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# Supersolvability of complementary signed-graphic hyperplane arrangements

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**Abstract**

We study a class of hyperplane arrangements associated to complementary signed graphs in which the positive part and the negative part are complementary to each other in $K_n$, the complete graph on $n$ vertices. These arrangements form a subclass of the $D_n$ arrangement but do not contain the $A_{n-1}$ arrangement. The main result says that the arrangement $\mathcal{A}(G)$ of a complementary signed graph $G$ is supersolvable if and only if the graph $G$ is switching equivalent to a complementary signed graph with negative part a star. We also prove that if $G$ is not switching equivalent to $\bar{G}$ and $\mathcal{A}(G)$ is supersolvable, then $\mathcal{A}(\bar{G})$ is not supersolvable, where $\bar{G}$ is the opposite graph of $G$.

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# 1 Introduction

Much of the motivation for the study of hyperplane arrangements comes from the study of Coxeter arrangements. The present work is motivated by the works of P. H. Edelman and V. Reiner [2] in which they characterized the freeness and supersolvability of subarrangements between $A_{n-1}$ and $B_n$ combinatorially. Bailey [1] studied the arrangements corresponding to the subgraphs of $B_n$ with $G^+ = G^-$. These results are formulated by relating these arrangements to signed graphs firstly by T. Zaslavsky [7].

We study the graphic arrangement associated to a signed graph $G$ consisting of a positive graph $G^+$ and a negative graph $G^-$, such that the union of $G^+$ and $G^-$ is exactly $K_n$, the complete graph on $n$ vertices. These graphs are called complementary signed graphs. For a complementary signed graph $G$ the following questions arise naturally. Is there any graph-theoretic criterion for $G$ to characterize the supersolvability of $\mathcal{A}(G)$? Is there any relation between the exponents of $\mathcal{A}(G)$ and the numerical invariants of $G$? If one switches the signs of all the edges of $G$ and thus obtains the opposite graph $\bar{G}$, will the supersolvability of the arrangement $\mathcal{A}(G)$ be changed? We answer these questions positively by proving the following main theorems. The notations and terminologies will be explained in §2.

**Theorem 1** *Let $G$ be a complementary signed graph. The arrangement $\mathcal{A}(G)$ is supersolvable if and only if $G$ is switching equivalent to a signed graph $\tilde{G}$ with $\tilde{G}^-$ a star. And in this case,*

$$exp(\mathcal{A}(G)) = (k, n-k-1, 1, 2, \ldots, n-3, n-2).$$

*where $k = |E(\tilde{G}^-)|$, the cardinality of $E(\tilde{G}^-)$ $(1 \leq k \leq n-1)$.*

**Theorem 2** *Let $G$ be a complementary signed graph and $\bar{G}$ the opposite graph. Assume that $G$ is not switching equivalent to $\bar{G}$. If the arrangement $\mathcal{A}(G)$ is supersolvable, $\mathcal{A}(\bar{G})$ is not supersolvable.*

We remark that the arrangements considered in Theorem 1 are subarrangements of $\mathcal{A}(D_n)$, but, in general, they do not contain $\mathcal{A}(A_{n-1})$. The converse of Theorem 2 is formulated as conjecture 13 and is proved in case $n = 4, 5$ in §3.

We are grateful to the referee for informing us about the paper [8], in which T. Zaslavsky obtained a result (among others) similar to Theorem 1 by using a matroid theory associated to biased graphs, a large theory developed by himself in a series of papers [9].

# 2 Preliminaries

## 2.1 Supersolvable arrangements

The standard reference on arrangements of hyperplanes is the book by P. Orlik and H. Terao [4]. All notations and terminologies not defined in this paper can be found in that book. A subarrangement $\mathcal{A}'$ of $\mathcal{A}$ is *a modular coatom* of $\mathcal{A}$ if

1) For all pairs of hyperplanes $H_1, H_2 \in \mathcal{A} \setminus \mathcal{A}'$ there exists a hyperplane $H_3 \in \mathcal{A}'$ such that $H_1 \cap H_2 \subset H_3$ (called *the inclusion condition*);

2) $\text{rank}\mathcal{A}' = \text{rank}\mathcal{A} - 1$ (called *the rank condition*).

An arrangement $\mathcal{A}$ is *supersolvable* [6, 2] if there exists a chain of arrangements (called M-chain) $\emptyset = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_r = \mathcal{A}$ such that $\mathcal{A}_j$ is a modular coatom of $\mathcal{A}_{j+1}$ for all $0 \leq j \leq r-1$, where $r$ is the rank of $\mathcal{A}$. This is equivalent to saying that the intersection lattice $L(\mathcal{A})$ is a supersolvable lattice (see [4, §3.2] and [2]).

## 2.2 Signed graphs

A graph $G = (V, E)$ is an ordered pair in which $V = V(G) = \{1, 2, \ldots n\} = [n]$, called the vertex set, and $E = E(G)$, called edge set of $G$, is a collection of two-element subset of $V$. Note that a graph may have loops and multiple edges. A graph is said to be simple if it does not have multiple edges. A graph $H = (U, F)$ is called a subgraph of $G$ if $U \subset V$ and $F \subset E$. For $U \subset V$, let $E(U) = \{\{i, j\} \mid i, j \in U, \text{ and } \{i, j\} \in E\}$. The graph $G_U = (U, E(U))$ is called the (vertex) induced subgraph of $G$ on $U$. A *cycle* $C$ in a graph $G$ is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that $\{v_j, v_{j+1}\} \in E$ for $0 \leq j \leq k-1$ and $v_i \neq v_j$ for all $i$ and $j$ except that $v_0 = v_k$. A *tree* is a simple graph without cycles. A *star* is a tree in which all the edges incident to one vertex (the *center*). A *chord* of $C$ is an edge $\{v_i, v_j\}$ where $v_i$ and $v_j$ are not consecutive vertices on the cycle $C$. A graph is said to be chordal if every cycle of length greater than three has a chord.

A *signed graph* $G = (G^+, G^-)$ (without loop) consists of a simple graph $G^+ = (V, E^+)$, a simple graph $G^- = (V, E^-)$ on the same vertex set $V$. If an edge $\{i, j\} \in E(G)$ belongs to $E^+$, it is denoted by $\{i, j\}$ and is pictured as a line segment connecting the vertices $i$ and $j$. If an edge $\{i, j\} \in E(G)$ belongs to $E^-$, it is denoted by $\overline{\{i, j\}}$ and is pictured as a dashed line segment connecting the vertices $i$ and $j$.

