $\lambda_2 K_{v_2}$ such that $|E(H)|=\lambda_1=(s-1)v_2+t, t \le v_2, \Delta(H)-\delta(H) \le 2$ and the number of major vertices (of degree $\Delta(H)$) is $t \le v_2$. Then the edge-disjoint union of λ_1 K_2's where $V(K_2) \cap V(K_{v_2})=\emptyset$, H and $(s-1)K_{2,v_2} \cup K_{2,t}$ (the first, second and third associate edges) has a 4-cycle decomposition.*

Proof. Since H is an even graph, H has an eulerian circuit. Let the circuit be denoted as $(u_1,u_2,u_3,\ldots,u_{\lambda_1})$ where $u_i \in V(\lambda_2 K_{v_2})$. Also, let $\{a,b\}$ be the vertex set of the first associate edges. Then $\{(a,u_i,u_{i+1},b) \pmod{\lambda_1} \mid i=1,2,\ldots,\lambda\}$ is the desired decomposition. Note that in the decomposition a(or b) is incident to a vertex $x \in V(\lambda_2 K_{v_2})$ either s or $s-1$ times depending on $deg_H(x) = 2s$(t of them) or $2(s-1)$. Therefore, the third associate edges induce $(s-1)K_{2,v_2} \cup K_{2,t}$. ■

3 The proof of Theorem 1.3

Before we go through the details of the proof, we first describe the idea of how to prove the theorem. Since $(\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$ can be written as the disjoint union of $\lambda_1 K_{v_1}, \lambda_2 K_{v_2}$ and $\lambda_3 K_{v_1,v_2}$, the decomposition of $(\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$ can be obtained by packing each of $\lambda_1 K_{v_1}, \lambda_2 K_{v_2}$, and $\lambda_3 K_{v_1,v_2}$ with 4-cycles and decomposing the remaining edges into 4-cycles. For instance, if $C_4 \mid \lambda_1 K_{v_1}$, $C_4 \mid \lambda_2 K_{v_2}$ and $C_4 \mid \lambda_3 K_{v_1,v_2}$, then clearly $C_4 \mid (\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$. These cases do happen when $v_1,\ v_2 \geq 4$. Therefore, we skip the proof of these cases and focus on the cases such that $C_4 \nmid \lambda_1 K_{v_1}$, $C_4 \nmid \lambda_2 K_{v_2}$ or $C_4 \nmid \lambda_3 K_{v_1,v_2}$. Note that, since $\lambda_3 v_1 v_2 \equiv 0 \pmod 2$ (follows from (a)), we have $\lambda_1\binom{v_1}{2} + \lambda_2\binom{v_2}{2} \equiv \lambda_3 v_1 v_2 \pmod 4$. Therefore, for $v_1,\ v_2 \geq 4$, we consider the cases when either $\lambda_1\binom{v_1}{2}$ or $\lambda_2\binom{v_2}{2}$ is not a multiple of 4.

On the other hand, if v_1 is less than 4, then we have to combine the edges of $\lambda_1 K_{v_1}$ with the edges of $\lambda_3 K_{v_1,v_2}$ or $\lambda_2 K_{v_2}$ to obtain the desired decomposition.

For convenience, we shall call a 4-cycle C of type (α, β, γ) if the 4-cycle is obtained from using α edges of $\lambda_1 K_{v_1}$, β edges of $\lambda_2 K_{v_2}$ and γ edges of $\lambda_3 K_{v_1,v_2}$. Clearly, both γ and $\alpha + \beta$ are even and $\alpha + \beta + \gamma = 4$

Now, if $v_1 < 4$, then $v_1 = 2$ or 3. In the case that $v_1 = 2$, then for all 4-cycles of type (α, β, γ) with $\alpha > 0$, we have $\alpha = 1$, $\beta = 1$ and $\gamma = 2$. To prove such cases, we shall use up the edges in $\lambda_1 K_{v_1}$ first (one edge of $\lambda_1 K_{v_1}$ together with one edge of $\lambda_2 K_{v_2}$ and two edges of $\lambda_3 K_{v_1,v_2}$). Note that if $v_2 = 2$, then $\lambda_1 = \lambda_2$; and if $v_2 = 3$, then $3\lambda_2 \geq \lambda_1$.

Finally, when $v_1 = 3$ and $v_2 \geq 4$ our strategy is to use up the edges of $\lambda_1 K_{v_1}$ with as many type $(2, 2, 0)$ 4-cycles as possible. Note that there are type $(1, 2, 1)$ 4-cycles if λ_1 is odd or λ_3 is odd. We remark finally that if $v_2 = 3$, then $3(\lambda_1 + \lambda_2) \leq 9\lambda_3$ by condition (c).

Proof of Theorem 1.3.

The proof is split into four cases: $v_1, v_2 \geq 4$; $v_1 = 2, v_2 \geq 4$; $v_1 = 3, v_2 \geq 4$; and $v_1, v_2 \leq 3$.

Case 1 : $v_1,\ v_2 \geq 4$.
Case 1.1: $v_1 + v_2$ is even.

Clearly, v_1 and v_2 are of the same parity. First, if both v_1 and v_2 are odd, by Lemma 2.3, $\lambda_1 K_{v_1}$ and $\lambda_2 K_{v_2}$ can be packed with leave either $\emptyset$ or a cycle of length 3, 5 or 6 as the case may be. (Note here, we can always select 4-cycles from the packings to put them with the leave from the packings to obtain "new" leaves.) On the other hand, if both v_1 and v_2 are even, then both λ_1 and λ_2 must also even, otherwise the degrees of the vertices may be odd which violates (a).

Moreover, since $\lambda_3 v_1 v_2$ is a multiple of 4, $\lambda_3 K_{v_1,v_2}$ can be packed with leave either $\emptyset$ or 4-cycles (by Lemma 2.6) with some adjustment of the leave. This implies that the packing of $(\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$ can be obtained by combining the three leaves (cycles) together to obtain the leave of the packing of $(\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$. Of course, if all the three leaves are $\emptyset$, then we are done. So, the possible cases are: (D, C_4, D) and $(C_6, C_4{}'s, C_6)$ where the double edges or cycles are from $\lambda_1 K_{v_1}$, $\lambda_3 K_{v_1,v_2}$ and $\lambda_2 K_{v_2}$ respectively. In each case, by Lemma 2.14, the graph has a decomposition into 4-cycles. This completes the proof in this case.

