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Decompositions of λK_v into k-circuits with one chord*

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Abstract

Let λK_v be the complete multigraph with v vertices, where any two distinct vertices x and y are joined by λ edges $\{x, y\}$. Let G be a finite simple graph. A G-design of λK_v, denoted by (v, G, λ)-GD, is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly λ blocks of $\mathcal{B}$. In this paper, the graphs discussed are $C_k^{(r)}$, i.e., one circle of length k with one chord, where r is the number of vertices between the ends of the chord, $1 \le r < \lfloor \frac{k}{2} \rfloor$. We give a unified method to construct $C_k^{(r)}$-designs. In particular, for $G = C_6^{(r)} (r = 1, 2)$, $C_7^{(r)} (r = 1, 2)$ and $C_8^{(r)} (r = 1, 2, 3)$, we completely solve the existence spectrum of (v, G, λ)-GD.

1 Introduction

A *complete multigraph* of order v and index λ, denoted by λK_v, is a graph with v vertices, where any two distinct vertices x and y are joined by λ edges $\{x, y\}$. A *t-partite graph* is one whose vertex set can be partitioned into t subsets $X_1, X_2, \cdots, X_t$, such that two ends of each edge lie in distinct subsets. Such a partition $(X_1, X_2, \cdots, X_t)$ is called a *t-partition* of the graph. A *complete t-partite graph* with replication λ is a t-partite graph with t-partition $(X_1, X_2, \cdots, X_t)$, in which each vertex of X_i is joined to each vertex of X_j by λ edges (where $i \ne j$). Such a graph is denoted by $\lambda K_{n_1, n_2, \cdots, n_t}$ if $|X_i| = n_i$ $(1 \le i \le t)$. We denote a path of k vertices by P_k and an undirected cycle of length m by C_m. By $C_m^{(r)}$ we mean one cycle of length m with one chord, where r is the number of vertices between the ends of the chord, $1 \le r < \lfloor \frac{m}{2} \rfloor$. In [3], Blinco introduced the so-called theta-graph, that is a graph which consists of three internally disjoint paths with common end points and lengths a, b and c with

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$a \leq\ b \leq\ c$ and $b \neq 1$. This graph is denoted by $\Theta(a, b, c)$. Obviously, the graph $C_m^{(r)}$ is just $\Theta(1, r+1, m-r-1)$.

Let G be a finite simple graph. A *G-design* of λK_v, denoted by (v, G, λ)-GD, is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly λ blocks of $\mathcal{B}$. It is well known that if there exists a (v, G, λ)-GD, then

$$\lambda v(v-1) \equiv 0 \pmod{2e(G)} \ \text{ and } \ \lambda(v-1) \equiv 0 \pmod{d},$$

where $e(G)$ denotes the number of edges in G and d is the greatest common divisor of the degrees of the vertices of G. For the path P_k and the star $K_{1,k}$, the existence problems of (v, P_k, λ)-GD and $(v, K_{1,k}, \lambda)$-GD have been solved (see [4] and [8]). For some graphs, which have fewer vertices and fewer edges, the problem of their graph designs has already been researched (see [1], [5]–[7], [9] and [11]–[19]).

Let $(X_1, X_2, \cdots, X_t)$ be the t-partition of $\lambda K_{n_1, n_2, \cdots, n_t}$, and $|X_i| = n_i$. Let $v = \sum_{i=1}^{t} n_i$ and $\mathcal{G}=\{X_1, X_2, \cdots, X_t\}$. For any given graph G, if the edges of $\lambda K_{n_1, n_2, \cdots, n_t}$ can be decomposed into edge-disjoint subgraphs $\mathcal{A}$, each of which is isomorphic to G and is called a *block*, then the system $(X, \mathcal{G}, \mathcal{A})$ is called a *holey G-design* with index λ, denoted by G-$HD_\lambda(T)$, where $T = n_1^1 n_2^1 \cdots n_t^1$ is the *type* of the holey G-design. Usually, the type is denoted by exponential form, for example, the type $1^i 2^r 3^k \cdots$ denotes i occurrences of 1, r occurrences of 2, etc. A G-$HD_\lambda(1^{v-w} w^1)$ is called an *incomplete G-design*, denoted by G-$ID_\lambda(v; w) = (V, W, \mathcal{A})$, where $|V| = v$, $|W| = w$ and $W \subset V$. Obviously, a (v, G, λ)-GD is a G-$HD_\lambda(1^v)$ or a G-$ID_\lambda(v; w)$ with $w = 0$ or 1. Let H_1, H_2 and W be three disjoint sets. A G-$IHD_\lambda(h_1, h_2; w)$ is a pair $((H_1, H_2, W), \mathcal{A})$, where $\mathcal{A}$ is a collection of subgraphs in $H_1 \cup H_2 \cup W$, called *blocks*, such that each block is isomorphic to G and any two distinct vertices x, y are joined in

$$\begin{cases} \text{exactly } \lambda \text{ blocks of } \mathcal{B} & \text{if } x, y \in H_1 \text{ or } x, y \in H_2 \text{ or } x \in H_1 \cup H_2,\ y \in W \\ \text{no block of } \mathcal{B} & \text{otherwise} \end{cases}.$$

For HD_λ, ID_λ and IHD_λ, the subscript can be omitted when $\lambda = 1$.

In this paper, the graphs discussed are $C_k^{(r)}$. We provide a method to construct $C_k^{(r)}$-designs. The general structures will be given. In particular, for $k = 6, 7, 8$ and any r, λ, we completely solve the existence spectrum of $(v, C_k^{(r)}, \lambda)$-GD, where $v \geq k$. Considering the results have been known to all when $\lambda = 1$ (see [2]–[3]), we do not want to mention our method of solving the problem when $\lambda = 1$. We solve the existence problem only for $\lambda > 1$ in this paper.

2 General structures

Theorem 2.1 *Let G be a simple graph. For positive integers h, λ, m and nonnegative w, if there exist G-$HD_\lambda(h^m)$, G-$ID_\lambda(h+w; w)$ and (w, G, λ)-GD (or $(h+w, G, \lambda)$-GD), then there exists $(mh+w, G, \lambda)$-GD, too.*

Proof. Let $X = (Z_h \times Z_m) \cup W$, where W is a w-set. Suppose there exist

$G\text{-}HD_\lambda(h^m) = (Z_h \times Z_m, \mathcal{A})$,

$G\text{-}ID_\lambda(h+w; w) = ((Z_h \times \{i\}) \cup W, \mathcal{B}_i),\ i \in Z_m$ or $i \in Z_m \setminus \{0\}$, and

$(w, G, \lambda)\text{-}GD = (W, \mathcal{C})$ or $(h+w, G, \lambda)\text{-}GD = ((Z_h \times \{0\}) \cup W, \mathcal{D})$,

then (X, Ω) is a $(mh+w, G, \lambda)\text{-}GD$, where

$$\Omega = \mathcal{A} \cup (\bigcup_{i=0}^{m-1} \mathcal{B}_i) \bigcup \mathcal{C} \ \text{ or } \ \mathcal{A} \bigcup (\bigcup_{i=1}^{m-1} \mathcal{B}_i) \bigcup \mathcal{D}.$$

Note that

$$|\Omega| = \frac{\lambda\binom{mh+w}{2}}{e(G)} = \begin{cases} \frac{\lambda\binom{m}{2}h^2}{e(G)} + m \times \frac{\lambda(\binom{h}{2}+wh)}{e(G)} + \frac{\lambda\binom{w}{2}}{e(G)} \\ \frac{\lambda\binom{m}{2}h^2}{e(G)} + (m-1) \times \frac{\lambda(\binom{h}{2}+wh)}{e(G)} + \frac{\lambda\binom{w+h}{2}}{e(G)} \end{cases}$$

$$= \begin{cases} |\mathcal{A}| + \sum_{i=0}^{m-1} |\mathcal{B}_i| + |\mathcal{C}| \\ |\mathcal{A}| + \sum_{i=1}^{m-1} |\mathcal{B}_i| + |\mathcal{D}| \end{cases}.$$

□

However, the theorem can not be used to construct all orders $mh + w$. For example, when $G\text{-}HD_\lambda(h^m)$ exist only for odd m (see Theorem 2.4), or $G\text{-}ID_\lambda(h + w; w)$ merely exist for smaller w. Thus we have to present other structures, such as IHD etc.