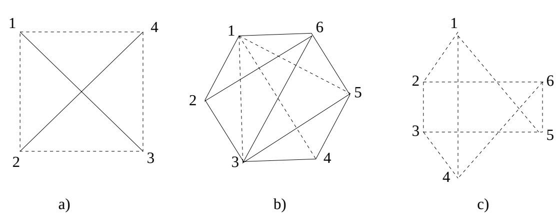


Figure 1:

Let $\mathbb{K}$ be the field of real or complex numbers. Given a signed graph $G = (G^+, G^-)$ with $V = [n]$, define an arrangement $\mathcal{A}(G)$ in $\mathbb{K}^n$ as follows.

$$x_i = x_j \in \mathcal{A}(G) \text{ if } \{i, j\} \in E(G^+), \text{ and } \quad x_i = -x_j \in \mathcal{A}(G) \text{ if } \overline{\{i, j\}} \in E(G^-).$$

The graph a) in Figure 1 consists of $E(G_a^+) = \{\{1,3\},\{2,4\}\}$ and $E(G_a^-) = \{\overline{\{1,2\}},\overline{\{2,3\}},\overline{\{3,4\}},\overline{\{1,4\}}\}$. The corresponding arrangement $\mathcal{A}(G_a) = \{x_1 = x_3, x_2 = x_4, x_1 = -x_2, x_2 = -x_3, x_3 = -x_4, x_1 = -x_4\}$.

For a signed graph $G = (G^+, G^-)$, we define the *negative degree* of a vertex $v$ to be the degree of $v$ in $G^-$ and denote it by $\mathrm{ndeg}(v)$.

### 2.3 Operations on signed graph

We consider the following operations on a signed graph $G$.

1) A permutation of the labels on the vertices of $G$.

2) For a vertex $v \in V$, switch the sign of the edge $\{v,i\}$ for all $i \in N_v := \{i \in [n] \mid \{v,i\} \in E(G)\}$, the neighborhood of $v$. We call this *the vertex switching (operation),* or, *switching v.*

3) Given a signed graph $G$, a signed graph $\bar{G}$ is called the *opposite graph* of $G$ if the two graphs have the same vertex set; $\{i,j\} \in E(G)$ if and only if $\overline{\{i,j\}} \in E(\bar{G})$, and $\overline{\{i,j\}} \in E(G)$ if and only if $\{i,j\} \in E(\bar{G})$.

Note that a coordinate transformation on the ambient vector space $\mathbb{K}^n$ does not affect the freeness and supersolvability of an arrangement. The first operation amounts to a permutation on the coordinates. This allows us to consider unlabeled graphs. For a vertex $v$, switching $v$ corresponds to switching the sign of the coordinate $x_v$.

Two signed graphs are *switching equivalent* if one can be obtained by vertex switching operations from the other. In this case, we also say that the corresponding arrangements are *switching equivalent.*

**Definition 3** Let $G = (G^+, G^-)$ be a signed graph. Denote by $\overline{E^-} = \{\{i,j\} \in E(G) \mid \overline{\{i,j\}} \in E^-\}$. A signed graph $G = (G^+, G^-)$ is called *complementary signed graph* on $n$ vertices if $V(G) = [n]$, $E \cap \overline{E^-} = \emptyset$, and $(V(G), E^+ \cup \overline{E^-}) = K_n$.

**Lemma 4** *By vertex switching operations, a complementary signed graph $G$ can be transformed into a signed graph $\tilde{G}$ in which $\mathrm{ndeg}(v) < \frac{n}{2}$ for each $v \in \tilde{G}^-$.*

*Proof* Note that switching a vertex $v$ of $G$ changes the cardinality of $|E^-|$ to

$$|\hat{E}^-| = |E^-| - \mathrm{ndeg}(v) + (n - 1 - \mathrm{ndeg}(v)) = |E^-| + n - (2\mathrm{ndeg}(v) + 1).$$

If $\mathrm{ndeg}(v) \geq \frac{n}{2}$, then

$$|\hat{E}^-| \leq |E^-| + n - (n+1) = |E^-| - 1.$$

We can repeat this for $\hat{G}$ if there exists a $v \in \hat{G}$ with $\mathrm{ndeg}(v) \geq \frac{n}{2}$. ◇

**Remark 5** *In many cases one can use vertex switching operations further to reduce the negative degree of all the vertices even when ndeg(v)* $< \frac{n}{2}$. *(See, e.g., Example 6.)*

**Example 6** *(A kite graph.) Let* $n = 2k+1$. *We construct a particular complementary signed graph* $G = (G^+, G^-)$ *called kite graph in the following way. The vertex set is* $[n] = \{1, 2, \ldots, n\}$. *Denote by* $G_1 = (G_1^+, G_1^-)$ *the induced complementary signed graph on the first* $n-1$ *vertices. The following three conditions are satisfied by* $G$.

1) $G_1^-$ *contains a rectangle* $\square$ *(the head) with vertices* $1, 2, 3, 4$ *as shown in Figure 1 a).*

2) *(The spars.) The edges of* $G_1^-$ *consist of the negative edges of* $\square$ *and*

$$\overline{\{1, 2i-1\}}, \overline{\{3, 2i-1\}}, \overline{\{2, 2i\}}, \overline{\{4, 2i\}} \quad \textit{and} \quad \overline{\{2i-1, 2j\}} \quad (3 \le i, j \le k).$$

3) *(The bridle legs.) All edges adjacent to the vertex* $n$ *are positive (belonging to* $G^+$*).*

*The graph c) in figure 1 shows* $G_1^-$ *in the case* $k = 3$.

*We claim that a kite graph* $G$ *with* $n$ *vertices is switching equivalent to a graph* $\hat{G} = (\hat{G}^+, \hat{G}^-)$ *with* $\hat{G}^-$ *a star and* $|E(\hat{G}^-)| = k$. *Hence, by Theorem 1,* $\mathcal{A}(G)$ *is supersolvable with*

$$exp(\mathcal{A}(G)) = (k, k, 1, 2, \ldots, 2k-2, 2k-1).$$

*Indeed, by vertex switching on (in this order)* $1, 3, 6, 8, 10, \ldots, 2k$, *all the negative edges in* $G_1$ *are switched into positive edges, the positive edges in* $G_1$ *are not changed, and the* $k$ *positive edges adjacent to the vertex* $n$ *are switched into negative ones. Thus the resulting graph has only* $k$ *negative edges adjacent to the vertex* $n$.