Case 1.2: $v_1 + v_2$ is odd.

Without loss of generality, let v_1 be odd and v_2 be even. (The other case is similar.) Now, if λ_2 is even, then the proof is similar to Case 1.1. Thus, we assume that λ_2 is odd and therefore λ_3 is also odd (by condition (a)). We consider the following two cases.

(i) $\lambda_1 > 0$. Let $\lambda_1' = \lambda_1 - 1$, $\lambda_2' = \lambda_2 - 1$ and $\lambda_3' = \lambda_3 - 1$. By Lemma 2.3, $K_{v_1+v_2}$ can be packed with leave an empty graph, a bowtie (or a 6-cycle), C_3 or C_5. Also, $\lambda_1' K_{v_1}$, $\lambda_2' K_{v_2}$ and $\lambda_3' K_{v_1,v_2}$ can be packed with leaves which are cycles of proper sizes. Thus, the decomposition is obtained by Lemma 2.14. (Now, we have four subgraphs, but we can combine the ones from the third associates first.)

(ii) $\lambda_1 = 0$. It is not difficult to see $\lambda_3 K_{v_1,v_2}$ can be written as the disjoint union of K_{1,v_2}, K_{v_1-1,v_2} and $\lambda_3' K_{v_1,v_2}$. The latter two graphs can be decomposed into 4-cycles by Lemma 2.6. Therefore, it remains to check $C_4 |\ K_{1,v_2} \cup \lambda_2 K_{v_2}$. But, $K_{1,v_2} \cup \lambda_2 K_{v_2}$ is the same as $K_{v_2+1} \cup (\lambda_2 - 1)K_{v_2}$. Since both graphs can be packed with leaves which are cycles and the number of edges in total is a multiple of 4, the proof follows by combining the two cycles together properly. This concludes the proof of Case 1.

Case 2: $v_1 = 2$ and $v_2 \geq 4$.

Since $v_1 = 2$, we can make a smaller table to depict the admissible values of v_2 modulo 8 for all possible λ_1, λ_2, and λ_3.

	$\lambda_1 \equiv 0 \pmod 4$				$\lambda_1 \equiv 1 \pmod 4$				$\lambda_1 \equiv 2 \pmod 4$				$\lambda_1 \equiv 3 \pmod 4$			
$\lambda_3 \setminus \lambda_2$	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
0	any	1	0,1,4,5	1	-	-	-	-	-	-	2,3,6,7	5	-	-	5	-
1	even	-	0,4	-	-	7	-	3	-	-	2,6	-	-	3	-	7
2	any	1	0,1,4,5	1	-	-	-	-	-	-	2,3,6,7	5	-	-	5	-
3	even	-	0,4	-	-	7	-	3	-	-	2,6	-	-	3	-	7
(mod 4)																

Table 4: Admissible $v_2 \pmod 8$ for $v_1 = 2$. (- : Not possible.)

Before we start the proof, it is worth of mentioning again that we shall reduce λ_1 first.

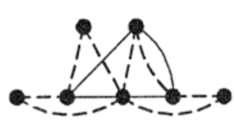
By condition (d), we have $\lambda_2 \binom{v_2}{2} \geq \lambda_1$ and $\lambda_3 v_2 \geq \lambda_1$. Therefore, in order to use up the first associate edges, we also need λ_1 edges in $\lambda_2 K_{v_2}$ and the graph H induced by these λ_1 edges is also important. This is due to the fact that a part of the third associate edges will be used to obtain the 4-cycles. Basically, we shall make H as regular as possible, i.e., if H has more than v_2 edges, we let the graph be a spanning subgraph of $\lambda_2 K_{v_2}$ which contains $\lambda_2 \cdot \frac{v_2-1}{2}$ 2-factors. Note here that we can view these 2-factors as the graphs which are induced by the differences in $V(\lambda_2 K_{v_2}) = \mathbb{Z}_{v_2}$.

Since $\lambda_1 \leq \lambda_2\binom{v_2}{2}$, let $\lambda_1 = (s-1)v_2 + t$ where $0 < s \leq \frac{\lambda_2(v_2-1)}{2}$ and $0 < t \leq v_2$. Also, since $\lambda_3 v_2 \geq \lambda_1,\ s \leq \lambda_3$. Note that λ_1 and λ_3 have the same parity when v_2 is odd.

First, consider the case when $\lambda_1 \leq v_2$. If $\lambda_1 \leq 2$, then the 4-cycle decomposition of $(\lambda_1, \lambda_2, \lambda_3)K_{2,v_2}$ can be obtained directly. If $\lambda_1 = 1$, then use the fact that $K_2 \vee C_5$ has a 4-cycle decomposition. Now, since λ_3 and v_2 must be odd, the decomposition follows by the 4-cycle decomposition of $K_{2,v_2-5} \cup (\lambda_3 - 1)K_{2,v_2} \cup (\lambda_2 K_{v_2} - C_5)$. On the other hand, if $\lambda_1 = 2$, the proof follows by using $C_4|(K_2 \cup K_2) \vee B$(bowtie), and the packing of the other two parts $\lambda_3 K_{2,v_2} - K_{2,5}$ and $\lambda_2 K_{v_2} - B$. Hence, in what follows, we consider only the case where $\lambda_1 \geq 3$.

Case 2.1: $3 \leq \lambda_1 \leq v_2$.

(i) λ_3 is even. Now, $\lambda_2(v_2 - 1)$ must be even, so is λ_1. Let H be a 2-regular graph of size λ_1 such that H contains either all 4-cycles or possibly one 6-cycle in $\lambda_2 K_{v_2}$. It is not difficult to see that $\lambda_1 K_2 \vee H$ can be decomposed into 4-cycles. Hence, the edges (3rd associates) used to obtain $C_4|\lambda_1 K_2 \vee H$ induced K_{2,λ_1}which is a subgraph of $\lambda_3 K_{2,v_2}$. By direct counting, $\lambda_3 K_{2,v_2} - K_{2,\lambda_1}$ has even number of edges and thus can be packed with 4-cycles with leave either an empty graph or $D \cup C_4$. In the first case, $4|\lambda_2\binom{v_2}{2}$; thus $\lambda_2 K_{v_2} - H$ can be decomposed into 4-cycles (Theorem 2.13). This implies $C_4|(\lambda_1, \lambda_2, \lambda_3)K_{v_1,v_2}$. On the other hand, we have a $D \cup C_4$ left from the third associate edges. But, now $\lambda_2\binom{v_2}{2} \equiv 2 \pmod 4$. Hence, $\lambda_2 K_{v_2} - H$ can be packed with 4-cycles and the leave is a bowtie (by Theorem 2.13). By arranging these two leaves together properly as in the following figure, we have a 4-cycle decomposition. This concludes the proof of this case.