Theorem 2.2 *Let G be a simple graph. For positive integers h, w, t and λ, if there exist $G\text{-}HD_\lambda(h^{2t+1})$, $G\text{-}IHD_\lambda(h, h; w)$ and $(h + w, G, \lambda)\text{-}GD$, then $((2t + 1)h + w, G, \lambda)\text{-}GD$ exists.*

Proof. Let $X = (Z_h \times Z_{2t+1}) \bigcup W$, where $|W| = w$. Suppose there exist

$G\text{-}HD_\lambda(h^{2t+1}) = (Z_h \times Z_{2t+1}, \mathcal{A})$,

$G\text{-}IHD_\lambda(h, h; w) = ((Z_h \times \{2i\}, Z_h \times \{2i+1\}, W), \mathcal{B}_i)$ for $0 \le i \le t-1$,

and

$(h+w, G, \lambda)\text{-}GD = ((Z_h \times \{2t\}) \bigcup W, \mathcal{C})$,

then $(X, \mathcal{A} \bigcup (\bigcup_{i=0}^{t-1} \mathcal{B}_i) \bigcup \mathcal{C})$ forms a $((2t+1)h + w, G, \lambda)\text{-}GD$. In fact, we have

$$|\mathcal{A}| + t|\mathcal{B}_i| + |\mathcal{C}| = \frac{\lambda\binom{2t+1}{2}h^2}{e(G)} + \frac{\lambda t(2hw + h(h-1))}{e(G)} + \frac{\lambda\binom{w+h}{2}}{e(G)} = \frac{\lambda\binom{(2t+1)h+w}{2}}{e(G)}.$$ □

Theorem 2.3 *There exist $C_{2k}^{(r)}\text{-}HD((2k+1)^t)$ for $t \ge 2$ and even r.*

Proof. Let $X = Z_{2k+1} \times Z_t = \bigcup_{x \in Z_t} V_x$, where $V_x = Z_{2k+1} \times \{x\}$. For $x \ne y \in \{1, 2, \cdots, t\}$ and $a_i, b_i \in Z_{2k+1}$, define a $2k$-circuit C as follows:

$$((a_0, x), (b_0, y), (a_1, x), (b_1, y), \cdots, (a_{k-1}, x), (b_{k-1}, y)),$$

where

$$a_i = \begin{cases} i, & i = 0, 1 \\ i+2, & 2 \le i \le \lfloor \frac{k}{2} \rfloor \\ \frac{k}{2} + 5 \text{ or } \frac{1-k}{2}, & i = \lfloor \frac{k}{2} \rfloor + 1 \ (k \text{ even or odd}) \\ i-k, & \lfloor \frac{k}{2} \rfloor + 2 \le i \le k-1 \end{cases},$$

$$b_i = \begin{cases} 3, & i=0 \\ -(i-1), & 1 \le i \le \lfloor \frac{k}{2} \rfloor - 1 \\ \frac{k}{2}+3 \text{ or } \frac{3-k}{2}, & i = \lfloor \frac{k}{2} \rfloor \ (k \text{ even or odd}) \\ \frac{k}{2}+2 \text{ or } -\frac{3+k}{2}, & i = \lfloor \frac{k}{2} \rfloor + 1 \ (k \text{ even or odd}) \\ k+3-i, & \lfloor \frac{k}{2} \rfloor + 2 \le i \le k-1 \end{cases}.$$

It is easy to see that, for odd or even k, the $2k$ vertices in C are different. Furthermore, the $2k$ edges in C just correspond to all mixed differences $\pm d_{xy}(1 \le d \le k)$. The remaining mixed difference 0_{xy} may correspond to any chord $((a_i,x),(a_i,y))$ or $((b_i,x),(b_i,y))$. Thus, the chord of $C_{2k}^{(r)}$ can be chosen as $((a_i,x),(a_i,y))$ if $r = 4i-2$ or $((b_i,x),(b_i,y))$ if $r = 4i$. Clearly, for all $x \ne y \in Z_t$, C modulo $(2k+1,-)$ gives the expected G-$HD((2k+1)^t)$. □

Theorem 2.4 *There exist* $C_{2k}^{(r)}$-$HD((2k+1)^{2t+1})$ *for* $t \ge 1$ *and odd* r.

Construction. Let $X = Z_{2k+1} \times Z_{2t+1}$ and $k = p+q$, where p and q are positive integers. For any $x \in \{1,2,\cdots,t\}$ define the following $2k$-circuit over X:

$$A_x = (a_0, a_1, a_2, \cdots, a_{2p}, b_{2q-1}, b_{2q-2}, \cdots, b_1),$$

where $a_0(=b_0)$ and $a_{2p}(=b_{2q})$ will become the ends of the unique chord of $C_{2k}^{(r)} = A_x + a_0a_{2p}$. These vertices a_i and b_i are defined as follows:

$$\text{when } p \text{ odd} \quad \begin{cases} a_{2j} = \begin{cases} (-j,0) & 0 \le j \le \frac{p-1}{2} \\ (p-j,2x) & \frac{p+1}{2} \le j \le p \end{cases} \\ a_{2j-1} = (j,x) \quad 1 \le j \le p \end{cases};$$

$$\text{when } p \text{ even} \quad \begin{cases} a_{2j} = \begin{cases} (-j,0) & 0 \le j \le \frac{p}{2}-1 \\ (j-p,x) & \frac{p}{2} \le j \le p-1 \\ (0,2x) & j=p \end{cases} \\ a_{2j-1} = \begin{cases} (j,-x) & 1 \le j \le \frac{p}{2} \\ (-j,0) & \frac{p}{2}+1 \le j \le p \end{cases} \end{cases};$$

$$\text{when } q \text{ odd} \quad \begin{cases} b_{2j} = \begin{cases} (j,0) & 0 \le j \le \frac{q-1}{2} \\ (j-q,2x) & \frac{q+1}{2} \le j \le q \end{cases} \\ b_{2j-1} = (-(p+j),x) \quad 1 \le j \le q \end{cases};$$

$$\text{when } q \text{ even} \quad \begin{cases} b_{2j} = \begin{cases} (j,0) & 0 \le j \le \frac{q}{2}-1 \\ (q-j,x) & \frac{q}{2} \le j \le q-1 \\ (0,2x) & j=q \end{cases} \\ b_{2j-1} = \begin{cases} (-(p+j),-x) & 1 \le j \le \frac{q}{2} \\ (p+j,0) & \frac{q}{2}+1 \le j \le q \end{cases} \end{cases}.$$

If $r \equiv 3 \bmod 4$ (say $r = 4n-1$) then take $p = 2n$ and $q = k-2n$. If $r \equiv 1 \bmod 4$ (say $r = 4n+1$) then take "$p = 2n+1$ and $q = k-2n-1$" (when k even) or "$q = 2n+1$ and $p = k-2n-1$" (when k odd). The blocks $\{A_x + a_0a_{2p} : 1 \le x \le t\}$

module $(2k+1, 2t+1)$ will be the desired $C_{2k}^{(r)}$-$HD((2k+1)^{2t+1})$.

Proof. First, by the given construction, we can list the differences $\langle d, d'\rangle$ corresponding to the edges $((a,b),(a',b'))$ in A_x, where $d = a'-a$ and $d' = b'-b$, $a, a' \in Z_{2k+1}$, $b, b' \in Z_{2t+1}$.

From the following table, it is easy to see that, for any p and q, the differences corresponding to all edges of A_x are just $\langle \pm d, d'\rangle$, where $1 \le d \le p+q$, $d' = x$ or $2x$. Furthermore, we have

$$\{\pm x : \ x \in \{1, 2, \cdots, t\}\} = \{\pm 2x : \ x \in \{1, 2, \cdots, t\}\} = Z_{2t+1}^*$$

and the chord $a_0 a_{2p} = b_0 b_{2q}$ corresponds to the difference $\langle 0, 2x\rangle$. Therefore, the blocks $\{A_x + a_0 a_{2p} : \ x \in \{1, 2, \cdots, t\}\}$ module $(2k+1, 2t+1)$ cover exactly all the edges of $K_{2k+1,\cdots,2k+1}$ with $2t+1$ parts. In this table, the symbol $[m, n]_2$ represents the set $\{m, m+2, \cdots, n-2, n\}$, where $m \equiv n \pmod 2$. And, the rows in this table are separated into four parts: odd p, even p, odd q and even q, in order down.

