AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **31** (2005), Pages 189–200

Vertex-disjoint copies of K_4^- in graphs

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Abstract

Let K_4^- denote the graph obtained from K_4 by removing one edge. Let k be an integer with $k \geq 2$. Kawarabayashi conjectured that if G is a graph of order $n \geq 4k+1$ with $\sigma_2(G) \geq n+k$, then G has k vertex-disjoint copies of K_4^-. In this paper, we settle this conjecture affirmatively.

1 Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and we let $d_G(x) := |N_G(x)|$. For a noncomplete graph G, let $\sigma_2(G) :=\min\{d_G(x) + d_G(y) |\ xy \notin E(G)\}$; if G is a complete graph, let $\sigma_2(G) := \infty$. Let K_4^- be the graph obtained from K_4 by removing one edge, and let S denote the graph obtained from K_4 by removing two edges which have a common vertex.

For a given connected graph F, a spanning subgraph of a graph G is called an F-factor if all of its components are isomorphic to F. Thus the statement that G has an F-factor is equivalent to the statement that $|V(G)|$ is a multiple of $|V(F)|$ and G contains $\frac{|V(G)|}{|V(F)|}$ vertex-disjoint copies of F. There are many results concerning degree conditions for the existence of vertex-disjoint copies of F. As for the existence of an F-factor, Alon and Yuster [1] proved the general result that under the assumption that $|V(G)|$ is a multiple of $|V(F)|$, the condition that $\delta(G) \geq (\frac{\chi(F)-1}{\chi(F)} + o(1))|V(G)|$ guarantees the existence of an F-factor in G, where $\chi(F)$ denotes the chromatic number of F and o refers to order of magnitude as $|V(G)|$ tends to infinity. In the case $F = K_l$, Hajnal and Szemerédi [3] proved that if $|V(G)|$ is a multiple of l and $\delta(G) \geq \frac{l-1}{l}|V(G)|$, then G has a K_l-factor.

In this paper, we are concerned with conditions on $\sigma_2(G)$ for the existence of vertex-disjoint subgraphs. As examples of results concerning such conditions, we mention that it is proved in Justesen [4] that a graph G of order at least $3k$ with

$\sigma_2(G) \geq |V(G)| + k$ contains k vertex-disjoint triangles (see also Theorem 2 below), and it is proved in Enomoto [2] and in Wang [7] that a graph G of order at least $3k$ with $\sigma_2(G) \geq 4k-1$ contains k vertex-disjoint cycles.

The main purpose of this paper is to prove the following theorem, which was conjectured by Kawarabayashi in [5] and [6].

Theorem 1 *Let k be an integer with $k \geq 2$, and let G be a graph of order $n \geq 4k+1$ with $\sigma_2(G) \geq n+k$. Then G contains k vertex-disjoint copies of K_4^-.*

In Theorem 1, the condition on $\sigma_2(G)$ is best possible in the following sense. Let k, n be integers with $k \geq 2$ and $n \geq 4k+1$ such that $n-k$ is odd, and let $G = \overline{K_{k-1}} + (\overline{K_{\frac{n-k+1}{2}}} + \overline{K_{\frac{n-k+1}{2}}})$. Then $\sigma_2(G) = n+k-1$, but G contains at most $k-1$ vertex-disjoint triangles, and hence G does not contain k vertex-disjoint K_4^-. Note also that the conclusion of Theorem 1 does not hold when $n = 4k$. To see this, let $k \geq 4$ be an integer and let $l = \lceil \frac{8k-2}{3} \rceil$, and let $G = (K_1 \cup K_{l-1}) + \overline{K_{4k-l}}$. Then $\sigma_2(G) = 8k-l-2 \geq 5k = |V(G)|+k$, but G does not contain k vertex-disjoint K_4^-.

In the proof of Theorem 1, we make use of the following theorem.

Theorem 2 ([6]) *Let k be an integer with $k \geq 2$, and let G be a graph of order $n \geq 4k$ with $\sigma_2(G) \geq n+k$. Then G contains k vertex-disjoint copies of S.*

Our notation is standard except possibly for the following. Let G be a graph. For a subset A of $V(G)$, the subgraph induced by A is denoted by $\langle A \rangle$. For a subgraph H of G, we let $G - H = \langle V(G) - V(H) \rangle$ and, for a vertex x of G, we let $G - x = \langle V(G) - \{x\} \rangle$. For disjoint subsets A and B of $V(G)$, we let $E(A, B)$ denote the set of edges of G joining a vertex in A and a vertex in B. When A or B consists of a single vertex, say $A = \{x\}$ or $B = \{y\}$, we write $E(x, B)$ or $E(A, y)$ for $E(A, B)$. For a subgraph H of G and for a vertex x of G with $x \notin V(H)$, we let $N_H(x) = N_G(x) \cap V(H)$; thus $|N_H(x)| = |E(x, V(H))|$.

2 Preparation for the proof of Theorem 1

Let G be a graph of order $n \geq 4k+1$ with $\sigma_2(G) \geq n+k$. Since G has k vertex-disjoint copies of S by Theorem 2, we can choose k vertex-disjoint induced subgraphs $S_1, \ldots, S_k$ such that for each $1 \leq i \leq k'$, S_i contains K_4^- as a spanning subgraph and, for each $k'+1 \leq i \leq k$, $S_i \cong S (0 \leq k' \leq k)$. Let $H := \langle \cup_{i=1}^{k} V(S_i) \rangle$. For $i = 1, \ldots, k$, write $V(S_i) = \{a_i, b_i, c_i, d_i\}$ so that $d_{S_i}(a_i) \geq d_{S_i}(b_i) \geq d_{S_i}(c_i) \geq d_{S_i}(d_i)$. Note that $d_{S_i}(a_i) = d_{S_i}(b_i) = 3$ and $d_{S_i}(c_i) = d_{S_i}(d_i) \geq 2$ for each $1 \leq i \leq k'$, and $d_{S_i}(a_i) = 3, d_{S_i}(b_i) = d_{S_i}(c_i) = 2$ and $d_{S_i}(d_i) = 1$ for each $k'+1 \leq i \leq k$. If $k' = k$, then the desired conclusion holds. Thus we may assume that $k' \leq k-1$. We may also assume that $S_1, \ldots, S_k$ are chosen so that
(a) k' is maximum; and,
(b) subject to (a), $\sum_{j=1}^{k} |E(S_j)|$ is maximum.

We start with easy lemmas.