## 3 Classifications of complementary signed graphs

### 3.1

In this section, we first prove the "if" part of Theorem 1. Then we make some preparations for proving the "only if" part of Theorem 1 by proving a few facts and classifications of complementary signed graphs with 4 and 5 vertices.

**Theorem 7** *Let* $G = (G^+, G^-)$ *be a complementary signed graph with* $V = [n]$. *If* $G^-$ *is a star, then* $\mathcal{A}(G)$ *is supersolvable. Moreover, if the cardinality* $|E(G^-)| = k$ $(1 \le k \le n-1)$, *then*

$$exp(\mathcal{A}(G)) = (k, n-k-1, 1, 2, \ldots, n-3, n-2).$$

*Proof* Let the center of $G^-$ be the vertex $n$, $k < n-1$, and $E^- = E(G^-) = \{\{1,n\},\ldots,\{k,n\}\}$. For the case of $n=6, k=2$ see Figure 2 a). We have

$$\mathcal{A}(G) \setminus \mathcal{A}(G^+) = \{x_1 = -x_n, \ldots, x_k = -x_n\}$$

and for all $1 \le i < j \le k$, the intersection $\{x_i = -x_n\} \cap \{x_j = -x_n\}$ is contained in the hyperplane $x_i = x_j$, which is an element of $\mathcal{A}(G^+)$.

It is obvious that the rank $r(\mathcal{A}(G^+)) = n-1 = r(\mathcal{A}(G)) - 1$. Thus, $\mathcal{A}(G^+)$ is a coatom of $\mathcal{A}(G)$. By [2, Theorem 3.3], $\mathcal{A}(G^+)$ is supersolvable since $G^+$ is a chordal graph with elimination order $n, n-1, \ldots, 2, 1$. By [2, Lemma 3.4], $\exp\mathcal{A}(G^+) = (n-k-1, n-2, n-3, \ldots, 2, 1)$. Hence, $\exp\mathcal{A}(G) = (k, n-k-1, n-2, n-3, \ldots, 2, 1)$. If $k = n-1$, by vertex switching on $n$, $G$ is switching equivalent to $K_n$. ◇

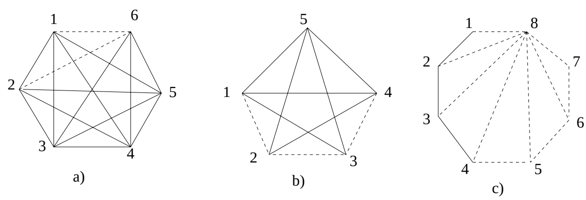


Figure 2:

**Lemma 8** *Let $G = (G^+, G^-)$ be a complementary signed graph with $V = [n]$, $n \ge 4$. Assume that $E(G^-)$ consists of all the $n-1$ edges from the vertex $n$, and the $k$ edges $\{n-1, n-2\}$, ..., $\{n-k, n-k-1\}$.*

1) *If $0 \le k \le 2$, then $\mathcal{A}(G)$ is supersolvable with exp$(\mathcal{A}(G)) = (k, n-k-1, n-2, n-3, \ldots, 2, 1)$;*

2) *If $2 < k < n-1$ $(n \ge 5)$, then $\mathcal{A}(G)$ is not free.*

*Proof* By vertex switching on $n$, we obtain the switching equivalent graph $\tilde{G}$. If $k=0$, the $\tilde{G} = K_n$. The conclusion is obvious. If $k > 0$, we have the switching equivalent arrangement $\mathcal{A}(\tilde{G})$ which is supersolvable by Theorem 7 in case $1 \le k \le 2$.

Let $k > 2$. (See graph c) in Figure 2 for the case $n = 8$, where some of the positive edges are omitted.) $\mathcal{A}(\tilde{G})$ has a localization with signed graph $G'$ shown in Figure 2 b). We will prove in Lemma 12 that $\mathcal{A}(G')$ is not free. ◇

**Theorem 9** *Let $G$ be a signed graph consisting of $K_n$, and $k$ negative edges starting from vertex 1, say $\{\{1,2\},\ldots,\{1,k\}\}$, and $m$ positive edges starting from the extra vertex $(n+1)$. Then $\mathcal{A}(G)$ is supersolvable with exp$(\mathcal{A}(G)) = (m, k, 1, 2, \ldots, n-1)$.*

*Proof* Remove all the $m$ positive edges starting from the vertex $n+1$. (See graph a) in Figure 3 for the case $n = 5, k = 2, m = 3$.) By [2, Theorem 4.15] the remained graph $G_0$ corresponds to a supersolvable arrangement with

$$\exp(\mathcal{A}(G_0)) = (k, 1, 2, \dots n - 1).$$

It is obvious that $\mathcal{A}(G_0)$ is a modular coatom of $\mathcal{A}(G)$ with $|\mathcal{A}(G) \backslash \mathcal{A}(G_0)| = m$. ◇

**Remark 10** *Theorem 9 improves [3, Theorem 2.2] in which it was proved that $\mathcal{A}(G)$ is free for the case $k = 0$, and an explicit basis for $D(\mathcal{A}(G))$ was constructed.*

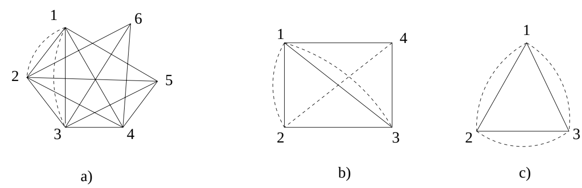


Figure 3:

**Theorem 11** *Let $G^+$ be a chordal graph with $n$ vertices which has only positive edges. Let $G$ be the graph obtained from $G^+$ by adding some negative edges, say, $\overline{\{1, u_1\}}, \dots, \overline{\{1, u_k\}}$, where $U = \{u_1, \dots, u_k\} \subset [n]$. If $G_U^+$ is a complete graph of $k$ vertices, then $\mathcal{A}(G)$ is supersolvable.*