edges in A_x	differences $\langle d, d'\rangle$		range of d
$((j,x),(-j,0))$	$\langle 2j, x\rangle$	$1 \le j \le \frac{p-1}{2}$	$[2, p-1]_2$
$((j,x),(p-j,2x))$	$\langle p-2j, x\rangle$	$\frac{p+1}{2} \le j \le p$	$[-p, -1]_2$
$((-j,0),(j+1,x))$	$\langle 2j+1, x\rangle$	$0 \le j \le \frac{p-1}{2}$	$[1, p]_2$
$((p-j,2x),(j+1,x))$	$\langle p-2j-1, x\rangle$	$\frac{p+1}{2} \le j \le p-1$	$[-(p-1), -2]_2$
$((j,-x),(-j,0))$	$\langle -2j, x\rangle$	$1 \le j \le \frac{p}{2}-1$	$[-(p-2), -2]_2$
$((\frac{p}{2},-x),(-\frac{p}{2},x))$	$\langle -p, 2x\rangle$	$j = \frac{p}{2}$	$-p$
$((-j,0),(j-p,x))$	$\langle 2j-p, x\rangle$	$\frac{p}{2}+1 \le j \le p-1$	$[2, p-2]_2$
$((-p,0),(0,2x))$	$\langle p, 2x\rangle$	$j = p$	p
$((-j,0),(j+1,-x))$	$\langle -2j-1, x\rangle$	$0 \le j \le \frac{p}{2}-1$	$[-(p-1), -1]_2$
$((j-p,x),(-j-1,0))$	$\langle 2j+1-p, x\rangle$	$\frac{p}{2} \le j \le p-1$	$[1, p-1]_2$
$((-p-j,x),(j,0))$	$\langle -p-2j, x\rangle$	$1 \le j \le \frac{q-1}{2}$	$[-(p+q-1), -(p+2)]_2$
$((-p-j,x),(j-q,2x))$	$\langle p-q+2j, x\rangle$	$\frac{q+1}{2} \le j \le q$	$[p+1, p+q]_2$
$((-p-j-1,x),(j,0))$	$\langle -p-1-2j, x\rangle$	$0 \le j \le \frac{q-1}{2}$	$[-(p+q), -(p+1)]_2$
$((j-q,2x),(-p-j-1,x))$	$\langle p-q+1+2j, x\rangle$	$\frac{q+1}{2} \le j \le q-1$	$[p+2, p+q-1]_2$
$((-p-j,-x),(j,0))$	$\langle p+2j, x\rangle$	$1 \le j \le \frac{q}{2}-1$	$[p+2, p+q-2]_2$
$((-p-\frac{q}{2},-x),(\frac{q}{2},x)$	$\langle p+q, 2x\rangle$	$j = \frac{q}{2}$	$p+q$
$((p+j,0),(q-j,x))$	$\langle q-p-2j, x\rangle$	$\frac{q}{2}+1 \le j \le q-1$	$[-(p+q-2), -(p+2)]_2$
$((p+q,0),(0,2x))$	$\langle -p-q, 2x\rangle$	$j = q$	$-(p+q)$
$((j,0),(-p-j-1,-x))$	$\langle p+2j+1, x\rangle$	$0 \le j \le \frac{q}{2}-1$	$[p+1, p+q-1]_2$
$((q-j,x),(p+j+1,0))$	$\langle q-p-1-2j, x\rangle$	$\frac{q}{2} \le j \le q-1$	$[-(p+q-1), -(p+1)]_2$

However, in some cases (for example p odd and q even) A_x does not form a circuit. In fact, we have the values of the vertices in A_x as follows.

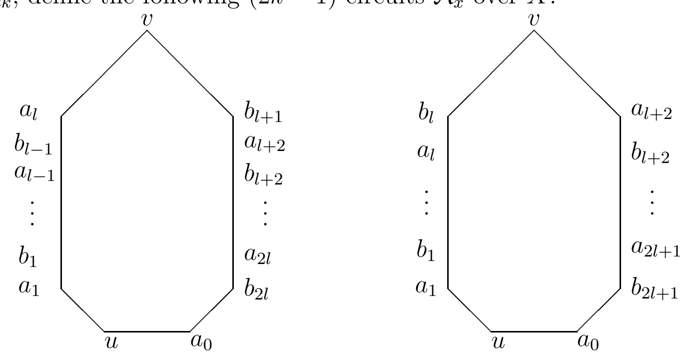
vertices		$(y,0)$	(y,x)	$(y,-x)$	$(y,2x)$
y	p odd	$[-\frac{p-1}{2}, 0]$	$[1, p]$		$[0, \frac{p-1}{2}]$
	p even	$[-p, 0] \setminus \{-\frac{p}{2}\}$	$[-\frac{p}{2}, -1]$	$[1, \frac{p}{2}]$	0
	q odd	$[0, \frac{q-1}{2}]$	$[-(p+q), -(p+1)]$		$[-\frac{q-1}{2}, 0]$
	q even	$[0, \frac{q}{2}-1] \cup [p+\frac{q}{2}+1, p+q]$	$[1, \frac{q}{2}]$	$[-(p+\frac{q}{2}), -(p+1)]$	0

Note that $a_0 = b_0 = (0,0)$ and $a_{2p} = b_{2q} = (0, 2x)$ for any p and any q. It is easy to verify that the values of all vertices in A_x are distinct for the following cases:

(1) p even and q even; (2) p odd and q odd; (3) p even and q odd. However, when p odd and q even, the values of vertices in the form (y,x) will be repeated. It is the reason that we take different p and q for $r = 4n+1$ in our construction. □

Theorem 2.5 *There exist $C_{2k-1}^{(r)}$-$HD((2k)^{2t+1})$ for $k \geq 3,\ t \geq 1$ and $1 \leq r \leq k-2$.*

Proof. Let $X = Z_{2k} \times Z_{2t+1} = \bigcup_{x \in Z_{2t+1}} V_x$, where $V_x = Z_{2k} \times \{x\}$. For $1 \leq x \leq t$ and $a_i, b_i \in Z_{2k}$, define the following $(2k-1)$-circuits $\mathcal{A}_x$ over X:



$(k = 2l+1)$ $(k = 2l+2)$

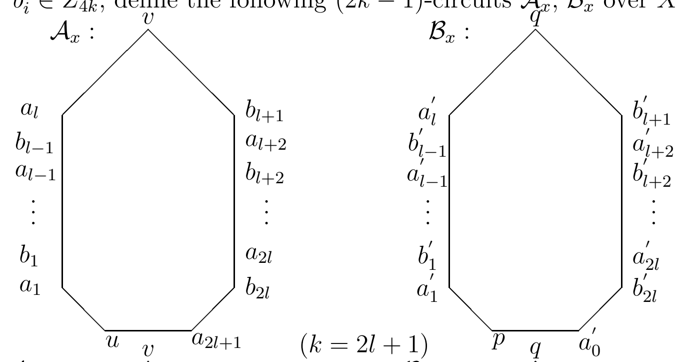
The vertices u, v, a_i, b_i are defined as follows:

$$\begin{cases} u = (0, -x), & \\ v = (-\frac{k-1}{2}, 0) \text{ or } (-\frac{k-4}{2}, 0), & \text{k even or odd} \\ a_i = (i, x), & 0 \leq i \leq k-1 \text{ and } i \neq \lfloor \frac{k+1}{2} \rfloor, \\ b_i = (-i, 2x), & 1 \leq i \leq k-1 \text{ and } i \neq \lfloor \frac{k}{2} \rfloor; \end{cases}$$

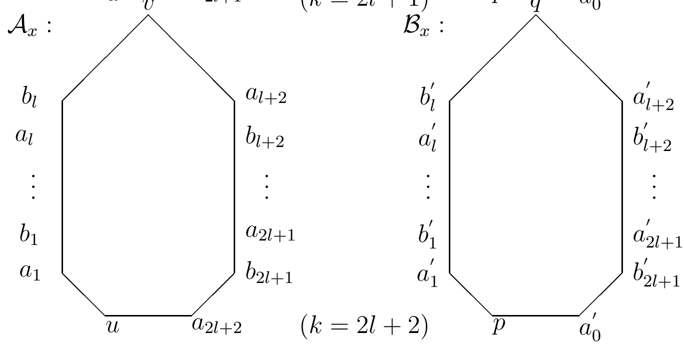
In each $\mathcal{A}_x$, $1 \leq x \leq t$, the $2k-1$ vertices are distinct obviously. The edge (u, a_0) just corresponds to the mixed difference 0_{2x}. The mixed differences $1_{2x}, -1_{2x}, (k-1)_x$ and $-(k-1)_x$ correspond to the edges $(u, a_1), (v, b_{l+1}), (v, a_l)$ and (a_0, b_{2l}), when $k = 2l+1$, or the edges $(u, a_1), (v, b_l), (v, a_{l+2})$ and (a_0, b_{2l+1}), when $k = 2l+2$. Other edges just correspond to the mixed differences $\pm d_x$ $(2 \leq d \leq k-2)$, and the remaining mixed difference k_x may correspond to any chord (a_i, b_{2l+1-i}), $1 \leq i \leq 2l$ and $i \neq l+1$, when $k = 2l+1$, or (a_i, b_{2l+2-i}), $1 \leq i \leq 2l+1$ and $i \neq l+1$, when $k = 2l+2$. Thus, the chord of $C_{2k-1}^{(r)}$ can be chosen as (a_i, b_{2l+1-i}), $\forall\ 1 \leq i \leq 2l$ and $i \neq l+1$, when $k = 2l+1$ or (a_i, b_{2l+2-i}), $\forall\ 1 \leq i \leq 2l+1$ and $i \neq l+1$, when $k = 2l+2$. Clearly, when $k = 2l+1$, let $C_{2k-1}^{(r)} = \mathcal{A}_x + (a_i, b_{2l+1-i})$, the blocks $\{\mathcal{A}_x + (a_i, b_{2l+1-i}) :\ 1 \leq x \leq t,\}$ $(\forall\ 1 \leq i \leq 2l$ and $i \neq l+1)$ mod $(2k, 2t+1)$ give the expected $C_{2k-1}^{(r)}$-$HD((2k)^{2t+1})$; when $k = 2l+2$, let $C_{2k-1}^{(r)} = \mathcal{A}_x + (a_i, b_{2l+2-i})$, the blocks $\{\mathcal{A}_x + (a_i, b_{2l+2-i}) :\ 1 \leq x \leq t,\}$ $(\forall\ 1 \leq i \leq 2l+1$ and $i \neq l+1)$ module $(2k, 2t+1)$ give the expected $C_{2k-1}^{(r)}$-$HD((2k)^{2t+1})$. □