(ii) λ_3 is odd. Now, λ_1 and v_2 have the same parity. First, let λ_1 be odd and let H be defined as in (i) except the last cycle is either a 3-cycle or a 5-cycle. Since $K_{2,v_2} = K_{2,\lambda_1} \cup K_{2,v_2-\lambda_1}$, $C_4|\lambda_1 K_2 \vee H$, and $C_4|K_{2,v_2-\lambda_1}$, we conclude that $\lambda_1 K_2 \cup K_{2,v_2} \cup H$ can be decomposed into 4-cycles. By the facts that $C_4|(\lambda_3 - 1)K_{2,v_2}$ and $C_4|\lambda_2 K_{v_2} - H$, we have the proof of the case when λ_1 is odd. On the other hand, let λ_1 and v_2 be even. Clearly, λ_2 is also even. Due to the reason that $C_4|K_{2,v_2-\lambda}$ and $C_4|\lambda_2 K_{v_2} - H$ (by Lemma 2.10), we have the proof of (ii) and thus Case 2.1.

Case 2.2: $\lambda_1 > v_2$.

So $\lambda_1 = (s-1)v_2 + t$ where $s \geq 2$ and $0 < t \leq v_2$. For convenience, we consider the following two cases.

Case 2.2.1: v_2 is odd.

Note that again, in this case λ_1 and λ_3 are of the same parity, i.e., λ_1 is odd if and only if λ_3 is odd. By observation $\lambda_2 K_{v_2}$ contains $\lambda_2 \cdot \frac{v_2-1}{2}$ 2-factors which are induced by the differences $1, 2, \ldots, \frac{v_2-1}{2}$ in $\mathbb{Z}_{v_2}$. (Each difference occurs λ_2 times.) Also, $s \leq \lambda_2 \cdot \frac{v_2-1}{2} = \lambda$. Our goal is to partition $\lambda_2 K_{v_2}$ into two parts such that we use one of them with exactly λ_1 edges to pair with the first associate edges and the other part has a maximum packing with 4-cycles.

So the plan is to reserve a set of $\lambda - 4k$ differences for the first part such that we have $4k$ differences left for the second part where k is as large as possible, i.e., $s > \lambda - 4(k+1)$. It is worth noting that the reserved differences are chosen depending on the relationship between s and $\lambda - 4k$.

(i) $s = \lambda - 4k$: The differences 1 and 2 are reserved for the first part if $t \leq v_2 - 3$, and if $t = v_2 - 1$ or $v_2 - 2$, we also reserve the differences 3, 4, 5, and 6 for the second part.(Note that $k \geq 1$.)

(ii) $s = \lambda - 4k - 1$ or $\lambda - 4k - 2$: The differences 1, 2 and 3 are reserved for the first part.

(iii) $s = \lambda - 4k - 3$: The differences $\frac{v_2-3}{2}$, $\frac{v_2-1}{2}$, 1, 2 and 3 are reserved for the first part.

Now we are ready for the proof. In (i), if $t \leq v_2 - 3$, then by Lemma 2.18, $G(1,2)$ contains a subgraph H of size $v_2 + t$ such that $G(1,2) - H$ has a maximum packing with 4-cycles. But, if $t = v_2 - 2$ or $v_2 - 1$, then we have to add an extra 4-cycle from $G(3,4,5,6)$, let the 4-cycle be (0, 3, 9, 4). Let $H = G(1,2) - (0,1,2) - (2,3,5,4) + (0,2,3,5,4)$. Then $|E(H)| = 2v_2 - 2$ and $G(3,4,5,6)$ becomes $G(3,4,5,6) - (0,3,9,4) + (9,4,2,1,0,3)$ which is a 4-cycle packing with leave a 6-cycle. By exchanging (0, 2, 3, 5, 4) and (9, 4, 2, 1, 0, 3), we have an H of size $2v_2 - 1$ and $G(3,4,5,6) - (0,3,9,4) + (0,2,3,5,4)$ which is clearly a 4-cycle packing with leave a C_5. By observation, the above situation occurs only at $k > 0$. For otherwise, the total number of edges won't be a multiple of 4, since the number the first and the third associate edges are of the same parity, so are λ_1 and λ_3.

After the graph H is obtained, we can use up all the first associate edges, the same number of second associate edges, and the third associate edges in $(s-1)K_{2,v_2} \cup K_{2,t}$. Since λ_1 and λ_3 have the same parity, $(\lambda_3 - s + 1)K_{2,v_2} \cup K_{2,v_2-t}$ must be an even graph. This is due to the reason that if λ_1 is odd, then $|E(H)|$ is odd. Therefore, each vertex in the partite set with two vertices is adjacent to an odd number of vertices in the other partite set $V(\lambda_2 K_{v_2})$. This implies the degree of this vertex left unused is $\lambda_3 v_2 - |E(H)|$ which is even. Now, it is a routine matter to pack $(\lambda_3 - s + 1)K_{2,v_2} \cup K_{2,v_2-t}$ with 4-cycles and its leave is either an empty graph or $D \cup C_4$ as mentioned Case 2.1. In both cases, we can combine this leave with the leave from the packing of $\lambda_2 K_{v_2} - H$ and the proof follows by the fact that total number of edges is a multiple of 4.

The cases (ii) and (iii) can be obtained by the same idea as above except the differences which are reserved for use are different. In (ii), we use $G(1,2,3)$ to construct H' which has $v_2 + t$ (or t) edges and H is obtained by taking the union of H' and an edge-disjoint union of the graph induced by $s-2$ (or $s-3$) differences. Then, apply Lemma 2.21 to obtain the decomposition. Finally, in (iii), we use $G(\frac{v_2-3}{2}, \frac{v_2-1}{2}, 1, 2, 3)$ to find an H' with $v_2 + t$ edges (Lemma 2.23) and then H is obtained by taking the union of H' and the graph induced by $s-5$ differences. This concludes the proof of the case when v_2 is odd.