*Proof* Since $\mathcal{A}(G) \backslash \mathcal{A}(G^+) = \{x_1 = -x_{u_1}, \dots, x_1 = -x_{u_k}\}$ and $G_U^+$ is complete graph, it follows that the intersection of any two hyperplanes $x_1 = -x_{u_i}$ and $x_1 = -x_{u_j}$ is contained in $x_{u_i} = x_{u_j}$, which is a hyperplane in $\mathcal{A}(G_U^+) \subset \mathcal{A}(G^+)$. The rank condition is obvious. By [2, Theorem 3.3] $\mathcal{A}(G^+)$ is supersolvable. Hence, $\mathcal{A}(G)$ is supersolvable. (See graph b) in Figure 1.) ◇

**Lemma 12** *Let $G_1$ be the signed graph in Figure 3 b), and $G'$ the signed graph b) in Figure 2. Then $\mathcal{A}(G_1)$ and $\mathcal{A}(G')$ are not free.*

*Proof* For $\mathcal{A}(G_1)$, we use Theorem 9 and deletion-restriction with respect to $\{x_2 + x_4 = 0\}$. For $\mathcal{A}(G')$, we use Theorem 11 and deletion-restriction with respect to $\{x_1 + x_2 = 0\}$. ◇

### 3.2 Classification of complementary signed graphs with four vertices

Let $G = (G^+, G^-)$ be a complementary signed graph with $V = \{1, 2, 3, 4\}$. Then $G$ is switching equivalent to one of the three graphs in Figure 4. The corresponding arrangements $\mathcal{A}(K_4)$ and $\mathcal{A}(K_4^1)$ are supersolvable by Theorem 7. It is easy to see that $\mathcal{A}(K_4^2)$ is not free.

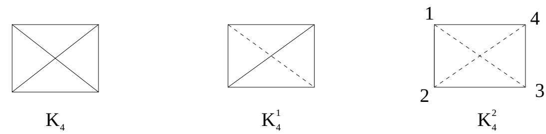


Figure 4:

### 3.3 Classification of complementary signed graphs with five vertices

We do this by considering the number $|E^-| = |E(G^-)|$. We find that there are two switching equivalent classes of non-supersolvable arrangements. The negative part of each graph in the first class is not connected or the associated arrangement has $\mathcal{A}(K_4^2)$ as a localization. The second class has a representative in Figure 2 b), where $G^-$ is not a star. In the classification we find a kind of duality.

**Duality** *For a complementary signed graph $G = (G^+, G^-)$ on $n$ vertices ($n = 4, 5$), let $\bar{G} = (\overline{G^-}, \overline{G^+})$ be the opposite graph. If $G$ and $\bar{G}$ are not switching equivalent, then $\mathcal{A}(G)$ is free if and only if $\mathcal{A}(\bar{G})$ is not free.*

**Conjecture 13** *If $G$ and $\bar{G}$ are not switching equivalent, then for all $n \geq 4$, one and only one arrangement from $\mathcal{A}(G)$ and $\mathcal{A}(\bar{G})$ is free (resp. supersolvable).*

## 4 Proof of the Main Theorems

### 4.1

The idea for the proof of Theorem 1 is to reduce the number of negative edges of the given complementary graph to the minimal possibility, then try to find a subgraph with less vertices which gives a non-supersolvable arrangement.

**Lemma 14** *The arrangement $\mathcal{A}$ corresponding to the graph a) in Figure 5 is not supersolvable.*

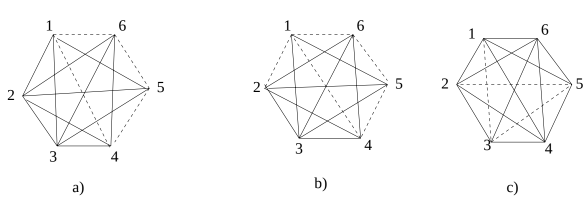


Figure 5:

*Proof* Suppose that $\mathcal{A}$ is supersolvable. By § 2.1, there exists an $M$-chain:

$$\emptyset = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \mathcal{A}_4 \subset \mathcal{A}_5 \subset \mathcal{A}_6 = \mathcal{A}.$$

Denote by $b_j = |\mathcal{A}_j \setminus \mathcal{A}_{j-1}|$ $(1 \leq j \leq 6)$. Then $\exp(\mathcal{A}) = (b_1, b_2, b_3, b_4, b_5, b_6)$. Since $\mathcal{A}$ is irreducible, $\exp(\mathcal{A})$ has six possibilities: $\exp(\mathcal{A}) = (1,2,2,2,2,6)$, $\exp(\mathcal{A}) = (1,2,2,2,3,5)$, $\exp(\mathcal{A}) = (1,2,2,2,4,4)$, $\exp(\mathcal{A}) = (1,2,2,3,3,4)$, or $\exp(\mathcal{A}) = (1,2,3,3,3,3)$.

We prove that $b_6 \notin \{1,2,3,4,5,6\}$, which implies that $b_6$ cannot be in $\exp(\mathcal{A})$, a contradiction. We carry this out by showing that, for each subarrangement $\mathcal{B} \subset \mathcal{A}$ with $|\mathcal{A} \setminus \mathcal{B}| = k \in \{1,2,3,4,5,6\}$, $\mathcal{B}$ cannot be a modular coatom of $\mathcal{A}$.

We only prove the case $k = 5$ since the proof for other cases are similar.
1) Assume the five edges taken away have a common end point. If the five edges contain a negative edge, the arrangement $\mathcal{B}$ corresponding to the remaining graph is not a modular coatom of $\mathcal{A}$, since the rank condition does not hold. If the five edges consist of only positive edges, the arrangement $\mathcal{B}$ corresponding to the remaining graph is not a modular coatom of $\mathcal{A}$, since the inclusion condition does not hold.
2) Assume the five edges taken away do not have a common end point. If the five edges contain the four negative edges, the arrangement $\mathcal{B}$ corresponding to the remaining graph is not a modular coatom of $\mathcal{A}$, since the inclusion condition does not hold. If the five edges do not contain all the four negative edges, the arrangement $\mathcal{B}$ corresponding to the remaining graph is not a modular coatom of $\mathcal{A}$, since the rank condition does not hold. ◇

**Lemma 15** *The arrangement $\mathcal{A}$ corresponding to the graph b) in Figure 5 is not free.*

*Proof* By switching the vertex 1 first, then switching the vertex 5, the graph b) is transformed into graph c) in Figure 5. Localizing the arrangement corresponding to graph c) to the arrangement corresponding to the induced subgraph on vertices $\{1,2,3,4,5\}$, we obtained a non-free arrangement by § 3.3 or Lemma 12. ◇

## 4.2 Outlines of the proof of Theorem 1

By Theorem 7, we only need to prove the "only if" part. The cases for $n = 4, 5$ have been treated in §3.2 and §3.3. We assume $n \geq 6$ and consider the complementary signed graphs of $K_n$.