Theorem 2.6 *There exist $C_{2k-1}^{(r)}$-$HD((4k)^{2t+1})$ for $k \geq 3,\ t \geq 1$ and $1 \leq r \leq k-2$.*

Proof. Let $X = Z_{4k} \times Z_{2t+1} = \bigcup_{x \in Z_{2t+1}} V_x$, where $V_x = Z_{4k} \times \{x\}$. For $1 \le x \le t$ and $a_i,\ b_i,\ a_i^{'},\ b_i^{'} \in Z_{4k}$, define the following $(2k-1)$-circuits $\mathcal{A}_x$, $\mathcal{B}_x$ over X:



$(k = 2l+1)$



$(k = 2l+2)$

The vertices $u, v, p, q, a_i, b_i, a_i^{'}, b_i^{'}$ are defined as follows:

$$\begin{cases} u = (k, 0), & \\ v = (\frac{k+1}{2}, 0) \text{ or } (\frac{k+4}{2}, 0), & k \text{ odd or even} \\ p = (-(2k-1), x), & \\ q = (-\frac{k+3}{2}, 0), \text{ or } (-\frac{k}{2}, 0), & k \text{ odd or even} \\ a_i = (i, x), & 1 \le i \le k \text{ and } i \ne \lfloor \frac{k+1}{2} \rfloor, \\ b_i = (-i, 2x), & 1 \le i \le k-1 \text{ and } i \ne \lfloor \frac{k}{2} \rfloor, \\ a_i^{'} = (i, 2x), & 0 \le i \le k-1 \text{ and } i \ne \lfloor \frac{k+1}{2} \rfloor, \\ b_i^{'} = (-i, x), & 1 \le i \le k-1 \text{ and } i \ne \lfloor \frac{k}{2} \rfloor; \end{cases}$$

In each $\mathcal{A}_x$ or $\mathcal{B}_x$, $1 \le x \le t$, the $2k-1$ vertices are distinct obviously.

For odd k, say $k = 2l+1$, we can verify that the edge (u, a_{2l+1}) in $\mathcal{A}_x$ and the edge $(p, a_1^{'})$ in $\mathcal{B}_x$ just correspond to the mixed differences 0_x and $(2k)_x$. The mixed differences $(2l+2)_{2x}$ and $-(2l+2)_{2x}$ correspond to the edge $(a_l^{'}, q)$ in $\mathcal{B}_x$ and the edge (v, b_{l+1}) in $\mathcal{A}_x$ respectively. Other edges in $\mathcal{A}_x$ and $\mathcal{B}_x$ just correspond to the mixed differences $\pm d_x$ ($1 \le d \le 4l+1$ and $d \ne 2l+1$), and the remaining mixed differences $(2l+1)_x$ and $-(2l+1)_x$ may correspond to any chord $(a_i^{'}, b_{2l+1-i}^{'})$ in $\mathcal{B}_x$ and any chord (a_i, b_{2l+1-i}) in $\mathcal{A}_x$, where $1 \le i \le 2l$ and $i \ne l+1$. Thus, the chord of $C_{2k-1}^{(r)}$ can be chosen as $(a_i^{'}, b_{2l+1-i}^{'})$ and (a_i, b_{2l+1-i}), $\forall\ 1 \le i \le 2l$ and

$i \neq l+1$. Clearly, let $C_{2k-1}^{(r)} = \mathcal{A}_x + (a_i, b_{2l+1-i})$ or $\mathcal{B}_x + (a_i^{'}, b_{2l+1-i}^{'})$, the blocks $\{\mathcal{A}_x + (a_i, b_{2l+1-i}),\ \mathcal{B}_x + (a_i^{'}, b_{2l+1-i}^{'})\}$ $(1 \leq x \leq t,\ \forall\ 1 \leq i \leq 2l$ and $i \neq l+1)$ module $(4k, 2t+1)$ give the expected $C_{2k-1}^{(r)}$-$HD((4k)^{2t+1})$.

For even k, say $k = 2l+2$, we can verify that the edge (u, a_{2l+2}) in $\mathcal{A}_x$ and the edge $(p, a_1^{'})$ in $\mathcal{B}_x$ just correspond to the mixed differences 0_x and $(2k)_x$. The mixed differences $(2l+3)_{2x}$ and $-(2l+3)_{2x}$ correspond to the edge $(q, a_{l+2}^{'})$ in $\mathcal{B}_x$ and the edge (b_l, v) in $\mathcal{A}_x$ respectively. Other edges in $\mathcal{A}_x$ and $\mathcal{B}_x$ just correspond to the mixed differences $\pm d_x$ $(1 \leq d \leq 4l+3$ and $d \neq 2l+2)$, and the remaining mixed differences $(2l+2)_x$ and $-(2l+2)_x$ may correspond to any chord $(a_i^{'}, b_{2l+2-i}^{'})$ in $\mathcal{B}_x$ and any chord (a_i, b_{2l+2-i}) in $\mathcal{A}_x$, where $1 \leq i \leq 2l+1$ and $i \neq l+1$. Thus, the chord of $C_{2k-1}^{(r)}$ can be chosen as $(a_i^{'}, b_{2l+2-i}^{'})$ and (a_i, b_{2l+2-i}), $\forall\ 1 \leq i \leq 2l+1$ and $i \neq l+1$. Clearly, let $C_{2k-1}^{(r)} = \mathcal{A}_x + (a_i, b_{2l+2-i})$ or $\mathcal{B}_x + (a_i^{'}, b_{2l+2-i}^{'})$, the blocks $\{\mathcal{A}_x + (a_i, b_{2l+2-i}),\ \mathcal{B}_x + (a_i^{'}, b_{2l+2-i}^{'})\}$ $(1 \leq x \leq t,\ \forall\ 1 \leq i \leq 2l+1$ and $i \neq l+1)$ module $(4k, 2t+1)$ give the expected $C_{2k-1}^{(r)}$-$HD((4k)^{2t+1})$. □

Lemma 2.7 *If there exists a $C_{2k}^{(r)}$-$ID(2k+1+w; w)$ for odd r, then there are $\lfloor \frac{k+2}{3} \rfloor$ nonnegative integers $j_0, j_1, \cdots, j_{\lfloor \frac{k-1}{3} \rfloor}$ such that*

$$\sum_{i=0}^{\lfloor \frac{k-1}{3} \rfloor} j_i = w \quad and \quad \sum_{i=0}^{\lfloor \frac{k-1}{3} \rfloor} i j_i \leq min\{\frac{k}{2}, k^2 - w\}.$$

Proof. Suppose $(X \cup Y, \mathcal{B})$ be a $C_{2k}^{(r)}$-$ID(2k+1+w; w)$, where $|X| = 2k+1$, $|Y| = w$, $X \bigcap Y = \emptyset$ and $|\mathcal{B}| = k+w$. A vertex $y \in Y$ appearing in a block B of $\mathcal{B}$ may be 2-degree or 3-degree, denoted by $d(y, B) = 2$ or $d(y, B) = 3$ respectively. For any $y \in Y$, denote

$$m_s(y) = |\{B \in \mathcal{B} : y \in B,\ d(y, B) = s\}|, \quad s = 2,\ 3.$$

Then, the equation $2m_2(y) + 3m_3(y) = |X| = 2k+1$ will give solutions

$$m_2(y) = k - 3i - 1,\ m_3(y) = 2i + 1,\ 0 \leq i \leq \lfloor \frac{k-1}{3} \rfloor.$$