Lemma 2.1. *Let $v \in V(G-H)$.*

(i) *For each i with $S_i \cong K_4^-$, $|E(v, V(S_i))| \leq 3$, with equality only if $|E(v, \{a_i, b_i\})| = 1$.*

(ii) *For each i with $S_i \cong S$, $|E(v, V(S_i))| \leq 2$, with equality only if $vd_i \in E(G)$.*

Proof.
(i) If $|E(v, V(S_i))| \geq 3$ and $a_i, b_i \in N_G(v)$, then replacing S_i by $\langle\{v, a_i, b_i, c_i\}\rangle$ or $\langle\{v, a_i, b_i, d_i\}\rangle$, we get a contradiction to the maximality of $\sum_{j=1}^{k} |E(S_j)|$.
(ii) If $|E(v, \{a_i, b_i, c_i\})| \geq 2$, then replacing S_i by $\langle\{v, a_i, b_i, c_i\}\rangle$, we get a contradiction to the maximality of k'. □

For later reference, we restate the case $i = k$ of Lemma 2.1(ii) in the following form (see the first paragraph of Section 3).

Lemma 2.2. *Let $v \in V(G-H)$. Then precisely one of the following six statements holds:*

(1) $N_{S_k}(v) = \{a_k\}$;

(2) $N_{S_k}(v) = \emptyset$;

(3) $N_{S_k}(v) = \{d_k\}$:

(4) $N_{S_k}(v) = \{b_k\}$ *or* $N_{S_k}(v) = \{c_k\}$;

(5) $N_{S_k}(v) = \{b_k, d_k\}$ *or* $N_{S_k}(v) = \{c_k, d_k\}$; *or*

(6) $N_{S_k}(v) = \{a_k, d_k\}$. □

Lemma 2.3. *Let i be an integer with $1 \leq i \leq k-1$. Let S', X be subgraphs of $\langle V(S_i) \cup V(S_k) \cup V(G-H)\rangle$ such that $S' \cong K_3$, $X \cong K_4^-$ or K_4, and $V(S') \cap V(X) = \emptyset$. Then for each $x \in (V(S_i) \cup V(S_k) \cup V(G-H)) - V(X) - V(S')$, $|E(x, V(S'))| \leq 1$.*

Proof. Suppose not. Then $\langle\{x\} \cup V(S')\rangle \supset K_4^-$. Hence by replacing S_i and S_k by X and $\langle\{x\} \cup V(S')\rangle$, respectively, we get a contradiction to the maximality of k'. □

Lemma 2.4. *Let $v \in \{d_k\} \cup V(G-H)$. Let $1 \leq i \leq k-1$, and suppose that $S_i \cong K_4^-$ and $|E(v, V(S_i))| \geq 2$. Then $|E(c_i, \{a_k, b_k, c_k\})| \leq 2$ and $|E(d_i, \{a_k, b_k, c_k\})| \leq 2$. Further if $N_G(v) \cap \{a_i, b_i\} \neq \emptyset$, then $|E(\{c_i, d_i\}, \{a_k, b_k, c_k\})| \leq 3$.*

Proof. If $|E(c_i, \{a_k, b_k, c_k\})| = 3$, then by replacing S_i and S_k by $\langle\{v\} \cup V(S_i - c_i)\rangle$ and $\langle\{c_i, a_k, b_k, c_k\}\rangle$, respectively, we get a contradiction to the maximality of $\sum_{j=1}^{k} |E(S_j)|$ because $E(v, V(S_i - c_i)) \neq \emptyset$. Thus $|E(c_i, \{a_k, b_k, c_k\})| \leq 2$ and, by symmetry, we similarly obtain $|E(d_i, \{a_k, b_k, c_k\})| \le 2$. Now assume that $N_G(v) \cap \{a_i, b_i\} \ne \emptyset$. Then we have $\langle \{v\} \cup V(S_i - c_i)\rangle \supset K_4^-$ or $\langle \{v\} \cup V(S_i - d_i)\rangle \supset K_4^-$. We may assume $\langle \{v\} \cup V(S_i - d_i)\rangle \supset K_4^-$. Then by applying Lemma 2.3 with $S' = \langle \{a_k, b_k, c_k\}\rangle$ and $X = \langle \{v\} \cup V(S_i - d_i)\rangle$, we obtain $|E(c_i, \{a_k, b_k, c_k\})| \le 1$, which implies $|E(\{c_i, d_i\}, \{a_k, b_k, c_k\})| \le 3$. □

3 Proof of Theorem 1

We continue with the notation of the preceding section. For convenience, when (m) of Lemma 2.2 holds for a vertex $v \in V(G - H)$, we say that the preference index of v is m, and write $\mathrm{pr}(v) = m$. Define $\alpha = \max\{\mathrm{pr}(v) |\ v \in V(G - H)\}$. We henceforth assume that we have chosen $S_1, \ldots, S_k$ so that α is as large as possible subject to conditions (a) and (b) stated at the end of the first paragraph of Section 2.

Lemma 3.1. *Let $v \in \{d_k\} \cup V(G - H)$. Let $1 \le i \le k - 1$, and suppose that $S_i \cong K_4$.*

(i) *If $|E(v, V(S_i))| \ge 3$, then $|E(V(S_i), \{a_k, b_k, c_k\})| \le 4$.*

(ii) *If $|E(v, V(S_i))| = 2$, then $|E(V(S_i), \{a_k, b_k, c_k\})| \le 8$ and $|E(V(S_i), \{b_k, c_k\})| \le 6$.*

(iii) *If $v \in V(G - H)$, $|E(v, V(S_i))| = 4$ and $\alpha \le 2$, then $E(V(S_i), \{b_k, c_k, d_k\}) = \emptyset$.*

(iv) *If $v \in V(G - H)$, $|E(v, V(S_i))| = 3$ and $\alpha \le 2$, then $|E(x, \{a_k, b_k, c_k\})| \le 1$ for each $x \in N_{S_i}(v)$ and, for the vertex z in $V(S_i) - N_G(v)$, we have $E(z, \{b_k, c_k, d_k\}) = \emptyset$ (so $|E(V(S_i), \{b_k, c_k\})| \le 3$).*

Proof. Assume that $|E(v, V(S_i))| \ge 2$. We first claim that $|E(x, \{a_k, b_k, c_k\})| \le 1$ for each $x \in V(S_i)$ such that $|E(v, V(S_i - x))| \ge 2$. To see this, take $x \in V(S_i)$ such that $|E(v, V(S_i - x))| \ge 2$. Then $\langle \{v\} \cup V(S_i - x)\rangle \supset K_4^-$. Hence by applying Lemma 2.3 with $S' = \{a_k, b_k, c_k\}$ and $X = \langle \{v\} \cup V(S_i - x)\rangle$, we obtain $|E(x, \{a_k, b_k, c_k\})| \le 1$, as claimed. Thus if $|E(v, V(S_i))| \ge 3$, then $|E(x, \{a_k, b_k, c_k\})| \le 1$ for each $x \in V(S_i)$, which proves (i) and the first assertion of (iv). Assume now that $|E(v, V(S_i))| = 2$. By symmetry, we may assume $N_{S_i}(v) = \{a_i, b_i\}$. Then by the above claim, $|E(\{c_i, d_i\}, \{a_k, b_k, c_k\})| \le 2$. Therefore, we obtain

$$|E(V(S_i), \{a_k, b_k, c_k\})| = |E(\{a_i, b_i\}, \{a_k, b_k, c_k\})| + |E(\{c_i, d_i\}, \{a_k, b_k, c_k\})| \le 6 + 2 = 8$$