Let $G = (G^+, G^-)$ be a complementary signed graph. By using vertex switching operations, $G$ can be transformed into $\tilde{G} = (\tilde{G}^+, \tilde{G}^-)$ such that $\text{ndeg}(v) < \frac{n}{2}$ for each vertex $v$. In particular, the following situation cannot happen to $\tilde{G}$: all the $n-1$ edges starting from one vertex are negative.

Let $\mathcal{A}(G)$ be supersolvable. We consider seven cases:

**Case I** If $\tilde{G}^-$ is disconnected, then $\mathcal{A}(\tilde{G})$ is not supersolvable. This is obvious.

**Case II** Let $\tilde{G}^-$ be a connected tree. If $\tilde{G}^-$ is a star, then there is nothing to prove.

**Case III** Let $\tilde{G}^-$ be a connected tree. If one of the branches in $\tilde{G}^-$ has length at least two and the tree has at least two branches. Then $\mathcal{A}(\tilde{G})$ is not supersolvable. (See § 4.3.)

**Case IV** Let $\tilde{G}^-$ be connected. If the graph $\tilde{G}^-$ contains a triangle $\triangle$, then $\mathcal{A}(\tilde{G})$ is not supersolvable. (See § 4.4.)

**Case V** Let $\tilde{G}^-$ be connected. There is no triangle in the graph $\tilde{G}^-$, and there is a rectangle $\square$. If $n = 2k$, then $\mathcal{A}(\tilde{G})$ is not supersolvable. (See § 4.5.)

**Case VI** Let $\tilde{G}^-$ be connected. There is no triangles in the graph $\tilde{G}^-$, and there is a rectangle $\square$. If $n = 2k + 1$, and $\tilde{G}$ is not a kite graph, then $\mathcal{A}(\tilde{G})$ is not supersolvable. (See § 4.6.)

**Case VII** Let $\tilde{G}^-$ be connected. There is no triangle and no rectangle $\square$ in the graph $\tilde{G}^-$, and there is a cycle of length at least five. Then $\mathcal{A}(\tilde{G})$ is not supersolvable. (See § 4.7.)

### 4.3 Proof of Case III

Let $\tilde{G}^-$ be a tree. Assume one of the branches in $\tilde{G}^-$ has length at least two and the tree has at least two branches. We take a subgraph of $\tilde{G}^-$ as indicated in the Figure 2 b)(the negative part). Since $\tilde{G}^-$ is a tree, we have in $G$ the positive edges $\{1,3\}, \{1,4\}, \{2,4\}$. Since $n \geq 6$, if there is a vertex 5 such that the edges $\{1,5\}, \{2,5\}, \{3,5\}, \{4,5\}$ are positive in $G$. The induced subgraph of $G$ on these five vertices is indicated in Figure 2 b), which gives a non-free arrangement. Hence $\mathcal{A}(G)$ is not free.

Assume that, for each $i = 5, 6, \ldots, n$, the four edges $\{1,i\}, \{2,i\}, \{3,i\}, \{4,i\}$ are not all positive. Since $\tilde{G}^-$ is a tree, then there are exactly one negative edge and three positive edges in $\{1,i\}, \{2,i\}$, $\{3,i\}$, $\{4,i\}$. Since the negative degree of each vertex in $\tilde{G}^-$ is at most $k-1$ if $n = 2k$ (or $k$ if $n = 2k+1$), these $n-4$ negative edges can not go to the same vertex among 1,2,3 and 4. Thus in case $k \geq 3$, there are two vertices, say, 5 and 6 such that the two negative edges from them go to two of the four vertices 1,2,3 and 4. It is easy to prove that the corresponding arrangements are not supersolvable no matter which sign the edge $\{5,6\}$ has (positive or negative). Hence $\mathcal{A}(\tilde{G})$ is not supersolvable.

### 4.4 Proof of case IV

Let $\tilde{G}^-$ be connected and contain a triangle $\triangle$. We prove that $\mathcal{A}(\tilde{G})$ is not supersolvable. We label the three vertices of this cycle $\triangle$ with 1,2 and 3.

If there is a vertex $v$ such that the three edges $\{1,v\}, \{2,v\}, \{3,v\}$ are all positive (resp. all negative), then these four vertices induce a subgraph of $\tilde{G}$, which is (resp. switching equivalent to) the graph $K_4^2$. The corresponding arrangement is not free by § 3.2.

If for each $v \in \{4, \ldots, n\}$, the three edges $\{1,v\}, \{2,v\}, \{3,v\}$ are not all positive (resp. not all negative). Assume that there are exactly $t$ vertices $v_i$ ($i = 1, \ldots, t$) in $\{4, \ldots, n\}$ such that there are two positive edges and one negative edge in the three edges $\{v_i, 1\}, \{v_i, 2\}, \{v_i, 3\}$. Then for each $u \in U = \{4, 5, \ldots, n\} \setminus \{v_1, \ldots, v_t\}$, there are two negative edges and one positive edge in $\{1, u\}, \{2, u\}, \{3, u\}$. Therefore there are $2(n-t-3)$ negative edges from $U$ to $\triangle$ pairwise. Altogether there are $2(n-t-3)+t = 2n-t-6$ negative edges starting from the vertices $4, \dots, n$ and ending at the three vertices 1,2 and 3.

Note that the negative degree of each vertex is less than $\frac{n}{2}$ (Lemma 4) and each of the three vertices 1,2 and 3 has already negative degree 2. In order that ndeg$v < \frac{n}{2}$ for each $v \in \{1, 2, 3\}$, the following inequality should hold.

$$0 \geq 2n - t - 6 - 3(\frac{n}{2} - 2) = \frac{n}{2} - t.$$

Since $n \geq 6$, we have $t \geq 3$. This inequality also implies that the $t$ negative edges from $\{v_1, \dots, v_t\}$ cannot go to the same vertex of $\triangle$. Next we show the following lemma.