For $0 \leq i \leq \lfloor \frac{k-1}{3} \rfloor$, denote

$$j_i = |\{y \in Y : m_2(y) = k - 3i - 1,\ m_3(y) = 2i + 1\}|,$$

then $\sum_{i=0}^{\lfloor \frac{k-1}{3} \rfloor} j_i = |Y| = w$. Let $N = \sum_{i=0}^{\lfloor \frac{k-1}{3} \rfloor} i j_i$, then the total number of 2-degree vertices and 3-degree vertices belonging to Y is respectively

$$M_2 = \sum_{i=0}^{\lfloor \frac{k-1}{3} \rfloor} (k - 3i - 1) j_i = (k-1)w - 3N, \text{ and}$$

$$M_3 = \sum_{i=0}^{\lfloor \frac{k-1}{3} \rfloor} (2i+1) j_i = w + 2N.$$

Since Y is a hole for the incomplete graph design, any vertices of Y can not be adjacent in any block. Thus, for any block B of $\mathcal{B}$, there are two cases:

(1) The block B contains one 3-degree Y-vertex and at most p 2-degree Y-vertices, where

$$p = \lfloor \frac{r}{2} \rfloor + \lfloor \frac{2k-2-r}{2} \rfloor = k-1+\lfloor \frac{r}{2} \rfloor - \lceil \frac{r}{2} \rceil = \begin{cases} k-1 & (r \ \text{even}) \\ k-2 & (r \ \text{odd}) \end{cases}.$$

(2) The block B contains no 3-degree Y-vertex and at most q 2-degree Y-vertices, where

$$q = \lceil \frac{r}{2} \rceil + \lceil \frac{2k-2-r}{2} \rceil = k-1-\lfloor \frac{r}{2} \rfloor + \lceil \frac{r}{2} \rceil = \begin{cases} k-1 & (r \ \text{even}) \\ k & (r \ \text{odd}) \end{cases}.$$

Therefore, we have the following conditions

$$\begin{cases} M_3 \le |\mathcal{B}|, \quad \text{i.e.} \ \ N \le \frac{k}{2} \\ M_2 \le pM_3 + q(|\mathcal{B}| - M_3) = \begin{cases} (w+k)(k-1) & (r \quad \text{even}) \\ (w+k)k - 2(w+2N) & (r \quad \text{odd}) \end{cases} \end{cases}.$$

When r even the second condition is $k(k-1)+3N \ge 0$, which always holds. As for odd r, the second condition is $N \le k^2 - w$. Thus, for odd r, a necessary condition to exist $C_{2k}^{(r)}$-$ID(2k+1+w;w)$ is $N = \sum_{i=0}^{\lfloor \frac{k-1}{3} \rfloor} ij_i \le min\{\frac{k}{2}, k^2-w\}$. □

Corollary 2.8 *There exists no* $C_{2k}^{(r)}$-$ID(2k+1+w;w)$ *for the following parameters:* $(k,r) = (2,1)$ *and* $5 \le w \le 9$*;* $(k,r) = (3,1)$ *and* $10 \le w \le 13$*;* $(k,r) = (4,1),(4,3)$ *and* $w = 17$*.*

Proof. When r is odd and $w > k^2$, we have $\min\{\frac{k}{2},\ k^2-w\} < 0$. Since $N = \sum_{i=0}^{\lfloor \frac{k-1}{3} \rfloor} ij_i \ge 0$, there exists no $C_{2k}^{(r)}$-$ID(2k+1+w;w)$ by Lemma 2.7. For our constructing method stated in Theorem 2.1, the needed $C_{2k}^{(r)}$-$ID(2k+1+w;w)$ are only for

$$3 \le w \le 2k \quad (r \ \ \text{even}) \quad \text{and} \quad 3 \le w \le 4k+1 \quad (r \ \ \text{odd}).$$

However, $k^2 > 4k+1$ when $k \ge 5$. So, when r is odd, the non-existence of the needed $C_{2k}^{(r)}$-$ID(2k+1+w;w)$ happens only for $2 \le k \le 4$ and $k^2 < w \le 4k+1$, i.e., it is impossible that the following incomplete $C_{2k}^{(r)}$-$ID(2k+1+w;w)$ exist for the parameters listed in the Collorary. □

Lemma 2.9 *There exists no* $C_{2k-1}^{(r)}$-$ID(2k+w;w)$ *for any* $w \ge 0$*.*

Proof. The graph $C_{2k-1}^{(r)}$ consists of $2k$ edges. A $C_{2k-1}^{(r)}$-$ID(2k+w;w)$ will cover $k(2k-1)+2kw$ pairs, which is not a multiple of $2k$. So there exists no $C_{2k-1}^{(r)}$-$ID(2k+w;w)$ for any w. □

3 $C_6^{(1)}$ and $C_6^{(2)}$

The necessary conditions for the existence of $(v, C_6^{(r)}, \lambda)$-GD are $\lambda v(v-1) \equiv 0 \pmod{14}$ and $v \geq 6$, i.e., $v \equiv 0, 1 \pmod 7$ for any λ, and $v \equiv 2, 3, 4, 5, 6 \pmod 7$ for $\lambda \equiv 0 \pmod 7$. For convenience, we denote $C_6^{(1)}$ (or $C_6^{(2)}$) by (a, b, c, d, e, f), where the edges on C_6 are ab, bc, cd, de, ef, fa and the chord is ac (or ad). It is enough to discuss the cases only for $\lambda = 1$ and 7. Because the results for $\lambda = 1$ are known (see [2,3]), we only need to solve the cases for $\lambda = 7$. By Theorem 2.1 or Theorem 2.2 and the following tables, we only need to give the constructions of ID or IHD, GD for the pointed orders.

(Table 3.1) For $C_6^{(1)}$

v (mod 14)	HD	ID	IHD	GD $\lambda = 7$
2	7^{2t-1}	$(16; 9)$		9
3	7^{2t-1}		$(7, 7; 10)$	17
4	7^{2t-1}		$(7, 7; 11)$	18
5	7^{2t-1}		$(7, 7; 12)$	19
6	7^{2t-1}		$(7, 7; 13)$	20
9	7^{2t+1}	$(9; 2)$		9
10	7^{2t+1}	$(10; 3)$		10
11	7^{2t+1}	$(11; 4)$		11
12	7^{2t+1}	$(12; 5)$		12
13	7^{2t+1}	$(13; 6)$		6

(Table 3.2) For $C_6^{(2)}$

v (mod 7)	HD	ID	GD $\lambda = 7$
2	7^t	$(9; 2)$	9
3	7^t	$(10; 3)$	10
4	7^t	$(11; 4)$	11
5	7^t	$(12; 5)$	12
6	7^t	$(13; 6)$	6

3.1 Incomplete $C_6^{(r)}$-designs

Lemma 3.1 *There exist* $C_6^{(1)}$-$ID(7+w; w)$ *for* $2 \leq w \leq 6$ *and* $w = 9$.

Proof. Let $X = Z_7 \cup \{\infty_1, \infty_2, \cdots, \infty_w\}$ and $C_6^{(1)}$-$ID(w+7; w)$=$(X, \mathcal{B})$, where $|\mathcal{B}| = w + 3$. The family $\mathcal{B}$ consists of the blocks listed in Appendix A(L3.1). □

Lemma 3.2 *There exist* $C_6^{(1)}$-$IHD(7, 7; h)$ *for* $10 \leq h \leq 13$.

Proof. Let $X = Z_7 \cup \bar{Z}_7 \cup \{\infty_1, \infty_2, \cdots, \infty_h\}$ and $C_6^{(1)}$-$IHD(7, 7; h) = (X, \mathcal{B})$, where $|\mathcal{B}| = 2(w + 3)$. The family $\mathcal{B}$ consists of the blocks listed in Appendix A(L3.2). □

Lemma 3.3 *There exist $C_6^{(2)}$-$ID(7+w;w)$ for $2 \le w \le 6$.*

Proof. Let $X = Z_7 \cup \{\infty_1, \infty_2, \cdots, \infty_w\}$ and $C_6^{(2)}$-$ID(w+7;w)$=$(X, \mathcal{B})$, where $|\mathcal{B}| = w+3$. The family $\mathcal{B}$ consists of the blocks listed in Appendix A(L3.3). □

3.2 Graph designs

Lemma 3.4 *There exist $(w, C_6^{(1)}, 7)$-GD for $w = 6, 9, 10, 11, 12, 17, 18, 19, 20$.*

Proof. For each order w, the corresponding base blocks under the automorphism group Z_m are listed in Appendix B(L3.4), where the vertex-set X is Z_m or $Z_m \cup \{\infty\}$. □

Theorem 3.5 *There exist $(v, C_6^{(1)}, \lambda)$-GD if and only if $\lambda v(v-1) \equiv 0$ $(mod\ 14)$ and $v \ge 6$.*

Proof. By Theorems 2.1, 2.2 and Lemmas 3.1, 3.2, 3.4 and the result for $\lambda = 1$ in [19]. □