and, since we clearly have $|E(\{c_i, d_i\}, \{b_k, c_k\})| \le |E(\{c_i, d_i\}, \{a_k, b_k, c_k\})|$, we also obtain $|E(V(S_i), \{b_k, c_k\})| = |E(\{a_i, b_i\}, \{b_k, c_k\})| + |E(\{c_i, d_i\}, \{b_k, c_k\})| \le 4 + 2 = 6$. Thus (ii) holds. Finally assume that $v \in V(G - H)$, $|E(v, V(S_i))| \ge 3$ and $\alpha \le 2$, and take $z \in V(S_i)$ such that $|E(v, V(S_i - z))| = 3$. If $E(z, \{b_k, c_k, d_k\}) \ne \emptyset$, then replacing S_i by $\langle \{v\} \cup V(S_i - z)\rangle$ and v by z, we get a contradiction to the maximality of α. Thus $E(z, \{b_k, c_k, d_k\}) = \emptyset$, which proves (iii) and the second assertion of (iv). □

Lemma 3.2. *Let $v \in V(G-H)$. Let $1 \le i \le k-1$, and suppose that $S_i \cong K_4^-$ and $|E(v, V(S_i))| \ge 3$. Then the following hold.*

(i) *We have $|E(V(S_i), \{a_k, b_k, c_k\})| \le 6$ and $|E(V(S_i), \{b_k, c_k\})| \le 5$.*

(ii) *If $\alpha \le 4$, then $|E(V(S_i), V(S_k))| \le 7$.*

(iii) *If $\alpha \le 2$, then $|E(u, V(S_i))| \le 1$ for each $u \in \{b_k, c_k, d_k\}$.*

Proof. By Lemma 2.1(i) and by symmetry, we may assume $N_{S_i}(v) = \{a_i, c_i, d_i\}$. Then for each $x \in \{b_i, c_i, d_i\}$, we get $|E(x, \{a_k, b_k, c_k\})| \le 1$ by applying Lemma 2.3 with $S' = \langle\{a_k, b_k, c_k\}\rangle$ and $X = \langle\{v\} \cup V(S_i - x)\rangle$. Hence we may deduce that $|E(\{b_i, c_i, d_i\}, \{a_k, b_k, c_k\})| \le 3$. Consequently,

$$|E(V(S_i), \{a_k, b_k, c_k\})| \le |E(a_i, \{a_k, b_k, c_k\})| + |E(\{b_i, c_i, d_i\}, \{a_k, b_k, c_k\})| \le 3 + 3 = 6$$

and, since we clearly have $|E(\{b_i, c_i, d_i\}, \{b_k, c_k\})| \le |E(\{b_i, c_i, d_i\}, \{a_k, b_k, c_k\})|$, we similarly obtain $|E(V(S_i), \{b_k, c_k\})| \le 2 + 3 = 5$. Now assume that $\alpha \le 4$. Take $x \in \{b_i, c_i, d_i\}$. If $|E(x, V(S_k))| \ge 2$, then replacing S_i by $\langle\{v\} \cup V(S_i - x)\rangle$ and v by x, we get a contradiction to the maximality of α. Thus $|E(x, V(S_k))| \le 1$ for each $x \in \{b_i, c_i, d_i\}$, and hence $|E(V(S_i), V(S_k))| \le 4 + 3 = 7$. Finally assume that $\alpha \le 2$, and let $u \in \{b_k, c_k, d_k\}$. Take $x \in \{b_i, c_i, d_i\}$. If $xu \in E(G)$, then as above, we get a contradiction to the maximality of α. Thus $xu \notin E(G)$ for each $x \in \{b_i, c_i, d_i\}$, which implies $|E(u, V(S_i))| \le 1$. □

Now fix $v \in V(G-H)$ such that $\mathrm{pr}(v) = \alpha$. We divide the proof of Theorem 1 into four cases according to the value of $\alpha = \mathrm{pr}(v)$.
Case 1: The case where $N_{S_k}(v) = \{a_k, d_k\}$ (i.e., $\alpha = 6$)

Lemma 3.3. *For each i with $1 \le i \le k-1$, $|E(\{b_k, c_k, d_k, v\}, V(S_i))| \le 10$.*

Proof. By way of contradiction, suppose that

$$|E(\{b_k, c_k, d_k, v\}, V(S_i))| \ge 11. \qquad \text{(A)}$$

We consider three subcases separately according as $S_i \cong K_4, K_4^-$ or S.

Subcase 1: $S_i \cong K_4$.

We first claim that if $N_{S_i}(d_k) \cap N_{S_i}(v) \ne \emptyset$, then $|E(x, V(S_i - y))| \le 1$ for each $y \in N_{S_i}(d_k) \cap N_{S_i}(v)$ and each $x \in \{b_k, c_k\}$. To see this, let $y \in N_{S_i}(d_k) \cap N_{S_i}(v)$ and $x \in \{b_k, c_k\}$. Then $\langle\{y, a_k, d_k, v\}\rangle \supset K_4^-$. Hence applying Lemma 2.3 with $S' = S - y$ and $X = \langle\{y, a_k, d_k, v\}\rangle$, we obtain $|E(x, V(S_i - y))| \le 1$, as claimed. Now suppose that $N_{S_i}(b_k) \cap N_{S_i}(c_k) \ne \emptyset$ and $N_{S_i}(d_k) \cap N_{S_i}(v) \ne \emptyset$. Then by the above claim, $|E(x, V(S_i))| \le 2$ for each $x \in \{b_k, c_k\}$ and, by the symmetry of the roles of $\{b_k, c_k\}$ and $\{d_k, v\}$, we similarly obtain $|E(x, V(S_i))| \le 2$ for each $x \in \{d_k, v\}$. Hence $|E(\{b_k, c_k, d_k, v\}, V(S_i))| \le 8$, which contradicts (A). Thus we have $N_{S_i}(b_k) \cap N_{S_i}(c_k) = \emptyset$ or $N_{S_i}(d_k) \cap N_{S_i}(v) = \emptyset$. We may assume

$N_{S_i}(b_k) \cap N_{S_i}(c_k) = \emptyset$. Then $|N_{S_i}(b_k)| + |N_{S_i}(c_k)| \le 4$, and hence $|N_{S_i}(d_k)| + |N_{S_i}(v)| \ge 7$ by (A), which implies $|N_{S_i}(d_k) \cap N_{S_i}(v)| \ge 3$. Therefore it follows from the claim made at the beginning of this subcase that $|E(x, V(S_i))| \le 1$ for each $x \in \{b_k, c_k\}$, and hence $|E(\{b_k, c_k\}, V(S_i))| \le 2$. Consequently $|E(b_k, c_k, d_k, v\}, V(S_i))| = |E(\{b_k, c_k\}, V(S_i))| + |E(\{d_k, v\}, V(S_i))| \le 2 + 8$, which contradicts (A).