**Lemma 16** *There is (at least) one negative edge starting from each of the vertices 1,2 and 3 to one of the vertices* $\{v_1, \dots, v_t\}$.

*Proof* Suppose that there exists an $s$ with $t \geq s > 0$ such that the negative edges $\overline{\{1, v_1\}}, \dots, \overline{\{1, v_s\}}$ go to 1 in $\triangle$ and the negative edges $\overline{\{2, v_{s+1}\}}, \dots, \overline{\{2, v_t\}}$ go to 2 in $\triangle$. We prove this is impossible.

We only prove case $n = 2k$ since the proof for $n = 2k+1$ is similar. The negative degree of each vertex is not larger than $k-1$. Each vertex of $\triangle$ has at most $k-3$ negative edges from the $T = 2(n-t-3)+t = 4k-t-6$ negative edges, thus $t \leq 2(k-3)$ and

$$0 \geq T - 3(k-3) = k - t + 3 \Longrightarrow t \geq k+3.$$

Hence, we have a contradiction if $3 \leq k \leq 8$. Assume $k \geq 9$. Among the $P = 2(2k-t-3)$ negative edges which appear pairwise, there are at most $T_{12} = 2(k-3)-(t-s)-s$ negative edges going to the vertices 1 and 2. Hence there are at most $T_{12}$ negative edges going to vertex 3 since these negative edges appear pairwise, e.g., if one negative edge goes to 1, then another edge should go to 2 or 3. Even if for each pair of edges, one goes to 1 or 2, the other goes to 3, the vertex 3 can have at most $T_{12}$ negative edges. Hence, the total negative edges going to $\triangle$ is $T' = T_{12} + T_{12} + t = 4k - t - 6$. From this we have $T - T' = 6 > 0$. This contradicts to the assumption that the negative degree of each vertex of $\triangle$ is at most $k-1$. $\diamond$

Under the assumption that $n \geq 6$, we have shown that there are at least three vertices, say $4, 5, 6$, such that there is exactly one negative edge and two positive edges such that each of them is connected to 1,2,3, and the three negative edges go to 1,2,3 respectively.

Hence the subgraph $G'$ induced from $\tilde{G}$ on the vertices 1,2,3,4,5,6 is as in Figure 6 a). To complete the graph a) in Figure 6, one needs to determine the signs of the three edges $\{4, 5\}, \{4, 6\}, \{5, 6\}$.

**Lemma 17** *For each choice of the signs of the three edges* $\{4, 5\}, \{4, 6\}, \{5, 6\}$, *the corresponding arrangement is not supersolvable.*

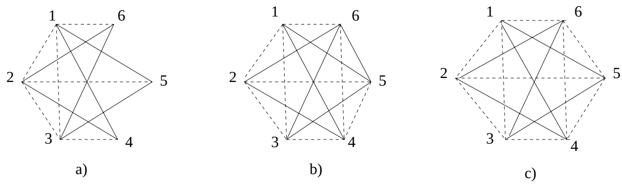


Figure 6:

*Proof* One needs to consider only four cases among the eight possible choices of the signs due to the symmetry of the graph. Here we prove the lemma only for the case where all the three edges are negative. In this case the graph is shown in Figure 6 c). By vertex switching operations (first 1, then 5, and 4), we obtain a graph isomorphic to the graph a) in Figure 5. By Lemma 14, the corresponding arrangement is not supersolvable. ◇

### 4.5 Proof of Case V

Now we consider the case that there is no triangle in the graph $\tilde{G}^-$ and there is a rectangle □. We label the four vertices of □ by 1,2,3,4 as indicated in Figure 1 a), and label the other vertices of $\tilde{G}^-$ by $5, 6, \ldots, n$. We see that for each $v \in \{5, 6, \ldots, n\}$, there are at most two negative edges in the four edges $\{1, v\}, \{2, v\}, \{3, v\}, \{4, v\}$. And if there are two negative edges, they are not consecutive, i.e., if $\{1, v\}$ is negative then $\{2, v\}$ and $\{4, v\}$ are not negative. Otherwise, there would be a triangle in $\tilde{G}^-$. The following two cases are special.

1) There are two vertices, say 5 and 6, such that the eight edges from 5 and 6 to the vertices of □ are all positive. By Lemma 14, the arrangement $\mathcal{B}$ corresponding to the subgraph induced from $\tilde{G}$ on the vertices 1,2,3,4,5,6 is not free if $\{5, 6\}$ is positive. If $\{5, 6\}$ is negative, $\mathcal{B}$ is obviously not free.

2) If $\{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}$ are positive, and there is exactly one negative edge in the four edges $\{1, 6\}, \{2, 6\}, \{3, 6\}, \{4, 6\}$, then the arrangement $\mathcal{B}$ corresponding to the subgraph induced from $\tilde{G}$ on the vertices 1,2,3,4,5,6 is not free by Lemma 15.

We consider the case $n = 2k$ in this subsection.

**Lemma 18** *If $n = 2k$, there are at most $n - 6$ vertices, say $\{7, 8, \ldots, n\}$ such that for each $v \in \{7, 8, \ldots, n\}$, there are exactly two positive edges in $\{1, v\}, \{2, v\}, \{3, v\}, \{4, v\}$, and the other two are negative edges which are not consecutive.*

*Proof* If there are $m$ vertices from $\{5, 6, \ldots, n\}$ such that each of them have exactly two non-consecutive negative edges. Then we have at least $2m$ negative edges going to □. Since the negative degree of each vertex in $\tilde{G}^-$ is not larger than $k - 1$, and there are already two negative edges adjacent to each vertex within □, we have $2m \leq 4(k - 3)$. Hence $m \leq n - 6$. ◇

Let $V' = \{v_5, \dots, v_t\} \subset \{5, 6, \dots, n\}$ such that for each $v \in V'$, there is exactly one negative edge in $\{\{1, v\}, \{2, v\}, \{3, v\}, \{4, v\}\}$, and for each $u \in U' = \{5, 6, \dots, n\} \backslash V'$, there are exactly two negative edges in $\{\{1, u\}, \{2, u\}, \{3, u\}, \{4, u\}\}$ which are not consecutive. Then the $2(2k - t)$ negative edges starting from the vertices in $U'$ go to the vertices 1 and 3, or, 2 and 4 in $\Box$ pairwise.