Lemma 3.6 *There exists $(7, C_6^{(2)}, \lambda)$-GD for any $\lambda \ge 2$.*

Proof. $(7, C_6^{(2)}, 2)$-GD: $X = (Z_3 \times Z_2) \cup \{\infty\}$
$(0_0, \infty, 0_1, 1_0, 1_1, 2_1)$ mod $(3, 2)$.
$(7, C_6^{(2)}, 3)$-GD: $X = (Z_3 \times Z_2) \cup \{\infty\}$
$(\infty, 0_0, 1_1, 1_0, 2_0, 0_1)$ mod $(3, 2)$.
$(0_0, 2_0, 1_0, 0_1, 2_1, 1_1)$ mod $(3, -)$.
Obviously, there are nonnegative integers m and n such that $\lambda = 2m + 3n$ for any $\lambda \ge 2$. Thus, we may assert that $(7, C_6^{(2)}, \lambda)$-GD exists for any $\lambda \ge 2$. □

Lemma 3.7 *There exist $(w, C_6^{(2)}, 7)$-GD for $w = 6, 9, 10, 11$ and 12.*

Proof. For each order w, the corresponding base blocks under the automorphism group Z_m are listed in Appendix B(L3.7), where the vertex-set X is Z_m or $Z_m \cup \{\infty\}$. □

Theorem 3.8 *There exist $(v, C_6^{(2)}, \lambda)$-GD if and only if $\lambda v(v-1) \equiv 0$ $(mod\ 14)$, $v \ge 6$ and $(v, \lambda) \ne (7, 1)$.*

Proof. By Theorem 2.1 and Lemmas 3.3, 3.6, 3.7 and the result for $\lambda = 1$ in [2, 3]. □

4 $C_7^{(1)}$ and $C_7^{(2)}$

For convenience, we denote $C_7^{(1)}$ and $C_7^{(2)}$ by (a, b, c, d, e, f, g), where the edges on C_7 are $ab, bc, cd,$
de, ef, fg, ga and the chord is ac (or ad). It is clear that the necessary conditions for the existence of $(v, C_7^{(r)}, \lambda)$-GD are $\lambda v(v-1) \equiv 0 \pmod{16}$ and $v \geq 7$, that is

(i) $v \equiv 0$ or $1 \pmod{16}$ and any λ;
(ii) $v \equiv 8$ or $9 \pmod{16}$ and $\lambda \equiv 0 \pmod 2$;
(iii) $v \equiv 4, 5, 12$ or $13 \pmod{16}$ and $\lambda \equiv 0 \pmod 4$;
(iv) $v \equiv 2, 3, 6, 7, 10, 11, 14$ or $15 \pmod{16}$ and $\lambda \equiv 0 \pmod 8$.

When $\lambda = 1$, the results are known in [2, 3], so by Theorem 2.1 or Theorem 2.2 and the following table, we only need to construct ID, GD, and IHD for the pointed orders.

(Table 4.1) For $C_7^{(r)} (r = 1, 2)$

v (mod 16)	HD	ID	IHD	GD $\lambda = 2$	GD $\lambda = 4$	GD $\lambda = 8$
2	8^{2t-1}		$(8, 8; 10)$			18
3	8^{2t-1}		$(8, 8; 11)$			19
4	8^{2t-1}		$(8, 8; 12)$		20	
5	8^{2t-1}		$(8, 8; 13)$		21	
6	16^{2t+1}	$(38; 22), (22, 6)$				22
7	16^{2t+1}	$(39; 23), (23, 7)$				$7, 23$
8	16^{2t+1}	$(40; 24), (24, 8)$		$8, 24$		
9	16^{2t+1}	$(41; 25), (25, 9)$		$9, 25$		
10	8^{2t+1}		$(8, 8; 2)$			10
11	8^{2t+1}		$(8, 8; 3)$			11
12	8^{2t+1}		$(8, 8; 4)$		12	
13	8^{2t+1}		$(8, 8; 5)$		13	
14	8^{2t+1}		$(8, 8; 6)$			14
15	8^{2t+1}		$(8, 8; 7)$			15

4.1 Incomplete $C_7^{(r)}$-designs

By Lemma 2.9, there exists no $C_7^{(r)}$-ID$(8+w; w)$ for $w \geq 0$ and $r = 1, 2$.

Lemma 4.1 *There exist* $C_7^{(1)}$-$IHD(8, 8; h)$ *for* $2 \leq h \leq 7$ *and* $10 \leq h \leq 13$.

Proof. Let $X = (Z_8 \times Z_2) \cup \{\infty_1, \infty_2, \cdots, \infty_h\}$ and $C_7^{(1)}$-$IHD(8, 8; h) = (X, \mathcal{B})$, where $|\mathcal{B}| = 7+2h$. The block set $\mathcal{B}$ consists of the blocks listed in Appendix C(L4.1). □

Lemma 4.2 *There exist* $C_7^{(1)}$-$ID(16+w; w)$ *for* $6 \leq w \leq 9$ *and* $22 \leq w \leq 25$.

Proof. Let $X = Z_{16} \cup \{\infty_1, \infty_2, \cdots, \infty_w\}$ and $C_7^{(1)}$-$ID(16+w; w)$=$(X, \mathcal{B})$, where $|\mathcal{B}| = 2w + 15$ and $6 \leq w \leq 9$. The block set $\mathcal{B}$ consists of the blocks listed in Appendix C(L4.2).

For $22 \le w \le 25$, let $C_7^{(1)}$-$IHD(8,8;w-12) = (X,\ \mathcal{B}_0)$, where $X = (Z_8 \times Z_2) \cup \{\infty_1, \infty_2, \cdots,$
$\infty_{w-12}\}$, $\mathcal{B}_0$ is from Lemma 4.1 and $|\mathcal{B}_0| = 2w-17$. Then $C_7^{(1)}$-$ID(16+w;w)$=$(Y, \mathcal{B}_0 \cup \mathcal{B}_1)$, where $Y = X \cup \{\infty_{w-11}, \infty_{w-10}, \cdots, \infty_w\}$ and $|\mathcal{B}_1| = 32$, so $|\mathcal{B}_0| + |\mathcal{B}_1| = (2w-17)+32 = 2w+15$. The family $\mathcal{B}_1$ consists of the following blocks:

$$\underline{w=22}: \left.\begin{array}{ll} (0_0,\infty_{22},0_1,\infty_{20},1_0,2_1,\infty_{21}), & (0_0,\infty_{19},2_1,\infty_{17},1_0,4_1,\infty_{18}), \\ (0_0,\infty_{16},4_1,\infty_{14},1_0,6_1,\infty_{15}), & (0_0,\infty_{13},6_1,\infty_{11},1_0,0_1,\infty_{12}). \end{array}\right\} \pmod 8$$

$$\underline{w=23}: \left.\begin{array}{ll} (0_0,\infty_{22},0_1,\infty_{20},1_0,2_1,\infty_{21}), & (0_0,\infty_{19},2_1,\infty_{17},1_0,4_1,\infty_{18}), \\ (0_0,\infty_{16},4_1,\infty_{14},1_0,6_1,\infty_{15}), & (0_0,\infty_{13},6_1,\infty_{23},1_0,0_1,\infty_{12}). \end{array}\right\} \pmod 8$$

$$\underline{w=24}: \left.\begin{array}{ll} (0_0,\infty_{22},0_1,\infty_{20},1_0,2_1,\infty_{21}), & (0_0,\infty_{19},2_1,\infty_{17},1_0,4_1,\infty_{18}), \\ (0_0,\infty_{16},4_1,\infty_{14},1_0,6_1,\infty_{15}), & (0_0,\infty_{13},6_1,\infty_{23},1_0,0_1,\infty_{24}). \end{array}\right\} \pmod 8$$

$$\underline{w=25}: \left.\begin{array}{ll} (0_0,\infty_{22},0_1,\infty_{20},1_0,2_1,\infty_{21}), & (0_0,\infty_{19},2_1,\infty_{17},1_0,4_1,\infty_{18}), \\ (0_0,\infty_{16},4_1,\infty_{14},1_0,6_1,\infty_{15}), & (0_0,\infty_{23},6_1,\infty_{24},1_0,0_1,\infty_{25}). \end{array}\right\} \pmod 8$$

□

Lemma 4.3 *There exist* $C_7^{(2)}$-$IHD(8,8;h)$ *for* $2 \le h \le 7$ *and* $10 \le h \le 13$.

Proof. Let $X = (Z_8 \times Z_2) \cup \{\infty_1, \infty_2, \cdots, \infty_h\}$ and $C_7^{(2)}$-$IHD(8,8;h) = (X, \mathcal{B})$, where $|\mathcal{B}| = 7+2h$. The block set $\mathcal{B}$ consists of the blocks listed in Appendix C(L4.3). □

Lemma 4.4 *There exist* $C_7^{(2)}$-$ID(16+w;w)$ *for* $6 \le w \le 9$ *and* $22 \le w \le 25$.