Case 2.2.2: v_2 is even.

Since λ_2 must be even, instead of using the differences in K_{v_2}, we use the differences in $2K_{v_2}$. Hence, we have the differences $1, 2, \ldots, \frac{v_2-2}{2}$ and $\frac{v_2}{2}$ each occur twice. Except for $\frac{v_2}{2}$, each difference induces a 2-factor in $\lambda_2 K_2$. The difference $\frac{v_2}{2}$ gives a 1-factor. Therefore, by using two differences $\frac{v_2}{2}$, we also have a 2-factor. Thus, in total, we have $v_2 - 1$ 2-factors in $2K_{v_2}$, and we have $\frac{\lambda_2}{2}$ 2-factors which are induced by the difference $\frac{v_2}{2}$. So in what follows, when $\frac{v_2}{2}$ is used, we put two of them together.

So the proof follows in a similar way. We start with the construction of H depending on λ_1. Since the difference $\frac{v_2}{2}$ (two occurrences) with any other difference i less than $\frac{v_2}{2}$ can induce a graph which has a 4-cycle decomposition (Lemma 2.24), the proof is simpler. Instead of $4k$ differences left (in Case 2.2.1) we only need even differences left. Therefore, by reserving 1 and 2 for the first part and using Lemma 2.26, we are done.

Finally, after H is constructed, we use λ_1 first associate edges, $2\lambda_1$ third associate edges (induce $(s-1)K_{2,v_2} \cup K_{2,t}$) and H to obtain λ_1 4-cycles (by Lemma 2.27). Since λ_1 must be even, t is also even. Hence $(\lambda_3 - s + 1)K_{2,v_2} \cup K_{2,v_2-t}$ has a 4-cycle decomposition, so is $\lambda_2 K_{v_2} - H$. This concludes the proof of this case and the case $v_2 = 2$ and $v_2 \geq 4$.

Case 3: $v_1 = 3$ and $v_2 \geq 4$.

For clarity, we list the possible values for v_2 (mod 8) as in the following table.

	$\lambda_1 \equiv 0$ (mod 4)				$\lambda_1 \equiv 1$ (mod 4)				$\lambda_1 \equiv 2$ (mod 4)				$\lambda_1 \equiv 3$ (mod 4)			
$\lambda_3 \setminus \lambda_2$	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
0	any	1	0,1,4,5	1	-	7	-	3	-	-	2,3,6,7	5	-	3	5	7
1	-	0	-	0	-	6	-	2	-	-	-	4	-	2	4	6
2	even	5	0,3,4,7	5	-	3	-	7	odd	-	1,2,5,6	1	-	7	1	3
3	-	0	-	0	-	6	-	2	-	-	-	4	-	2	4	6

(mod 4)

Table 5: Admissible v_2 (mod 8) for $v_1 = 3$.
- : Not admissible.

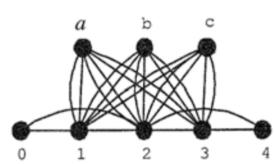
Observe that no matter which case we consider, we shall use up the first associate edges first.

Case 3.1.: λ_3 is even.

We split the proof of Case 3.1 into four subcases depending λ_1. First, we observe that $\lambda_3 K_{3,v_2}$ can be packed with $2K_{3,2}$'s such that its leave is either $2K_{3,3}$ or an empty graph depending on whether $3\lambda_3 v_2 \equiv 6$ or $0 \pmod{12}$.

(i) $\lambda_1 \equiv 0 \pmod 4$.

This implies that either both $3\lambda_3 v_2$ and $\lambda_2\binom{v_2}{2}$ are congruent to 0 modulo 4 or both are congruent to 2 modulo 4. In case that $3\lambda_3 v_2 \equiv 0 \pmod 4$, $\lambda_3 K_{3,v_2}$ can be decomposed into $2K_{3,2}$'s. By the fact $C_4|(4K_3, 2K_{3,2}, \emptyset)$ and $C_4|2K_{3,2}$, we can decompose $\lambda_1 K_3 \cup \lambda_3 K_{3,v_2}$ into 4-cycles. Hence, by condition (a) and (b), $4|\lambda_2\binom{v_2}{2}$ and $\lambda_2 K_{v_2}$ is an even graph, we have $C_4|\lambda_2 K_{v_2}$. Thus, the 4-cycle decomposition of $(\lambda_1, \lambda_2, \lambda_3)K_{3,v_2}$ is obtained. On the other hand, if $3\lambda_3 v_2 \equiv 2 \pmod 4$, then $\lambda_3 K_{3,v_2}$ can be decomposed into $2K_{3,2}$'s and one $2K_{3,3}$. Since $3\lambda_1 \leq 3\lambda_3 v_2$, $\lambda_1 K_3 \cup \lambda_3 K_{3,v_2}$ can be packed with 4-cycles such that the leave of the packing is $2K_{3,3}$. Now, since $\lambda_2\binom{v_2}{2} \equiv 2 \pmod 4$ and $\lambda_2 K_{v_2}$ is an even graph, $\lambda_2 K_{v_2}$ can be packed with 4-cycles such that its leave is a bowite. By decomposing $(\emptyset, 2K_{3,3}, B)$ into 4-cycles, we have the desired decomposition. See the following figure for the decomposition.



4-cycles are: $(a, 2, 4, 3), (b, 1, 2, 3), (c, 2, 0, 1), (a, 2, b, 1), (b, 3, c, 2), (a, 3, c, 1)$.

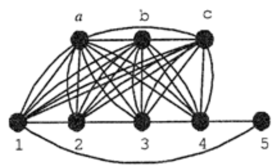
(ii) $\lambda_1 \equiv 2 \pmod 4$.