**Lemma 19** *The $t - 4$ negative edges starting from the vertices in $V'$ cannot go to the same vertex of $\Box$. And $t \geq 8$.*

*Proof* Suppose that all the negative edges considered go to vertex 1. Then at most $k-3-(t-4) = k-t+1$ other negative edges can go to 1. There are at most $2(k-t+1)$ negative edges from $U'$ to 1 and 3 pairwise. The other negative edges from $U'$ should go to 2 and 4 pairwise, the number should be $2(2k-t)-2(k-t+1) = 2k-2 > 2(k-3)$, a contradiction.

Since the total number of negative edges from $\{5, 6, \dots, n\}$ is $t+2(2k-t) = 4k-t$ and each of the four vertices has already two negative edges. We have $4k - t \leq 4(k - 3) = 4k - 8$. $\diamond$

By Lemma 19, we may assume that there is exactly one negative edge from each of 5 and 6 to $\Box$, and the two negative edges go to different vertices of $\Box$. Then the subgraph $\tilde{G}_1$ induced from $\tilde{G}$ on vertices 1,2,3,4,5,6 has two possibilities. It is easy to see that no matter which sign the edge $\{5, 6\}$ takes in $\tilde{G}$, the corresponding arrangement is not free.

### 4.6 Proof of Case VI

Let $n = 2k+1$. We continue to use notations and assumptions in the first 3 paragraphs of § 4.5. There are two cases to be treated.

**Case 1)** There is no vertex in $\{5, 6, 7, \dots, n\}$ having four positive edges going to $\Box$.

**Case 2)** There is exactly one vertex, say $n$, such that the four edges $\{1, n\}$, $\{2, n\}$, $\{3, n\}$, $\{4, n\}$ are positive.

We consider case 1).

**Lemma 20** *Let $n = 2k+1$. Assume that there is no vertex in $\{5, 6, 7, \dots, n\}$ having four positive edges going to $\Box$. If there is a subset $V' = \{v_5, \dots, v_t\} \subset \{5, 6, 7, \dots, n\}$ such that for each $i = 5, \dots, t$, only one edge in $\{1, v_i\}, \{2, v_i\}, \{3, v_i\}, \{4, v_i\}$ is negative, then $t \geq 6$.*

*Proof* Let $v_5, \dots, v_t$ be the vertices such that for each $i = 5, \dots, t$, there is only one negative edge in $\{1, v_i\}, \{2, v_i\}, \{3, v_i\}, \{4, v_i\}$. Then the total number of the negative edges from $\{5, 6, 7, \dots, n\}$ to $\Box$ is $T = 2(n - t) + (t - 4) = 4k - t - 2$. Since the negative degree of each vertex is less than or equal to $k$ and there are already two negative edges connecting to each vertex of $\Box$, we have $T \leq 4(k - 2) = 4k - 8$, i.e. $t \geq 6$. $\diamond$

**Lemma 21** *Let $n = 2k+1$. Assume that there is no vertex in $\{5, 6, 7, \dots, n\}$ having four positive edges going to $\Box$. The $t-4$ negative edges starting from the vertices in $V'$ can not go to the same vertex of $\Box$.*

*Proof* Suppose that $\{1, v_5\}, \dots, \{1, v_t\}$ are negative. One can put at most $k-2-(t-4) = k-t+2$ extra negative edges to 1. There are $T = 2(2k+1-t) = 4k-2t-2$ negative edges from $U' = \{5, 6, 7, \dots, n\} \setminus V'$ to $\Box$. Since the negative edges from $U'$ are in pairs and each pair of the edges can not go to the consecutive vertices of $\Box$, there are at most $2(k-t+2)$ of them go to 1 and 3, and the $T-2(k-t+2) = 2k-2$ negative edges should go to 2 and 4. This would make the negative degree of 2 and 4 to be $k+1$, a contradiction. ◇

*Conclusion for case 1)* Under the assumptions of Lemmas 20 and 21, we know that there are two vertices, say 5 and 6, such that only one edge in $\{1, i\}, \{2, i\}, \{3, i\}, \{4, i\}$ is negative ($i = 5, 6$) and the two negative edges go to different vertices of $\Box$. Hence the induced subgraph of $\tilde{G}$ has two possibilities, which give non-free arrangements by the discussion in § 4.5.

Now we treat case 2).

Let $n = 2k+1$ and let there be exactly one vertex in $\{5, 6, 7, \dots, n\}$ having four positive edges going to $\Box$. We assume this vertex is $n$. By 1) and 2) at the beginning of §4.5, we may assume that there are two positive edges and two negative edges in $\{1, i\}, \{2, i\}, \{3, i\}, \{4, i\}$ ($i = 5, \dots 2k$). Since these negative edges go to $\Box$ pairwise, and not consecutively, and their negative degrees are not larger than $k$, we can assume

$$\{1, 2i-1\}, \{3, 2i-1\}, \{2, 2i\}, \{4, 2i\} \ (i = 3, \dots k) \text{ are negative,}$$

$$\{2, 2i-1\}, \{4, 2i-1\}, \{1, 2i\}, \{3, 2i\} \ (i = 3, \dots k) \text{ are positive.}$$

We need to consider the following three possibilities:

(1) There is a vertex $v \in V_1 = \{5, 6, \dots, 2k\}$ such that the edge $\{v, n\}$ is negative. Assume that $v = 2j$. (If $v$ is an odd number, the argument is similar.) Then there is $u = 2j-1 \in V_1$ such that the edge $\{u, v\}$ in $\tilde{G}$ is positive. Indeed, suppose that the $k-2$ edges $\{5, v\}, \{7, v\}, \{9, v\} \dots, \{2k-1, v\}$ are all negative. Since $\{2, v\}, \{4, v\}, \{n, v\}$ are negative, $\mathrm{ndeg}(v) = k+1 > k$.