Subcase 2: $S_i \cong K_4^-$.

By (A) and Lemma 2.1 and by symmetry, we may assume that $|E(v, V(\ S_i))| = |E(b_k, V(S_i))| = 3$. Then $|E(\{b_k, c_k\}, V(S_i))| \le 5$ by Lemma 3.2(i) and, by symmetry, we similarly obtain $|E(\{v, d_k\}, V(S_i))| \le 5$. Thus, $|E(\{b_k, c_k, d_k, v\}, V(S_i))| \le 10$, which contradicts (A).

Subcase 3: $S_i \cong S$.

By Lemma 2.1 and by symmetry, $|E(x, V(S_i))| \le 2$ for each $x \in \{b_k, c_k, d_k, v\}$, which implies that $|E(\{b_k, c_k, d_k, v\}, V(S_i))| \le 8$, a contradiction. □

Now for each $x \in V(G - H - v)$, Lemma 2.1(ii) implies that $|E(\{b_k, c_k\}, x)| \le 1$, and we also have $|E(\{d_k, v\}, x)| \le 1$ by symmetry. Hence by Lemma 3.3 , $d_G(b_k) + d_G(c_k) + d_G(d_k) + d_G(v) \le 10(k-1) + 2\{n - (4k+1)\} + 8 = 2n + 2k - 4$. On the other hand, $d_G(b_k) + d_G(c_k) + d_G(d_k) + d_G(v) \ge 2\sigma_2(G) \ge 2n + 2k$, which is a contradiction. This completes the proof for Case 1.

Case 2: The case where $N_{S_k}(v) = \{b_k, d_k\}, \{c_k, d_k\}, \{b_k\}, \{c_k\}$ or $\{d_k\}$ (i.e., $3 \le \alpha \le 5$)

By the symmetry of the roles of b_k and c_k, we may assume $N_{S_k}(v) = \{b_k, d_k\}, \{b_k\}$ or $\{d_k\}$.

Lemma 3.4. *For each i with $1 \le i \le k-1$, $2|E(v, V(S_i))| + |E(\{a_k, b_k, c_k, d_k\}, V(S_i))| \le 15$.*

Proof. By way of contradiction, suppose that

$$2|E(v, V(S_i))| + |E(\{a_k, b_k, c_k, d_k\}, V(S_i))| \ge 16. \tag{B}$$

As in Lemma 3.3, we consider three subcases separately.

Subcase 1: $S_i \cong K_4$.

Suppose that $|E(v, V(S_i))| \ge 3$ or $|E(d_k, V(S_i))| \ge 3$. Then by Lemma 3.1(i), $|E(V(S_i), \{a_k, b_k, c_k\})| \le 4$. By assumption (B), this implies that $|E(v, V(S_i))| = 4$, $|E(V(S_i), \{a_k, b_k, c_k\})| = 4$ and $|E(d_k, V(S_i))| = 4$. From $|E(V(S_i), \{a_k, b_k, c_k\})| = 4$, we see that there exist $x, y \in V(S_i)$ such that $N_G(x) \cap \{a_k, b_k, c_k\} \cap N_G(y) \ne \emptyset$. Now by replacing S_i by $\langle V(S_i - x) \cup \{v\} \rangle$ and v by x, we see from the maximality of α that $\alpha = 5$, that is to say, $N_{S_k}(v) = \{b_k, d_k\}$. This implies $\langle \{d_k, v\} \cup (V(S_i) - \{x, y\}) \rangle \cong K_4$. Consequently by replacing S_i, S_k and v by $\langle \{d_k, v\} \cup (V(S_i) - \{x, y\}) \rangle$, $\langle \{x, b_k, c_k, a_k\} \rangle$ and y, we get a contradiction to the maximality of α. Thus $|E(v, V(S_i))| \le 2$ and $|E(d_k, V(S_i))| \le 2$. If $|E(v, V(S_i))| = 2$ or $|E(d_k, V(S_i))| = 2$, then $|E(V(S_i), \{a_k, b_k, c_k\})| \le 8$ by Lemma 3.1(ii), and hence $2|E(v,V(S_i))| + |E(\{a_k,b_k,c_k,d_k\},V(S_i))| \le 4+2+8 = 14$, a contradiction. Thus $|E(v,V(S_i))| \le 1$ and $|E(d_k,V(S_i))| \le 1$. Consequently, $2|E(v,V(S_i))| + |E(\{a_k,b_k,c_k,d_k\},V(S_i))| \le 2+1+12 = 15$, which contradicts (B).

Subcase 2: $S_i \cong K_4^-$.

By Lemma 2.1, $|E(v,V(S_i))| \le 3$. We divide the proof into two subcases according as $b_k \notin N_{S_k}(v)$ or $b_k \in N_{S_k}(v)$.

Subcase 2.1: $N_{S_k}(v) = \{d_k\}$.

First let $|E(v,V(S_i))| = 3$. Then by Lemma 3.2(ii), $|E(V(S_i),\{a_k, b_k,c_k,d_k\})| \le 7$. Consequently, $2|E(v,V(S_i))| + |E(\{a_k,b_k,c_k,d_k\},V(S_i))| \le 6+7$, which contradicts (B).

Next assume that $|E(v,V(S_i))| = 2$. If $|E(v,\{a_i,b_i\})| = 2$, then arguing as in the proof of Lemma 3.2(ii), we see from the maximality of α that $|E(\{c_i,d_i\},V(S_k))| \le 2$, and hence $2|E(v,V(S_i))|+|E(\{a_k,b_k,c_k,d_k\},V(S_i))| \le 4+8+2 = 14$, a contradiction. Thus $|E(v,\{a_i,b_i\})| \le 1$. Assume for the morment that $|E(v,\{a_i,b_i\})| = 1$, say, $va_i, vc_i \in E(G)$. Then by the maximality of α, $|E(d_i,V(S_k)| \le 1$. If $|E(b_i,V(S_k))| \ge 3$, then replacing S_i, S_k and v by $\langle\{b_i,a_k,b_k,c_k\}\rangle$, $\langle\{a_i,c_i,d_i,v\}\rangle$ and d_k, respectively, we get a contradiction to the maximality of α. Thus we also have $|E(b_i,V(S_k))| \le 2$. Consequently, $2|E(v,V(S_i))| + |E(\{a_k,b_k,c_k,d_k\},V(S_i))| \le 4+8+2+1 = 15$, a contradiction.