This implies that $3\lambda_1 \equiv 2 \pmod 4$, and therefore one of $3\lambda_3 v_2$ and $\lambda_2\binom{v_2}{2}$ is congruent to 2 modulo 4. In one hand, if $\lambda_2\binom{v_2}{2} \equiv 2 \pmod 4$, then $3\lambda_3 v_2 \equiv 0 \pmod{12}$. Now, since $\lambda_2 K_{v_2}$ is an even graph, by Lemma 2.3, $\lambda_2 K_{v_2} - D$ can be decomposed into 4-cycles. Therefore, following the idea of the above case and the fact $C_4|(2K_3, 2K_{3,2}, D)$, we have the desired decomposition of $(\lambda_1 K_{v_1}, \lambda_3 K_{3,v_2}, \lambda_2 K_{v_2})$. We remark here that if $3\lambda_3 v_2 \geq 3\lambda_1$, then $3\lambda_3 v_2 \geq 3\lambda_1 + 2$ (by the fact $3\lambda_3 v_2 \equiv 0 \pmod{12}$). On the other hand, if $\lambda_2\binom{v_2}{2} \equiv 0 \pmod 4$, then $3\lambda_3 v_2 \equiv 6 \pmod{12}$. The proof follows by the fact that $C_4|(2K_3, 2K_{3,3}, \emptyset)$.

(iii) $\lambda_1 \equiv 1 \pmod 4$ (The proof of the case $\lambda_1 \equiv 3 \pmod 4$ is similar.)

Again, since $3\lambda_3 v_2 \equiv 0$ or $2 \pmod 4$, $\lambda_2\binom{v_2}{2} \equiv 1$ or $3 \pmod 4$ correspondingly. First, let $3\lambda_3 v_2 \equiv 0 \pmod 4$. Since λ_3 is even, $\lambda_2(v_2 - 1)$ is also even. This implies that $C_4|(\lambda_2 K_{v_2} - C_5)$. Thus, the decomposition follows by the fact $C_4|(K_3, 2K_{3,2} \cup 2K_{3,2}, C_5)$ see the following figure. As to the case when $3\lambda_3 v_2 \equiv$

2 (mod 4), the decomposition follows from $C_4|(K_3, 2K_{3,3}, K_3)$ which is easy to check.



4-cycles are: $(a, 4, 5, 1)$, $(b, 2, 3, 4)$, $(a, c, 1, 2)$, $(3, a, b, c)$, $(b, 1, c, 3)$, $(b, 1, a, 3)$, $(a, 2, c, 4)$, $(b, 2, c, 4)$.

Case 3.2.: λ_3 is odd.

First, we observe that $\lambda_3 K_{3,v_2} = K_{3,v_2} \cup (\lambda_3 - 1)K_{3,v_2}$. Since v_2 must be even, $3(\lambda_3 - 1)v_2 \equiv 0 \pmod{12}$, $(\lambda_3 - 1)K_{3,v_2}$ can be decomposed into $(\lambda_3 - 1) \cdot \frac{v_2}{4}$ copies of $2K_{3,2}$. By Case 3.1, $C_4|(4K_3, 2K_{3,2}, \emptyset)$. Therefore, if $\lambda_1 \leq (\lambda_3 - 1)\frac{v_2}{4} \cdot 4$, we first pack $(K_3, K_{3,v_2}, K_{v_2}) = K_{3+v_2}$ by using 4-cycles with proper 2-regular leave H depending on $3 + v_2$ which is odd(Lemma 2.8), and then we deal with the graph $(\lambda_1 - 1, \lambda_2 - 1, \lambda_3 - 1)K_{3,v_2} \cup H$. Since $\lambda_3 - 1$ is even, the decomposition follows by a similar argument as in Case 3.1.(Note that if $\lambda_3 = 1$, then the above process is not necessary.)

Now, it is left to consider the case when $\lambda_3 v_2 - \lceil \frac{v_2}{6} \rceil \geq \lambda_1 > (\lambda_3 - 1)v_2$.

Since $\lambda_3 v_2 - \lceil \frac{v_2}{6} \rceil \geq \lambda_1 > (\lambda_3 - 1)v_2$, $\lambda_1 = (\lambda_3 - 1)v_2 + t$ where $v_2 - \lceil \frac{v_2}{6} \rceil \geq t > 0$. Thus, the number of first associate edges is $3(\lambda_3 - 1)v_2 + 3t$ which is between 0 and $3(\lambda_3 - 1)v_2 + 3v_2 - 3\lceil \frac{v_2}{6} \rceil$. So, our strategy is to find a collection of second and third associate edges to match with these $3\lambda_1$ edges properly to form 4-cycles. It is worth of nothing that we shall use as less second associate edges as possible. In fact, for $3(\lambda_3 - 1)v_2$ first associate edges, we shall use only the third associate edges and the rest of them, we will use a matching or a spanning odd forest in $\lambda_2 K_{v_2}$.

Now, if $\lambda_3 > 1$, by the reduction idea mentioned above, we have $C_4|(\lambda_3 - 1)v_2 K_3 \cup (\lambda_3 - 1)K_{3,v_2}$. Hence, we have K_{3,v_2} left which we can use it to form 4-cycles with $3t$ first associates and at most $\frac{v_2}{2}$ second associate edges. This shows that as long as we can handle the case $\lambda_3 = 1$ and $v_2 - \lceil \frac{v_2}{6} \rceil \geq t > 0$, we have the proof of the case when λ_3 is odd.

First, if $3\lambda_1 < \frac{v_2}{2}$, i.e. $\lambda_1 < \frac{v_2}{6}$, let $V(\lambda_2 K_{v_2}) = Y = Y_1 \cup Y_2$ where $|Y_1| = 6\lambda_1$ and $|Y_2| = v_2 - 6\lambda_1$. By Lemma 2.14, $C_4|(K_3, K_{3,6}, M_3)$ and thus $C_4|(\lambda_1 K_3, K_{3,6\lambda_1}, M_{3\lambda_1})$. Observe that $(\lambda_1, \lambda_2, \lambda_3)K_{3,v_2} = (\lambda_1 K_3, K_{3,6\lambda_1}, K_{6\lambda_1}) \cup K_{v_2-6\lambda_1} \cup K_{3+6\lambda_1, v_2-6\lambda_1} \cup (\lambda_2 - 1)K_{v_2} = (\lambda_1 K_3, K_{3,6\lambda_1}, M_{3\lambda_1}) \cup (K_{6\lambda_1} - M_{3\lambda_1}) \cup K_{2+6\lambda_1, v_2-6\lambda_1} \cup K_{v_2-6\lambda_1+1} \cup (\lambda_2 - 1)K_{v_2}$. Thus, by Lemma 2.3 and 2.4, the 4-cycle decomposition of $(\lambda_1, \lambda_2, \lambda_3)K_{3,v_2}$ can be obtained if $K_{v_2-6\lambda_1+1} \cup (\lambda_2 - 1)K_{v_2}$ has a 4-cycle decomposition. Since both $K_{v_2-6\lambda_1+1}$ and $(\lambda_2 - 1)K_{v_2}$ are even graphs, the decomposition can be obtained by packing them with 4-cycles respectively and them combine their leaves together. By the fact that the number of edges is a multiple of 4, the union does have a 4-cycle decomposition.