Proof. Let $X = Z_{16} \cup \{\infty_1, \infty_2, \cdots, \infty_w\}$ and $C_7^{(2)}$-$ID(16+w;w)$=$(X, \mathcal{B})$, where $|\mathcal{B}| = 2w+15$ and $6 \le w \le 9$. The block set $\mathcal{B}$ consists of the blocks listed in Appendix C(L4.4-1).

For $22 \le w \le 25$, let $X = (Z_8 \times Z_2) \cup \{\infty_1, \infty_2, \cdots, \infty_w\}$ and $C_7^{(2)}$-$ID(16+w;w)$=$(X, \mathcal{B})$, where $|\mathcal{B}| = 2w+15$. The family $\mathcal{B}$ consists of the blocks listed in Appendix C(L4.4-2). □

4.2 Graph designs for $C_7^{(r)}$

In this section, the symbol $(a,b,c,d,e,f,g) \times n$ means the block (a,b,c,d,e,f,g) occurs n times.

Lemma 4.5 *There exist* $(w, C_7^{(1)}, \lambda)$-*GD for*
(i) $\lambda = 2$ *and* $w = 8, 9, 24, 25$. (ii) $\lambda = 4$ *and* $w = 12, 13, 20, 21$.
(iii) $\lambda = 8$ *and* $w = 7, 10, 11, 14, 15, 18, 19, 22, 23$.

Proof. The constructions are listed in Appendix D (L4.5). □

Theorem 4.6 *There exist* $(v, C_7^{(1)}, \lambda)$-*GD if and only if* $\lambda v(v-1) \equiv 0 \pmod{16}$ *and* $v \ge 7$.

Proof. By Lemmas 4.1, 4.2, 4.5 and the result for $\lambda = 1$ in [2,3]. □

Lemma 4.7 *There exist* $(w, C_7^{(2)}, \lambda)$*-GD for*
(i) $\lambda = 2$ *and* $w = 8, 9, 24, 25$; (ii) $\lambda = 4$ *and* $w = 12, 13, 20, 21$;
(iii) $\lambda = 8$ *and* $w = 7, 10, 11, 14, 15, 18, 19, 22, 23$.

Proof. The constructions are listed in Appendix D (L4.7). □

Theorem 4.8 *There exist* $(v, C_7^{(2)}, \lambda)$*-GD if and only if* $\lambda v(v-1) \equiv 0$ $(mod\ 16)$ *and* $v \geq 7$.

Proof. By Lemmas 4.3, 4.4, 4.7 and the result for $\lambda = 1$ in [2,3]. □

5 $C_8^{(1)}$,$C_8^{(2)}$ and $C_8^{(3)}$

The necessary conditions for the existence of $(v, C_8^{(r)}, \lambda)$-GD are $\lambda v(v-1) \equiv 0$ (mod 18) and $v \geq 8$, i.e.,

(i) $v \equiv 0, 1$ (mod 9) for any λ,
(ii) $v \equiv 3, 4, 6, 7$ (mod 9) for $\lambda \equiv 0$ (mod 3),
(iii) $v \equiv 2, 5, 8$ (mod 9) for $\lambda \equiv 0$ (mod 9).

For convenience, we denote $C_8^{(1)}$ (or $C_8^{(2)}$, or $C_8^{(3)}$) by (a, b, c, d, e, f, g, h), where the edges on C_8 are ab, bc, cd, de, ef, fg, gh, ha and the chord is ac (or ad, or ae). The results for $\lambda = 1$ have been known in [2,3], so by Theorem 2.1 or Theorem 2.2 and the following tables, we only need to construct ID or IHD, and GD for the pointed orders.

(Table 5.1) For $C_8^{(1)}$ and $C_8^{(3)}$

v (mod 18)	HD	ID	IHD	GD $\lambda = 3$	GD $\lambda = 9$
2	9^{2t-1}	(20; 11)			11
3	9^{2t-1}	(21; 12)		12	
4	9^{2t-1}	(22; 13)		13	
5	9^{2t-1}	(23; 14)			14
6	9^{2t-1}	(24; 15)		15	
7	9^{2t-1}	(25; 16)		16	
8	9^{2t-1}		(9, 9; 17)		26
11	9^{2t+1}	(11; 2)			11
12	9^{2t+1}	(12; 3)		12	
13	9^{2t+1}	(13; 4)		13	
14	9^{2t+1}	(14; 5)			14
15	9^{2t+1}	(15; 6)		15	
16	9^{2t+1}	(16; 7)		16	
17	9^{2t+1}	(17; 8)			8,17

(Table 5.2) For $C_8^{(2)}$

v (mod 9)	HD	ID	GD $\lambda=3$	GD $\lambda=9$
2	9^t	(11; 2)		11
3	9^t	(12; 3)	12	
4	9^t	(13; 4)	13	
5	9^t	(14; 5)		14
6	9^t	(15; 6)	15	
7	9^t	(16; 7)	16	
8	9^t	(17; 8)		8

5.1 Incomplete designs for $C_8^{(r)}$

Lemma 5.1 *There exist $C_8^{(1)}$-$ID(9+w;w)$ for $2 \le w \le 8$ and $11 \le w \le 16$.*

Proof. $\underline{w=2}$: $X=(Z_3 \times Z_3) \cup \{x_1, x_2\}$

$$(x_1, 1_0, 0_1, 2_0, 2_2, 1_1, 0_2, 1_2),\ (x_2, 1_2, 1_1, 0_1, 0_0, 2_0, 0_2, 1_0) \quad \text{mod } (3,-).$$

When $w \ge 3$, let $X = Z_9 \cup \{x_1, x_2, \cdots, x_w\}$ and $C_8^{(1)}$-$ID(w+9;w)$=$(X,\mathcal{B})$, where $|\mathcal{B}| = w+4$. The block set $\mathcal{B}$ consists of the blocks listed in Appendix E(L5.1). □

Lemma 5.2 *There exists a $C_8^{(1)}$-$IHD(9,9;17)$.*

Proof. Let $X = Z_9 \cup \bar{Z}_9 \cup \{x_1, x_2, \cdots, x_{17}\}$ and $(X,\mathcal{B})$ be a $C_8^{(1)}$-$IHD(9,9;17)$, where $\mathcal{B} = \mathcal{B}_1 \cup \bar{\mathcal{B}_1} \cup \mathcal{B}_2$ and $\bar{\mathcal{B}_1}$ is obtained from $\mathcal{B}_1$ by replacing every $i \in Z_9$ with $\bar{i} \in \bar{Z}_9$. The families $\mathcal{B}_1$ and $\mathcal{B}_2$ consist of the following blocks:

$$\begin{array}{llll}
\mathcal{B}_1: & (x_1,1,0,x_3,3,2,x_2,8), & (x_6,4,5,x_3,7,0,x_4,6), & (x_{11},6,1,x_7,4,3,x_8,0),\\
& (x_2,5,1,x_5,8,7,x_1,6), & (x_7,7,6,x_3,8,2,x_4,5), & (x_{12},7,2,x_7,8,6,x_8,0),\\
& (x_3,1,2,x_1,5,3,x_2,4), & (x_8,4,7,x_5,5,2,x_6,1), & (x_{13},6,3,x_9,2,0,x_{11},4),\\
& (x_4,1,3,x_1,4,0,x_2,7), & (x_9,1,8,x_6,7,3,x_7,0), & (x_{14},1,4,x_9,5,0,x_{13},2),\\
& (x_5,2,4,x_4,8,0,x_6,3), & (x_{10},3,0,x_5,6,5,x_8,2), & (x_{15},8,5,x_{10},4,6,x_9,7).
\end{array}$$

$$\begin{array}{lll}
\mathcal{B}_2: & (5,x_{11},7,x_{16},0,x_{17},1,x_{13}), & (4,x_{16},8,x_{13},7,x_{17},5,x_{12}),\\
& (6,x_{14},0,x_{15},\bar{1},x_{17},3,x_{16}), & (7,x_{10},1,x_{12},6,x_{15},3,x_{14}),\\
& (8,x_{12},3,x_{11},2,x_{15},4,x_{17}), & (2,x_{17},6,x_{10},8,x_{14},5,x_{16}),\\
& (\bar{5},x_{11},\bar{7},x_{17},\bar{0},x_{16},\bar{1},x_{13}), & (\bar{4},x_{17},\bar{8},x_{13},\bar{7},x_{16},\bar{5},x_{12}),\\
& (\bar{6},x_{14},\bar{0},x_{15},1,x_{16},\bar{3},x_{17}), & (\bar{7},x_{10},\bar{1},x_{12},\bar{6},x_{15},\bar{3},x_{14}),\\
& (\bar{8},x_{12},\bar{3},x_{11},\bar{2},x_{15},\bar{4},x_{16}), & (\bar{2},x_{16},\bar{6},x_{10},\bar{8},x_{14},\bar{5},x_{17}).
\end{array}$$
□