Thus we are reduced to the case where $N_{S_i}(v) = \{c_i,d_i\}$. Suppose that $N_G(d_k) \cap \{a_i,b_i\} \ne \emptyset$. Then we see from the maximality of α that $|E(\{c_i,d_i\},\{a_k,b_k,c_k\})| \le 2$, and hence $2|E(v,V(S_i))| + |E(\{a_k,b_k,c_k,d_k\},V(S_i))| \le 4+8+2+2 = 16$. In view of (B), this implies that $|E(d_k,\{c_i,d_i\})| = 2$ and $|E(\{a_k,b_k,c_k,d_k\},\{a_i,b_i\})| = 8$. Consequently replacing S_i and S_k by $\langle\{v,d_k,c_i,d_i\}\rangle$ and $\langle\{b_i,a_k,b_k,c_k\}\rangle$, we get a contradiction to the maximality of k'. Thus $N_G(d_k) \cap \{a_i,b_i\} = \emptyset$, which implies that $|E(d_k,V(S_i))| = |E(d_k,\{c_i,d_i\})| \le 2$. Since we are assuming $N_{S_i}(v) = \{c_i,d_i\}$, we also have $|E(\{c_i,d_i\},\{a_k,b_k,c_k\})| \le 4$ by Lemma 2.4. By (B), it follows that $|E(d_k,\{c_i,d_i\})| = 2$ and $|E(\{a_k,b_k,c_k\},\{a_i,b_i\})| = 6$. Consequently by replacing S_i and S_k by $\langle\{v,d_k,c_i,d_i\}\rangle$ and $\langle\{b_i,a_k,b_k,c_k\}\rangle$, respectively, we get a contradiction to the maximality of k'. This concludes the discussion for the case where $|E(v,V(S_i))| = 2$.

Finally assume $|E(v,V(S_i))| \le 1$. If $|E(c_i,\{a_k,b_k,c_k\})| = 3$ or $|E(d_i,\{a_k,b_k,c_k\})| = 3$, then $|E(d_k,V(S_i))| \le 1$ by Lemma 2.4, and hence

$$2|E(v,V(S_i))| + |E(\{a_k,b_k,c_k,d_k\},V(S_i))| \le 2+1+12 = 15,$$

a contradiction. Thus $|E(c_i,\{a_k,b_k,c_k\})| \le 2$ and $|E(d_i,\{a_k,b_k,c_k\})| \le 2$. By (B), this implies that $|E(d_k,V(S_i))| = 4$ and $|E(c_i,\{a_k,b_k,c_k\})| = 2$. Therefore replacing S_i and S_k by $\langle\{d_k,a_i,b_i,d_i\}\rangle$ and $\langle\{c_i,a_k,b_k,c_k\}\rangle$, we get a contradiction to the maximality of k'.

Subcase 2.2: $N_{S_k}(v) = \{b_k,d_k\}$ or $N_{S_k}(v) = \{b_k\}$.

By the symmetry of v and d_k in $\langle V(S_k) \cup \{v\}\rangle$, we have $|E(d_k,V(S_i))| \le 3$ by Lemma 2.1. If $|E(v,V(S_i))| = 3$ or $|E(d_k,V(S_i))| = 3$, then $|E(V(S_i),\{a_k,b_k,c_k\})| \le 6$ by Lemma 3.2(i), and hence $2|E(v,V(S_i))| + |E(\{a_k,b_k,c_k,d_k\},V(S_i))| \le 6 +$

$3 + 6 = 15$, a contradiction. Thus $|E(v, V(S_i))| \leq 2$ and $|E(d_k, V(S_i))| \leq 2$. Hence by (B), $|E(V(S_i), \{a_k, b_k, c_k\})| \geq 10$. Now if $|E(c_i, \{a_k, b_k, c_k\})| = 3$ or $|E(d_i, \{a_k, b_k, c_k\})| = 3$, then $|E(v, V(S_i))| \leq 1$ and $|E(d_k, V(S_i))| \leq 1$ by Lemma 2.4, and hence $2|E(v, V(S_i))| + |E(\{a_k, b_k, c_k, d_k\}, V(S_i))| \leq 3 + 12 = 15$, a contradiction. Thus $|E(c_i, \{a_k, b_k, c_k\})| \leq 2$ and $|E(d_i, \{a_k, b_k, c_k\})| \leq 2$. Consequently $|E(c_i, \{a_k, b_k, c_k\})| = 2$, $|E(d_i, \{a_k, b_k, c_k\})| = 2$, $|E(\{a_i, b_i\}, \{a_k, b_k, c_k\})| = 6$ and $|E(v, V(S_i))| = |E(d_k, V(S_i))| = 2$. If $va_i \in E(G)$, we get a contradiction to the maximality of k' by replacing S_i and S_k either by $\langle\{a_i, b_i, c_i, v\}\rangle$ and $\langle\{a_k, b_k, c_k, d_i\}\rangle$ or by $\langle\{a_i, b_i, d_i, v\}\rangle$ and $\langle\{a_k, b_k, c_k, c_i\}\rangle$. Thus $va_i \notin E(G)$. Similarly $vb_i \notin E(G)$. Hence $N_{S_i}(v) = \{c_i, d_i\}$. By symmetry, we similarly obtain $N_{S_i}(d_k) = \{c_i, d_i\}$. Since $|E(c_i, \{a_k, b_k, c_k\})| = 2$, we have $c_i a_k \in E(G)$ or $c_i b_k \in E(G)$. By the symmetry of a_k and b_k in $\langle V(S_k) \cup \{v\}\rangle$, we may assume $c_i b_k \in E(G)$. But then replacing S_i and S_k by $\langle\{a_i, c_i, b_k, v\}\rangle$ and $\langle\{b_i, d_i, a_k, c_k\}\rangle$, we obtain a contradiction to the maximality of k'.

Subcase 3: $S_i \cong S$.

First assume that $|E(v, V(S_i))| \leq 1$. Then by (B), $|E(\{a_k, b_k, c_k, d_k\}, V(S_i))| \geq 14$. If $|E(d_k, \{a_i, b_i, c_i\})| + |E(d_i, \{a_k, b_k, c_k\})| \geq 4$, then $E(d_k, \{a_i, b_i, c_i\}) \neq \emptyset$ and $E(d_i, \{a_k, b_k, c_k\}) \neq \emptyset$ and we have $|E(d_k, \{a_i, b_i, c_i\})| \geq 2$ or $|E(d_i, \{a_k, b_k, c_k\})| \geq 2$, and hence we get a contradiction to the maximality of k' by replacing S_i and S_k by $\langle\{a_i, b_i, c_i, d_k\}\rangle$ and $\langle\{a_k, b_k, c_k, d_i\}\rangle$. Thus $|E(d_k, \{a_i, b_i, c_i\})| + |E(d_i, \{a_k, b_k, c_k\})| \leq 3$. Hence $|E(V(S_k), V(S_i))| = |E(d_k, d_i)| + (|E(d_k, \{a_i, b_i, c_i\})| + |E(\{a_k, b_k, c_k\}, d_i)|) + |E(\{a_k, b_k, c_k\}, \{a_i, b_i, c_i\})| \leq 1 + 3 + 9$, which contradicts the earlier assertion that $|E(V(S_k), V(S_i))| \geq 14$. Next assume that $|E(v, V(S_i))| \geq 2$. By the maximality of α, this forces $|E(v, V(S_i))| = 2$, $N_{S_k}(v) = \{b_k, d_k\}$, $d_i \in N_{S_i}(v)$ and $|\{b_i, c_i\} \cap N_{S_i}(v)| = 1$. Hence applying Lemma 2.1 with the roles of v and d_k interchanged we obtain $|E(d_k, \{a_i, b_i, c_i\})| \leq 1$. Further applying Lemma 2.1 to S_k with the roles of v and d_i interchanged, we obtain $|E(d_i, \{a_k, b_k, c_k\})| \leq 1$. Consequently $|E(V(S_k), V(S_i))| = |E(d_k, d_i)| + |E(d_k, \{a_i, b_i, c_i\})| + |E(\{a_k, b_k, c_k\}, d_i)| + |E(\{a_k, b_k, c_k\}, \{a_i, b_i, c_i\})| \leq 1+1+1+9$. In view of (B), this forces $d_k d_i \in E(G)$ and $|E(\{a_k, b_k, c_k\}, \{a_i, b_i, c_i\})| = 9$. Therefore $\langle\{a_i, d_i, v, d_k\}\rangle \supset S$ and $\langle\{b_i, a_k, b_k, c_k\}\rangle \cong K_4$, and hence we get a contradiction to the maximality of k' by replacing S_i and S_k by $\langle\{a_i, d_i, v, d_k\}\rangle$ and $\langle\{b_i, a_k, b_k, c_k\}\rangle$. □