Finally, consider the case $v_2 - \lceil \frac{v_2}{6} \rceil \geq \lambda_1 \geq \frac{v_2}{6}$. By direct counting, if $v_2 = 6k + k'$ where $k' = 0$, 2 or 4, then $\lceil \frac{v_2}{6} \rceil = k$ or $k+1$ and $v_2 - \lceil \frac{v_2}{6} \rceil = 5k$, $5k+1$ or $5k+3$. Therefore, for convenience, we split the proof of this part into three cases. (Basically, the idea of proof is similar.)

(i) $v_2 = 6k$

If k is even, then $\lambda_2 \binom{v_2}{2} \equiv 0$ or 2 (mod 4) depending on $k \equiv 0$ or 2 (mod 4). In either case, $|E(K_{3,v_2})| \equiv 0$ (mod 4). Hence, $\lambda_1 \equiv 0$ or 2 (mod 4) depending on k. First, if $k \equiv 0$ (mod 4), then by Lemma 2.14, $C_4|(K_3, K_{3,6}, M_3)$ for every 6-subset of $V(\lambda_2 K_{v_2})$, and thus $C_4|(kK_3, K_{3,v_2}, M_{\frac{v_2}{2}})$. This implies if $\lambda_1 = k$, we have the decomposition to use up all first associate edges and the third associate edges. The decomposition of $(\lambda_1, \lambda_2, \lambda_3)K_{3,v_2}$ follows by decomposing $\lambda_2 K_{v_2} - M_{\frac{v_2}{2}}$. Now, if $\lambda_1 > k$, then $\lambda_1 - k \equiv 0$ (mod 4), let $\lambda_1 = k + 4h$ where $1 \leq h \leq k$. Again, by Lemma 2.14, $C_4|(5K_3, K_{3,6}, M_3)$, we can use h disjoint 6-subsets of $V(\lambda_2 K_{v_2})$ which we have $C_4|(K_3, K_{3,6}, M_3)$ by $C_4|(5K_3, K_{3,6}, M_3)$ and conclude that $C_4|(\lambda_1 K_3, K_{3,v_2}, M_{\frac{v_2}{2}})$.

On the other hand, if $k \equiv 2$ (mod 4), then $\lambda_2 \binom{v_2}{2}$ (mod 4), and hence $\lambda_1 \equiv 2$ (mod 4). Since, we also start with k, the proof follows by a similar argument.

Now, if k is odd, then $\lambda_2 \binom{v_2}{2} \equiv 1$ or 3 (mod 4) depending on k and λ_2. Therefore, $\lambda_1 \equiv 3$ or 1 (mod 4) correspondingly, since $3v_2 \equiv 2$ (mod 4).

First, if $k \equiv 1$ (mod 4) and $\lambda_2 \equiv 1$ (mod 4), then $\lambda_2 \binom{v_2}{2} \equiv 3$ (mod 4), and $\lambda_1 \equiv 1$ (mod 4). Since $\lambda_1 \geq k$, the smallest possible λ_1 is k, and the other possible λ_1's are $k + 4h$, $1 \leq h \leq k$. So, the decomposition can be obtained easily. On the other hand, if $k \equiv 1$ (mod 4) and $\lambda_2 \equiv 3$ (mod 4), then $\lambda_2 \binom{v_2}{2} \equiv 1$ (mod 4), and $\lambda_1 \equiv 3$ (mod 4). Hence, the smallest possible λ_1 is $k+2$, and the others are $k + 2 + 4h$ where $1 \leq h \leq k - 1$. Now, let $F = K_{1,5} \cup M_3 \cup ... \cup M_3$. By using the decomposition of $(3K_3, K_{3,6}, K_{1,5})$ and $(K_3, K_{3,6}, M_3)$ in Lemma 2.15, we have a 4-cycle decomposition of $((k+2)K_3, K_{3,v_2}, F)$. Moreover, if we replace $C_4|(K_3, K_{3,6}, M_3)$ by $C_4|(5K_3, K_{3,6}, M_3)$, we are able to obtain a 4-cycle decomposition of $((k+2+4h)K_3, K_{3,v_2}, F)$ for each $1 \leq h \leq k-1$. Now, the 4-cycle decomposition of $(\lambda_1, \lambda_2, \lambda_3)K_{3,v_2}$ will be obtained following the 4-cycle decomposition of $\lambda_2 K_{v_2} - F$ which is by Lemma 2.3.

With the above idea in hand, the case when $k \equiv 3$ (mod 4) and $\lambda_2 \equiv 1$ or 3 (mod 4) can also be obtained similarly.

(ii) $v_2 = 6k + 2$. ($\lceil \frac{v_2}{6} \rceil = k + 1$)

If k is even, then $\lambda_2 \binom{v_2}{2} \equiv 1$ or 3 (mod 4) depending on $\lambda_2 \equiv 1$ or 3 (mod 4). Since $3v_2 \equiv 2$ (mod 4), $\lambda_1 \equiv 1$ or 3 (mod 4) as the case may be. By assumption, k is even and thus $k \equiv 0$ or 2 (mod 4). In the case where $\lambda_1 \equiv 1$ (mod 4) and $k \equiv 0$ (mod 4) or $\lambda_1 \equiv 3$ (mod 4) and $k \equiv 2$ (mod 4), the proof follows by using the 4-cycle decomposition of $((k+1)K_3, K_{3,v_2}, H)$ where $H = M_{\frac{v_2-8}{2}} \cup K_{1,3} \cup K_{1,3}$. Now, partition $V(\lambda_2 K_{v_2})$ into $\frac{v_2-8}{6}$ 6-subsets and one 8-subset such that $K_{1,3} \cup K_{1,3}$ defined on the 8-subset and the graph M_3 is defined on a 6-subset.