Lemma 5.3 *There exist $C_8^{(2)}$-$ID(9+w;w)$ for $2 \le w \le 8$.*

Proof. $\underline{w=2}$: $X=(Z_3 \times Z_3) \cup \{x_1, x_2\}$

$$(x_1, 1_0, 1_1, 0_1, 0_2, 1_2, 2_0, 2_2),\ (x_2, 0_0, 1_0, 0_1, 2_0, 0_2, 1_1, 2_2) \quad \text{mod } (3,-).$$

When $w \ge 3$, let $X = Z_9 \cup \{x_1, x_2, \cdots, x_w\}$ and $C_8^{(2)}$-$ID(w+9;w)$=$(X,\mathcal{B})$, where $|\mathcal{B}| = w+4$. The block set $\mathcal{B}$ consists of the blocks listed in Appendix E(L5.3). □

Lemma 5.4 *There exist $C_8^{(3)}$-$ID(9+w;w)$ for $2 \le w \le 8$ and $11 \le w \le 16$.*

Proof. $\underline{w=2}$: $X = (Z_3 \times Z_3) \cup \{x_1, x_2\}$

$$(x_1, 0_0, 0_1, 0_2, 1_2, 2_0, 1_1, 2_1),\ (x_2, 1_0, 2_0, 0_1, 1_2, 0_0, 0_2, 1_1) \mod (3, -).$$

When $w \ge 3$, let $X = Z_9 \cup \{x_1, x_2, \cdots, x_w\}$ and $C_8^{(3)}$-$ID(w+9;w)$=$(X, \mathcal{B})$, where $|\mathcal{B}| = w+4$. The block set $\mathcal{B}$ consists of the blocks listed in Appendix E(L5.4). □

Lemma 5.5 *There exists a $C_8^{(3)}$-$IHD(9, 9; 17)$.*

Proof. Let $X = Z_9 \cup \bar{Z}_9 \cup \{x_1, x_2, \cdots, x_{17}\}$ and $(X, \mathcal{B})$ be a $C_8^{(3)}$-$IHD(9, 9; 17)$, where $\mathcal{B} = \mathcal{B}_1 \cup \bar{\mathcal{B}_1} \cup \mathcal{B}_2$ and $\bar{\mathcal{B}_1}$ is obtained from $\mathcal{B}_1$ by replacing every $i \in Z_9$ with $\bar{i} \in \bar{Z}_9$. The families $\mathcal{B}_1$ and $\mathcal{B}_2$ consist of the following blocks:

$\mathcal{B}_1$: $(x_1, 4, 8, x_2, 0, 1, x_3, 7)$, $(x_{10}, 1, 4, x_6, 0, 2, x_7, 5)$, $(x_6, 8, x_5, 2, 5, x_4, 1, 3)$,
$(x_2, 6, x_3, 8, 1, x_1, 5, 4)$, $(x_{13}, 1, x_9, 7, 3, x_8, 8, 2)$, $(x_7, 7, 1, x_6, 6, 8, x_4, 4)$,
$(x_3, 0, 7, x_2, 2, x_1, 6, 3)$, $(x_{12}, 0, x_9, 3, 2, x_8, 5, 6)$, $(x_8, 1, 2, x_6, 7, x_5, 3, 0)$,
$(x_4, 0, 5, x_2, 3, x_1, 8, 7)$, $(x_{11}, 6, 7, x_{12}, 1, x_7, 0, 4)$, $(x_9, 5, x_5, 0, 8, x_7, 3, 4)$,
$(x_5, 6, x_4, 2, 4, x_3, 5, 1)$, $(x_{14}, 2, x_9, 6, 4, x_{10}, 3, 5)$, $(x_{15}, 2, 6, x_{13}, 5, 7, x_{11}, 3)$.

$\mathcal{B}_2$: $(6, x_{17}, 5, x_{11}, 0, x_{16}, 4, x_8)$, $(\bar{6}, x_{16}, \bar{5}, x_{11}, \bar{0}, x_{17}, \bar{4}, x_8)$,
$(1, x_{15}, 0, x_{14}, 6, x_{10}, 2, x_{17})$, $(\bar{1}, x_{15}, \bar{0}, x_{14}, \bar{6}, x_{10}, \bar{2}, x_{16})$,
$(7, x_{15}, 6, x_{16}, 2, x_{11}, 8, x_{10})$, $(4, x_{15}, 8, x_{14}, 7, x_{13}, 0, x_{17})$,
$(\bar{7}, x_{15}, \bar{6}, x_{17}, \bar{2}, x_{11}, \bar{8}, x_{10})$, $(\bar{4}, x_{15}, \bar{8}, x_{14}, \bar{7}, x_{13}, \bar{0}, x_{16})$,
$(8, x_{12}, \bar{3}, x_{17}, 3, x_{14}, 1, x_{16})$, $(5, x_{16}, 7, x_{17}, 8, x_{13}, 4, x_{12})$,
$(\bar{8}, x_{12}, 3, x_{16}, \bar{3}, x_{14}, \bar{1}, x_{17})$, $(\bar{5}, x_{17}, \bar{7}, x_{16}, \bar{8}, x_{13}, \bar{4}, x_{12})$. □

5.2 Graph designs

Lemma 5.6 *There exist $(w, C_8^{(1)}, 3)$-GD for w=12, 13, 15, 16.*

Proof. The blocks are listed in Appendix F(L5.6). □

Lemma 5.7 *There exist $(w, C_8^{(1)}, 9)$-GD for $w = 8$, 11, 14, 17, 26.*

Proof. For each order w, the corresponding base blocks under the automorphism group Z_m are listed in Appendix F(L5.7), where the vertex-set X is Z_m or $Z_m \cup \{\infty\}$ and one base block $B \times k$ will always mean that it is repeated k times. □

Theorem 5.8 *There exist $(v, C_8^{(1)}, \lambda)$-GD if and only if $\lambda v(v-1) \equiv 0 \pmod{18}$ and $v \ge 8$.*

Proof. By Lemmas 5.1, 5.2, 5.6, 5.7 and the result for $\lambda = 1$ in [19]. □

Lemma 5.9 *There exist $(w, C_8^{(2)}, 3)$-GD for w=12, 13, 15, 16.*

Proof. The blocks are listed in Appendix F(L5.9). □

Lemma 5.10 *There exist $(w, C_8^{(2)}, 9)$-GD for $w = 8$, 11, 14.*

Proof. For each order w, the corresponding base blocks under the automorphism group Z_m are listed in Appendix F(L5.10), where the vertex-set X is Z_m or $Z_m \cup \{\infty\}$. □

Theorem 5.11 *There exist $(v, C_8^{(2)}, \lambda)$-GD if and only if $\lambda v(v-1) \equiv 0 \pmod{18}$ and $v \geq 8$.*

Proof. By Lemmas 5.3, 5.9, 5.10 and the result for $\lambda = 1$ in [3]. □

Lemma 5.12 *There exist $(9, C_8^{(3)}, \lambda)$-GD for $\lambda \geq 2$.*

Proof. $(9, C_8^{(3)}, 2)$-GD: $X = Z_8 \cup \{\infty\}$, $(0, 1, \infty, 5, 2, 7, 6, 4) \mod 8$.
$(9, C_8^{(3)}, 3)$-GD: $X = Z_8 \cup \{\infty\}$, $(\infty, 3, 6, 1, 0, 4, 5, 7) \mod 8$;
$(0, 1, 3, 6, 4, 5, 7, 2)$, $(1, 2, 4, 7, 5, 6, 0, 3)$,
$(2, 3, 5, 0, 6, 7, 1, 4)$, $(3, 4, 6, 1, 7, 0, 2, 5)$.
Obviously, there are nonnegative integers m and n such that $\lambda = 2m + 3n$ for any $\lambda \geq 2$. Thus, we may assert that $(9, C_8^{(3)}, \lambda)$-GD exists for any $\lambda \geq 2$. □

Lemma 5.13 *There exist $(w, C_8^{(3)}, 3)$-GD for w=12, 13, 15, 16.*

Proof. The blocks are listed in Appendix F(L5.13). □

Lemma 5.14 *There exist $(w, C_8^{(3)}, 9)$-GD for $w = 8, 11, 14, 17, 26$.*

Proof. For each order w, the corresponding base blocks under the automorphism group Z_m are listed in Appendix F(L5.14), where the vertex-set X is Z_m or $Z_m \cup \{\infty\}$. □

Theorem 5.15 *There exist $(v, C_8^{(3)}, \lambda)$-GD if and only if $\lambda v(v-1) \equiv 0 \pmod{18}$, $v \geq 8$ and $(v, \lambda) \neq (9, 1)$.*

Proof. By Lemmas 5.4, 5.5, 5.12, 5.13, 5.14 and the result for $\lambda = 1$ in [3]. □

The electronic results in Appendices A, B, C, D, E, F are available on our website: `http://qdkang.hebtu.edu.cn` .

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