For each $x \in V(G - H - v)$, $2|E(v, x)| + |E(\{a_k, b_k, c_k, d_k\}, x)| \leq 4$ by Lemma 2.1 and, if equality holds, then $d_k, v \in N_G(x)$ by Lemma 2.1, and $N_{S_k}(v) = \{b_k, d_k\}$ (and $|N_G(x) \cap \{b_k, c_k\}| = 1$) by the maximality of $\mathrm{pr}(v)$. Thus if there exist two vertices $x, y \in V(G - H - v)$ such that $2|E(v, x)| + |E(\{a_k, b_k, c_k, d_k\}, x)| = 2|E(v, y)| + |E(\{a_k, b_k, c_k, d_k\}, y)| = 4$, then by replacing S_k by $\langle\{x, y, v, d_k\}\rangle$, we get a contradiction to the maximality of k'. Consequently, by Lemma 3.4, $2d_G(v) + d_G(a_k) + d_G(b_k) + d_G(c_k) + d_G(d_k) \leq 15(k-1) + 3(n - 4k - 1) + 1 + 14 = 3n + 3k - 3$. On the other hand, by the assumption that $\sigma_2(G) \geq n + k$, $2d_G(v) + d_G(a_k) + d_G(b_k) + d_G(c_k) + d_G(d_k) \geq 3n + 3k$. This is a contradiction, and this completes the proof for Case 2.

Case 3: The case where $N_{S_k}(v) = \emptyset$ (i.e., $\alpha = 2$)

Lemma 3.5. *For each i with $1 \le i \le k-1$,*

$$4|E(v, V(S_i))| + |E(\{a_k, b_k, c_k, d_k\}, V(S_i))| \le 20.$$

Proof. By way of contradiction, suppose that

$$4|E(v, V(S_i))| + |E(\{a_k, b_k, c_k, d_k\}, V(S_i))| \ge 21. \qquad \text{(C)}$$

Then $|E(v, V(S_i))| \ge 2$. By the maximality of α, this implies that $S_i \not\cong S$.

Subcase 1: $S_i \cong K_4$.

If $|E(v, V(S_i))| = 4$, then by Lemma 3.1(iii), $E(\{b_k, c_k, d_k\}, V(S_i)) = \emptyset$, which implies that $4|E(v, V(S_i))| + |E(\{a_k, b_k, c_k, d_k\}, V(S_i))| \le 16 + 4$, a contradiction. If $|E(v, V(S_i))| = 3$, then $|E(V(S_i), \{a_k, b_k, c_k\})| \le 4$ by Lemma 3.1(i), and hence $4|E(v, V(S_i))| + |E(\{a_k, b_k, c_k, d_k\}, V(S_i))| \le 12+4+4 = 20$, a contradiction. Finally if $|E(v, V(S_i))| = 2$, then $|E(V(S_i), \{a_k, b_k, c_k\})| \le 8$ by Lemma 3.1(ii), and hence $4|E(v, V(S_i))| + |E(\{a_k, b_k, c_k, d_k\}, V(S_i))| \le 8 + 4 + 8 = 20$, a contradiction.

Subcase 2: $S_i \cong K_4^-$.

By Lemma 2.1, $|E(v, V(S_i))| \le 3$. If $|E(v, V(S_i))| = 3$, then by Lemma 3.2(ii), $|E(V(S_i), \{a_k, b_k, c_k, d_k\})| \le 7$ and hence

$$4|E(v, V(S_i))| + |E(\{a_k, b_k, c_k, d_k\}, V(S_i))| \le 12 + 7 = 19,$$

a contradiction. Thus $|E(v, V(S_i))| = 2$. By Lemma 2.4, $|E(c_i, \{a_k, b_k, c_k\})| \le 2$ and $|E(d_i, \{a_k, b_k, c_k\})| \le 2$. Since we clearly have $|E(\{a_i, b_i\}, \{a_k, b_k, c_k\})| \le 6$, this together with (C) implies that we have $|E(c_i, \{a_k, b_k, c_k\})| = 2$ or $|E(d_i, \{a_k, b_k, c_k\})| = 2$, and $|E(d_k, V(S_i))| \ge 3$. By the symmetry of the roles of c_i and d_i, we may assume $|E(c_i, \{a_k, b_k, c_k\})| = 2$. Then replacing S_i and S_k by $\langle\{d_k, a_i, b_i, d_i\}\rangle$ and $\langle\{c_i, a_k, b_k, c_k\}\rangle$, we get a contradiction to the maximality of k'. □

Let $x \in V(G - H - v)$. By the maximality of $\mathrm{pr}(v)$, $N_{S_k}(x) \subset \{a_k\}$. Further if $xa_k, xv \in E(G)$, then by replacing S_k by $\langle\{x\} \cup V(S_k - d_k)\rangle$, we get a contradiction to the maximality of α. Hence $4|E(v, x)| + |E(\{a_k, b_k, c_k, d_k\}, x)| \le 4$. Consequently, by Lemma 3.5, $4d_G(v) + d_G(a_k) + d_G(b_k) + d_G(c_k) + d_G(d_k) \le 20(k-1) + 4(n - 4k - 1) + 8 = 4n + 4k - 16$. On the other hand, by the assumption that $\sigma_2(G) \ge n + k$, $4d_G(v) + d_G(a_k) + d_G(b_k) + d_G(c_k) + d_G(d_k) \ge 4n + 4k$. This is a contradiction, and this completes the proof for Case 3.