By Lemma 2.14, we have $C_4|(K_3, K_{3,6}, M_3)$ and $C_4|(2K_3, K_{3,8}, K_{1,3} \cup K_{1,3})$. Hence $((k+1)K_3, K_{3,v_2}, H)$ has a 4-cycle decomposition. Then the 4-cycle decomposition of $((k+1+4h)K_3, K_{3,v_2}, H)$ for each $1 \leq h \leq (k-1)$ can be obtained by replacing $C_4|(K_3, K_{3,6}, M_3)$ with $C_4|(5K_3, K_{3,6}, M_3)$. Furthermore, if we replace $C_4|(2K_3, K_{3,8}, K_{1,3} \cup K_{1,3})$ with two 4-cycle decompositions of $(3K_3, K_{3,4}, K_{1,3})$, we have $\lambda_1 = (6k+2) - (k+1) = 5k+1$. By the same technique as we have in the above case, we conclude the proof of this case.

Now, consider the case where $\lambda_1 \equiv 1 \pmod 4$ and $k \equiv 2 \pmod 4$ or $\lambda_1 \equiv 3 \pmod 4$ and $k \equiv 0 \pmod 4$. Clearly, $\lambda_1 = k+3$ in each case. Therefore, the decomposition starts at considering $((k+3)K_3, K_{3,v_2}, H)$ where H is a spanning odd forest and this can be done by using the 4-cycle decomposition of $(2K_3, K_{3,4}, M_2)$ and $(K_3, K_{3,6}, M_3)$. By replacing the 4-cycle decomposition of $(K_3, K_{3,6}, M_3)$ with the 4-cycle decomposition of $(5K_3, K_{3,6}, M_3)$ we have the 4-cycle decomposition of $((k+3+4h)K_3, K_{3,v_2}, M_{\frac{v_2}{2}})$ for each $1 \leq h \leq k-1$. (Here $H \cong M_{\frac{v_2}{2}}$.) Again, the 4-cycle decomposition of $\lambda_2 K_{v_2} - M_{\frac{v_2}{2}}$ takes care the remainding graph and we have the proof of this case.

On the other hand, if k is odd, then $\lambda_2\binom{v_2}{2} \equiv 0$ or $2 \pmod 4$ depending on $k \equiv 1$ or $3 \pmod 4$. First, let $k \equiv 1 \pmod 4$, then $\lambda_1 \equiv 0 \pmod 4$. This implies that we consider the decomposition of $((k+3)K_3, K_{3,v_2}, H)$ first and this is easy to get from the same idea as above. Finally, if $k \equiv 3 \pmod 4$, then $\lambda_2\binom{v_2}{2} \equiv 2 \pmod 4$. Thus, $\lambda_1 \equiv 2 \pmod 4$. Again, the decomposition will be started at $\lambda_1 = k+3$ and ended at $\lambda_1 = 5k-1$. The proof of this case is therefore concluded by a similar decomposition as above.

(iii) $v_2 = 6k+4$. ($\lceil \frac{v_2}{6} \rceil = k+1$ and $v_2 - \lceil \frac{v_2}{6} \rceil = 6k+3$.)

First, if k is even, then $\lambda_2\binom{v_2}{2}$ is also even. In the case where $k \equiv 2 \pmod 4$, then $\lambda_1 \equiv 0 \pmod 4$. It suffices to find the decomposition of $(\lambda_1 K_3, K_{3,v_2}, H)$ with $\lambda_1 \in \{k+2, k+6, \cdots, 5k+2\}$ and H is a spanning odd forest of $\lambda_2 K_{v_2}$. Clearly, $\lambda_1 = k+2$ can by obtained by using $H = M_3 \cup M_3 \cup \cdots \cup M_3 \cup M_2$ and the 4-cycle decompositions of $(K_3, K_{3,6}, M_3)$ and $(2K_3, K_{3,4}, M_2)$ respectively. Then, by replacing $(K_3, K_{3,6}, M_3)$ with $(5K_3, K_{3,6}, M_3)$, we have the 4-cycle decompositions of all possible $(\lambda_1 K_3, K_{3,v_2}, M_{\frac{v_2}{2}})$ for each $\lambda_1 \in \{k+2, k+6, \cdots, 5k+2\}$, and the rest of decompositions can be obtained similarly.

On the other hand, if $k \equiv 0 \pmod 4$, then $\lambda_1 \equiv 2 \pmod 4$. The decomposition also starts at the case $\lambda_1 = k+2$. Thus the proof is similar.

Finally, it is left to consider the case where k is odd. By direct counting, $\lambda_2\binom{v_2}{2} \equiv 1$ or $3 \pmod 4$ and $\lambda_1 \equiv 3$ or $1 \pmod 4$ correspondingly. ($3v_2 \equiv 2 \pmod 4$.) So, if $k \equiv 1 \pmod 4$ and $\lambda_2 \equiv 1 \pmod 4$, then $\lambda_2\binom{v_2}{2} \equiv 1 \pmod 4$ which implies that $\lambda_1 \equiv 3 \pmod 4$. Hence, the decomposition starts at $\lambda_1 = k+2$ and this is done as above, so is $\lambda_1 \in \{k+6, k+10, \cdots, 5k+2\}$. If $k \equiv 1 \pmod 4$ and $\lambda_2 \equiv 3 \pmod 4$, then $\lambda_1 \equiv 1 \pmod 4$. Since $\lambda_1 \geq k+1$, the decomposition starts at $\lambda_1 = k+4$. Let $H = K_{1,5} \cup M_3 \cup M_3 \cup \cdots \cup M_3 \cup M_2$. Then, by Lemma 2.14, $C_4|(3K_3, K_{3,6}, K_{1,5})$, $C_4|(K_3, K_{3,6}, M_3)$ or $C_4|(5K_3, K_{3,6}, M_3)$, and $C_4|(2K_3, K_{3,4}, M_2)$ we can handle the cases where $\lambda_1 = k+4, k+8, \cdots, 5k$. This gives the desired decomposition. On the other hand, if $k \equiv 1 \pmod 4$ and $\lambda_2 \equiv 3 \pmod 4$, then $\lambda_1 \equiv 1 \pmod 4$. Since $\lambda_1 \geq k+1$, the decomposition starts at $k+4$ which is exactly the same as the above case.