Let $v = 6$, $u = 5$. Then the induced subgraph $G'$ of $\tilde{G}$ on the vertices 1,2,3,4,5,6,$n$ is the graph a) in Figure 7 (some positive edges are not shown). If the edge $\{5, n\}$ is positive, then $G'$ is the graph b) in Figure 7. Localize the corresponding arrangement to the subgraph $G''$ of $G'$ on the vertices 3,4,5,6,$n$. Then $G''$ is switching equivalent to a graph whose negative part is not connected. Hence the corresponding arrangement is not free. If the edge $\{5, n\}$ is negative, one can prove similarly that the corresponding arrangement is not free.

(2) All the edges $\{j, n\}$ ($1 \le j \le 2k$) are positive, and there is a positive edge in $\{2i-1, 2j\}$ ($3 \le i, j \le k$), say $\{5, 6\}$ is positive. The induced subgraph $G'$ of $\tilde{G}$ on the vertices 1,2,3,4,5,6,$n$ is the graph a) in Figure 8. The induced subgraph

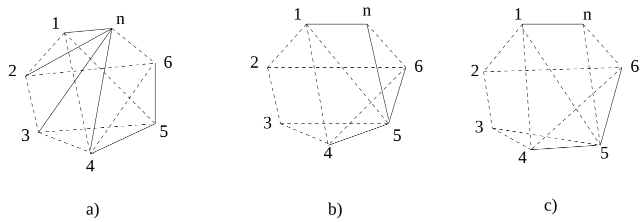


Figure 7:

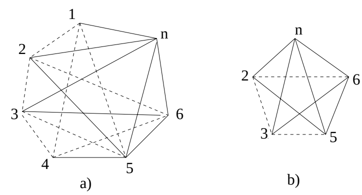


Figure 8:

$G''$ of $G'$ on the vertices 2,3,5,6,$n$ is the graph b) in Figure 8 which gives a non-free arrangement by Lemma 12.

(3) All the edges $\{j, n\}$ $(1 \leq j \leq 2k)$ are positive and all the edges in $\{2i-1, 2j\}$ $(3 \leq i, j \leq k)$ are negative. Then $\tilde{G}$ is a cone with the vertex $n$ as the conical point over the complementary signed graph $\tilde{G}_1 = (\tilde{G}_1^+, \tilde{G}_1^-)$ on $2k$ vertices. And the edges of $\tilde{G}_1^-$ consist of the edges of $\Box$ and

$$\overline{\{1, 2i-1\}}, \overline{\{3, 2i-1\}}, \overline{\{2, 2i\}}, \overline{\{4, 2i\}} \quad \text{and} \quad \overline{\{2i-1, 2j\}} \quad (3 \leq i, j \leq k).$$

Hence this graph is a *kite graph* which is switching equivalent to a graph $\hat{G} = (\hat{G}^+, \hat{G}^-)$ with $\hat{G}^-$ a star and $|E(\hat{G}^-)| = k$ by Example 6.

## 4.7 Proof of Case VII

Assume that $\tilde{G}^-$ has no triangles and rectangles, and has a cycle of length at least 5. Let one of the cycles be $v_1 v_2 \cdots v_k$ $(k \geq 5)$. Consider the subgraph $G_1$ of $\tilde{G}$ induced on the vertices $v_1 v_2 v_3 v_4 v_5$. Since there are no triangles and rectangles in $\tilde{G}^-$, $\mathcal{A}(G_1)$ has a localization $\mathcal{A}(K_4^2)$. Hence $\mathcal{A}(G_1)$ is not free.

## 4.8 Proof of Theorem 2

We only need to consider the case $n \geq 6$ and prove the following statement. Let $G$ be a complementary signed graph and $\bar{G}$ be the opposite graph of $G$. If $\mathcal{A}(G)$ is supersolvable, then $\mathcal{A}(\bar{G})$ is not supersolvable.

In fact, by Theorem 1, we can assume $G = (G^+, G^-)$ and $G^-$ is a star. Let the vertex $n$ be the center of the star. Let $\bar{G} = (\bar{G}^+, \bar{G}^-)$ be the opposite graph of $G$. If $|E(G^-)| = |E(\bar{G}^+)| \le 2$, there is a subgraph $\bar{G}_1$ of $\bar{G}^-$ such that the opposite graph of $\bar{G}_1$ is $K_4$. Hence $\mathcal{A}(\bar{G}_1)$ is not free, which implies that $\mathcal{A}(\bar{G}_1)$ is not supersolvable.

If $|E(G^-)| = |E(\bar{G}^+)| \ge 3$, in $\bar{G}^-$ there are three vertices, say, $1, 2, 3$ such that $\{1, n\}$, $\{2, n\}$, $\{3, n\}$ are positive and $\{1, 2\}, \{2, 3\}, \{3, 1\}$ are negative in $\bar{G}$. The induced subgraph of $\bar{G}$ on the four vertices $1, 2, 3, n$ is switching equivalent to the graph $K_4^2$ in Figure 4. From §3.2 we know that $\mathcal{A}(K_4^2)$ is not free.

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## References

[1] G.D. Bailey, Inductively factored signed-graphic arrangements of hyperplanes, Preprint.

[2] P.H. Edelman and V. Reiner, Free hyperplane arrangements between $A_{n-1}$ and $B_n$, *Math. Z.* **215** (1994), 347–365.

[3] T. Józefiak and B.E. Sagan, Basic derivations for subarrangements of Coxeter Arrangements, *J. Alg. Combinatorics.* **2** (1993), 291–320.

[4] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer-Verlag, Berlin, Heidelberg, 1992.

[5] H. Terao, Free arrangements of hyperplanes and unitary reflection groups, *Proc. Japan Acad. Ser.A* **56** (1980), 389–392.

[6] H. Terao, Modular elements of lattices and topological fibration, *Adv. Math.* **62** (1986), 135–154.

[7] T. Zaslavsky, The geometry of root system and signed graphes, *Amer. Math. Monthly* **88** (1981), 88–105.

[8] T. Zaslavsky, Supersolvable Frame-matroid and Graphic-lift Lattices, *Europ. J. Combinatorics* **22** (2001), 119–133.

[9] T. Zaslavsky, Biased graphs. I. Bias, balance, and gains; II. The three matroids; III. Chromatic and dichromatic invariants; IV. Geometrical realizations, *J. Comb. Theory. Ser. B*, **47**(1989), 32–52; **51** (1991), 46–72; **64** (1995), 17–88; to appear.

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