Case 4: The case where $N_{S_k}(v) = \{a_k\}$ (i.e.,$\alpha = 1$)

Lemma 3.6. *For each i with $1 \le i \le k-1$, $2|E(\{v, d_k\}, V(S_i))| + |E(\{b_k, c_k\}, V(S_i))| \le 16$, and equality holds only if $S_i \cong K_4^-$, $N_{S_i}(v) = N_{S_i}(d_k) = \{c_i, d_i\}$, $N_G(b_k) \supset V(S_i)$ and $N_G(c_k) \supset V(S_i)$.*

Proof. Suppose that

$$2|E(\{v, d_k\}, V(S_i))| + |E(\{b_k, c_k\}, V(S_i))| \ge 16. \qquad \text{(D)}$$

Then we have $|E(v, V(S_i))| \geq 2$ or $|E(d_k, V(S_i))| \geq 2$. By the symmetry of the roles of v and d_k, we may assume that $|E(v, V(S_i))| \geq 2$. Then by the maximality of α, $S_i \not\cong S$.

Subcase 1: $S_i \cong K_4$.

If $|E(v, V(S_i))| = 4$, then $E(V(S_i), \{b_k, c_k, d_k\}) = \emptyset$ by Lemma 3.1(iii), and hence $2|E(\{v, d_k\}, V(S_i))| + |E(\{b_k, c_k\}, V(S_i))| \leq 8$, a contradiction. Thus $|E(v, V(S_i))| \leq 3$ and, by symmetry, we similarly obtain $|E(d_k, V(S_i))| \leq 3$. If $|E(v, V(S_i))| = 3$, then $|E(V(S_i), \{b_k, c_k\})| \leq 3$ by Lemma 3.1(iv), and hence $2|E(\{v, d_k\}, V(S_i))| + |E(\{b_k, c_k\}, V(S_i))| \leq 12 + 3 = 15$, a contradiction. Thus $|E(v, V(S_i))| = 2$, and we similarly obtain $|E(d_k, V(S_i))| \leq 2$. Now by Lemma 3.1(ii), $|E(V(S_i), \{b_k, c_k\})| \leq 6$, and hence $2|E(\{v, d_k\}, V(S_i))| + |E(\{b_k, c_k\}, V(S_i))| \leq 8 + 6 = 14$, a contradiction.

Subcase 2: $S_i \cong K_4^-$.

By Lemma 2.1, $|E(v, V(S_i))| \leq 3$. By symmetry, we also have $|E(d_k, V(S_i))| \leq 3$. If $|E(v, V(S_i))| = 3$, then $|E(V(S_i), \{b_k, c_k\})| \leq 2$ by Lemma 3.2(iii), and hence $2|E(\{v, d_k\}, V(S_i))| + |E(\{b_k, c_k\}, V(S_i))| \leq 12 + 2 = 14$, a contradiction. Thus $|E(v, V(S_i))| = 2$. By symmetry, we similarly obtain $|E(d_k, V(S_i))| \leq 2$. In view of (D), this implies that $|E(d_k, V(S_i))| = 2$, $N_G(b_k) \supseteq V(S_i)$ and $N_G(c_k) \supseteq V(S_i)$. If $N_G(v) \cap \{a_i, b_i\} \neq \emptyset$, then $|E(\{c_i, d_i\}, \{b_k, c_k\})| \leq 3$ by Lemma 2.4, which contradicts the assertion that $N_G(b_k) \supseteq V(S_i)$ and $N_G(c_k) \supseteq V(S_i)$.Thus $N_{S_i}(v) = \{c_i, d_i\}$, and we similarly obtain $N_{S_i}(d_k) = \{c_i, d_i\}$. □

Note that for each $x \in V(G-H-v)$, $N_G(x) \cap \{v, b_k, c_k, d_k\} = \emptyset$ by the maximality of $\mathrm{pr}(v)$. Suppose that $2|E(\{v, d_k\}, V(S_i))| + |E(\{b_k, c_k\}, V(S_i))| \leq 15$ for each i with $1 \leq i \leq k-1$. Then $2(d_G(v) + d_G(d_k)) + d_G(b_k) + d_G(c_k) \leq 15(k-1) + 8 = 15k - 7$. On the other hand, by the assumption that $\sigma_2(G) \geq n + k$, $2(d_G(v) + d_G(d_k)) + d_G(b_k) + d_G(c_k) \geq 3(n+k) \geq 15k + 3$. This is a contradiction. Thus we may assume that $2|E(\{v, d_k\}, V(S_1))| + |E(\{b_k, c_k\}, V(S_1))| \geq 16$. Then by Lemma 3.6, $2|E(\{v, d_k\}, V(S_1))| + |E(\{b_k, c_k\}, V(S_1))| = 16$, $S_1 \cong K_4^-$, $N_{S_1}(v) = N_{S_1}(d_k) = \{c_1, d_1\}$, $N_G(b_k) \supset V(S_1)$ and $N_G(c_k) \supset V(S_1)$. If $a_k c_1 \in E(G)$, then by replacing S_1 and S_k by $\langle \{v, c_1, a_k, d_k\} \rangle$ and $\langle \{b_k, a_1, b_1, d_1\} \rangle$, we get a contradiction to the maximality of k'. Thus $a_k c_1 \notin E(G)$. Now we prove the following lemma.

Lemma 3.7. *For each i with $2 \leq i \leq k-1$,*
$$2|E(\{v, d_k\}, V(S_i))| + |E(\{a_k, c_1\}, V(S_i))| \leq 15.$$

Proof. By way of contradiction, suppose that
$$2|E(\{v, d_k\}, V(S_i))| + |E(\{a_k, c_1\}, V(S_i))| \geq 16. \tag{E}$$
By the symmetry of the roles of v and d_k, we may assume that $|E(v, V(S_i))| \geq |E(d_k, V(S_i))|$.

Subcase 1: $S_i \cong K_4$.