Finally, consider $k \equiv 3 \pmod 4$. First, if $\lambda_2 \equiv 1 \pmod 4$, then $\lambda_1 \equiv 3 \pmod 4$. Thus, the decomposition starts at $\lambda_2 = k+4$. On the other hand, if $\lambda_2 \equiv 3 \pmod 4$, then $\lambda_1 \equiv 1 \pmod 4$ and the decomposition starts at $\lambda_1 = k+2$. In both cases, we have similar decompositions mentioned above. Hence the proof follows by the same technique. This concludes the proof of the case $v_1 = 3$ and $v_2 \geq 4$.

Case 4: $v_1, v_2 \leq 3$.

Since the case where $v_1 = v_2$ and $\lambda_1 = \lambda_2$ has been proved in Theorem 1.2, it suffices to consider the case when $v_1 = 2$ and $v_2 = 3$ or $v_1 = v_2 = 3$ and $\lambda_1 < \lambda_2$. We shall use the idea mentioned in Case 2 to handle the first case. It is easy to see that we can reduce 3 at a time by using one triangle in K_{v_1} and one $K_{2,3}$ in $\lambda_3 K_{2,3}$. Therefore, reduce λ_2 and λ_3 by 1 respectively. Hence, it is left to consider $\lambda_1 \leq 2$. First, if $\lambda_1 = 1$, then $\lambda_2 \equiv 3 \pmod 4$ and λ_3 must be odd. By the 4-cycle decomposition of $(K_2, 3K_{2,3}, 3K_3)$ ($\lambda_3 = 1$ is not possible), we have the desired decomposition by combining $(\lambda_3 - 3)K_{2,3}$ and $4tK_3$. On the other hand, if $\lambda_1 = 2$, λ_2 must be even, so is λ_3. Hence we can reduce each associate by 2 using a 4-cycle system of order 5 with index 2 and then find a 4-cycle decomposition of $(\lambda_3 - 2)K_{2,3} \cup (\lambda_2 - 2)K_3$. Now, it is left to consider the case where $v_1 = v_2 = 3$ and $\lambda_1 < \lambda_2$.

By condition (b), λ_3 must be even and by condition (c) $3\lambda_3 \geq \lambda_1 + \lambda_2$. First, if λ_1 is even, then λ_2 is also even. Since it is not difficult to see both $(2K_3, C_6, \emptyset)$ and $(\emptyset, C_6, 2K_3)$ have 4-cycle decompositions, the proof follows by decomposing $\lambda_3 K_{3,3}$ into $\frac{\lambda_1+\lambda_2}{2}$ 6-cycles and 4-cycles. By the fact that $C_6|2K_{3,3}$, $\lambda K_{3,3}$ has a 6-cycle decomposition if and only if λ is even. So, if $\lambda_1 + \lambda_2 \equiv 0 \pmod 6$, $\frac{\lambda_1+\lambda_2}{3}K_{3,3}$ has a 6-cycle decomposition which contains exactly $\frac{\lambda_1+\lambda_2}{2}$ 6-cycles. By condition (a) $(\lambda_3 - \frac{\lambda_1+\lambda_2}{2})K_{3,3}$ must have $4t$ edges and thus $4|(\lambda_3 - \frac{\lambda_1+\lambda_2}{2})$ which implies that the graph has a 4-cycle decomposition. On the other hand, if $\lambda_1+\lambda_2 \equiv 2$ or $4 \pmod 6$, let $\lambda = \lambda_1+\lambda_2-2$ or $\lambda = \lambda_1+\lambda_2-4$ in corresponding cases. As mentioned above, $\frac{\lambda}{3}K_{3,3}$ has a 6-cycle decomposition which contains $\frac{\lambda}{2}$ 6-cycles. For the first case, it suffices to claim that $(\lambda_3 - \frac{\lambda}{3})K_{3,3} - C_6$ has a 4-cycle decomposition and $(\lambda_3 - \frac{\lambda}{3})K_{3,3} - 2K_6$ has a 4-cycle decomposition for the second case. By condition (a), $(\lambda_3 - \frac{\lambda}{3})K_{3,3} - C_6$ is an even graph, therefore $\lambda_3 - \frac{\lambda}{3}$ must be even. The condition (b) shows that $(\lambda_3 - \frac{\lambda}{3}) \equiv 2 \pmod 4$. Hence, by the decompositions $C_4|(\lambda_3 - \frac{\lambda}{3} - 2)K_{3,3}$ and $C_4|2K_{3,3} - C_6$, we have the proof of the first case. Similarly, if $\lambda = \lambda_1 + \lambda_2 - 4$, then $(\lambda_3 - \frac{\lambda}{3}) \equiv 0 \pmod 4$ and the proof follows by the decompositions $C_4|(\lambda_3 - \frac{\lambda}{3} - 4)K_{3,3}$ and $C_4|4K_{3,3} - 2C_6$.

Now, we have the case "λ_1 is odd" left. Hence, λ_2 is also odd. By the fact that $C_4|(K_3, C_6, K_3)$, it suffices to consider the decomposition of the graph $((\lambda_1 - 1)K_3, \lambda_3 K_{3,3} - C_6, (\lambda_2 - 1)K_3)$. Now, $\lambda_1 - 1$ and $\lambda_3 - 1$ are both even. Hence, by the same idea as above, the decomposition depends on the 4-cycle packing of $\lambda_3 K_{3,3} - C_6$

and this can be done similarly. Therefore, the proof of Case 4 is completed and the proof of the theorem is concluded. ■

Concluding Remarks

Since the notion of balanced bipartite block design $BBBD(v_1, v_2; k; \lambda_1, \lambda_2, \lambda_3)$ was introduced around 60 years ago, to determine all 6-tuples $(v_1, v_2; k; \lambda_1, \lambda_2, \lambda_3)$ such that a $BBBD$ exists becomes an interesting problem. But, so far, only partial results have been obtained. A substantial effort has been spent on the case $k = 3$ by Fu, Mishima and Rodger in recent years which handles quite a few possible 5-tuples $(v_1, v_2; \lambda_1, \lambda_2, \lambda_3)$. Unfortunately, there are too many tiny pieces to write and it is difficult to put them together in decent content. I wish this can be done soon.

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