First we consider the case $|E(v, V(S_i))| = 4$. By Lemma 3.1(iii), $E(d_k, V(S_i)) = \emptyset$. In view of (E), this forces $|E(a_k, V(S_i))| = 4$ and $|E(c_1, V(S_i))| = 4$. Hence by replacing S_1, S_i and S_k by $\langle \{a_1, b_1, b_k, c_k\} \rangle, \langle \{c_1, a_i, b_i, c_i\} \rangle$ and $\langle \{d_i, v, a_k, d_k\} \rangle$, respectively, we get a contradiction to the maximality of $\sum_{j=1}^k |E(S_j)|$. Next we consider the case $|E(v, V(S_i))| = 3$. Note that we have $N_{S_i}(v) \supset N_{S_i}(d_k)$ by Lemma 3.1(iv). Assume first that $|E(d_k, V(S_i))| = 3$. Then $N_{S_i}(v) = N_{S_i}(d_k)$. Suppose that $N_{S_i}(v) \cap N_{S_i}(a_k) \neq \emptyset$, and let $y \in N_{S_i}(v) \cap N_{S_i}(a_k)$. Then by replacing S_1, S_i, S_k and v by $\langle\{c_k, a_1, b_1, d_1\}\rangle, \langle\{d_k\} \cup V(S_i - y)\rangle, \langle\{y, v, a_k, b_k\}\rangle$ and c_1, respectively, we get a contradiction to the maximality of α. Thus $N_{S_i}(v) \cap N_{S_i}(a_k) = \emptyset$, which implies that $|E(a_k, V(S_i))| \leq 1$. If $|E(a_k, V(S_i))| = \emptyset$, then by replacing S_i by $\langle\{v\} \cup N_{S_i}(v)\rangle$ and v by the vertex in $V(S_i) - N_G(v)$, we get a contradiction to the maximality of α. Thus $|E(a_k, V(S_i))| = 1$. Also, by (E), we have $|E(c_1, V(S_i))| \geq 3$. Take $y \in N_{S_i}(v) \cap N_{S_i}(d_k) \cap N_{S_i}(c_1)$. Then by replacing S_1, S_i, S_k and v by $\langle\{c_k, a_1, b_1, d_1\}\rangle$, $\langle\{y, v, d_k, c_1\}\rangle$, $\langle\{a_k\} \cup V(S_i - y)\rangle$ and b_k, respectively, we get a contradiction to the maximality of α because $|E(a_k, V(S_i) - \{x\})| = 1$ and $a_k b_k \in E(G)$. Assume now that $|E(d_k, V(S_i))| \leq 2$. Then by (E), $1 \leq |E(d_k, V(S_i))| \leq 2$ and $|E(c_1, V(S_i))| + |E(a_k, V(S_i))| \geq 6$. Suppose that $N_{S_i}(v) \cap N_{S_i}(d_k) \cap N_{S_i}(c_1) \neq \emptyset$, and let $y \in N_{S_i}(v) \cap N_{S_i}(d_k) \cap N_{S_i}(c_1)$. Since $|E(c_1, V(S_i))| + |E(a_k, V(S_i))| \geq 6$, we have $N_{S_i}(a_k) - \{y\} \neq \emptyset$. Hence by replacing S_1, S_i, S_k and v by $\langle\{c_k, a_1, b_1, d_1\}\rangle, \langle\{y, v, d_k, c_1\}\rangle, \langle V(S_i - y) \cup \{a_k\}\rangle$ and b_k, respectively, we get a contradiction to the maximality of α. Thus $N_{S_i}(v) \cap N_{S_i}(d_k) \cap N_{S_i}(c_1) = \emptyset$. Since $N_{S_i}(v) \supset N_{S_i}(d_k)$, this together with (E) implies that $|E(d_k, V(S_i))| = |E(c_1, V(S_i))| = 2$ and $|E(a_k, V(S_i))| = 4$. Let $y \in N_{S_i}(v) \cap N_{S_i}(a_k) - N_G(c_1)$. Then by replacing S_1, S_i, S_k and v by $\langle\{a_1, b_1, b_k, c_k\}\rangle, \langle V(S_i - y) \cup \{c_1\}\rangle, \langle\{y, v, a_k, d_k\}\rangle$ and d_1, respectively, we get a contradiction to the maximality of α because $d_1 d_k, d_1 v \in E(G)$. This concludes the discussion for the case $|E(v, V(S_i))| = 3$. Finally we consider the case $|E(v, V(S_i))| \leq 2$. By (E), $|E(v, V(S_i))| = |E(d_k, V(S_i))| = 2, |E(\{a_k, c_1\}, V(S_i))| = 8$. Let $y \in N_{S_i}(v) \cap N_{S_i}(a_k)$. Then by replacing S_1, S_i and S_k by $\langle\{a_1, b_1, b_k, c_k\}\rangle, \langle V(S_i - y) \cup \{c_1\}\rangle$ and $\langle\{y, v, a_k, d_k\}\rangle$, respectively, we get a contradiction to the maximality of $\sum_{j=1}^k |E(S_j)|$.

Subcase 2: $S_i \cong K_4^-$.

By Lemma 2.1, $|E(v, V(S_i))| \leq 3$. Suppose that $|E(v, V(S_i))| = 3$. Then $c_i, d_i \in N_G(v)$ by Lemma 2.1. By Lemma 3.2(iii), we also have $|E(d_k, V(S_i))| \leq 1$. By (E), this forces $|E(a_k, V(S_i))| = |E(c_1, V(S_i))| = 4$. Consequently by replacing S_1, S_i and S_k by $\langle\{c_k\} \cup V(S_1 - c_1)\rangle, \langle\{c_1\} \cup V(S_i - c_i)\rangle$ and $\langle\{c_i, v, a_k, d_k\}\rangle$, we get a contradiction to the maximality of $\sum_{j=1}^k |E(S_j)|$. Thus $|E(v, V(S_i))| \leq 2$. By (E), this forces $|E(v, V(S_i))| = 2, |E(d_k, V(S_i))| = 2$, and $|E(\{a_k, c_1\}, V(S_i))| = 8$. Suppose that $N_{S_i}(v) \cap N_{S_i}(d_k) \neq \emptyset$, and take $y \in N_{S_i}(v) \cap N_{S_i}(d_k)$. Then replacing S_1, S_i and S_k by $\langle\{c_k, a_1, b_1, d_1\}\rangle$, $\langle\{a_k\} \cup V(S_i - y)\rangle$ and $\langle\{y, v, c_1, d_k\}\rangle$, we get a contradiction to the maximality of k'. Thus $N_{S_i}(v) \cap N_{S_i}(d_k) = \emptyset$. Since $|E(d_k, V(S_i))| = 2$ and $|E(v, V(S_i))| = 2$, we may assume $c_i d_k \in E(G)$ by symmetry. Then by replacing S_i by $\langle V(S_i - c_i) \cup \{v\}\rangle$ and v by c_i, we get a contradiction to the maximality of α.

Subcase 3: $S_i \cong S$.

By the maximality of α, $E(v, V(S_i)) \subset \{a_i v\}$ and $E(d_k, V(S_i)) \subset \{a_i d_k\}$, and hence $2|E(\{v, d_k\}, V(S_i))| + |E(\{a_k, c_1\}, V(S_i))| \leq 4 + 8 = 12$. This is a contradiction. □

For each $x \in V(G - H - v)$, $E(x, \{v, c_1, a_k, d_k\}) \subset \{xc_1, xa_k\}$ by the maximality of α, and hence $2|E(\{v, d_k\}, x)| + |E(\{a_k, c_1\}, x)| \leq 2$. Consequently, by Lemma 3.7, $2(d_G(v) + d_G(d_k)) + d_G(a_k) + d_G(c_1) \leq 15(k-2) + 2(n-4k-1) + 2(3+3) + 8 + 6 = 2n + 7k - 6$. On the other hand, by the assumption that $\sigma_2(G) \geq n + k$, $2(d_G(v)+d_G(d_k))+d_G(a_k)+d_G(c_1) \geq 3n+3k$. Since $n \geq 4k+1$, this is a contradiction.

This completes the proof of Theorem 1.

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(Received 17 May 2003; revised 10 Oct 2003)