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# Characterizing factor critical graphs and an algorithm

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**Abstract**

In this paper, we show a necessary and sufficient condition which characterizes all factor critical graphs. Using this necessary and sufficient condition, we develop a linear time algorithm to determine whether a graph is factor critical if one of its maximum matchings is given.

## 1 Terminology and introduction

All graphs considered in this paper are undirected, finite and simple. In general, we follow the terminology of [1].

Let $G$ be a connected graph and $w$ be a vertex not in $V(G)$. Then $H = G + w$ denotes the graph with vertex set $V(H) = V(G) \cup \{w\}$ and edge set $E(H) = E(G) \cup \{vw \mid v \in V(G)\}$.

A graph is said to be factor critical if $G - u$ has a perfect matching for any vertex $u$ in $G$. A graph $G$ is said to be bicritical if $G - \{u, v\}$ has a perfect matching for any two vertices $u$ and $v$ in $G$. A matching $M$ is called near perfect matching in $G$ if all vertices but one are incident with edges in $M$ . The vertex $u$ not incident with any edge in $M$ is said to be $M$-unsaturated. The concept of factor critical graphs is also generalized to $n$-critical graphs. Let $G$ be a connected graph with $\nu$ vertices and $n$ be an integer such that $0 \le n \le \nu - 2$ and $n \equiv \nu \pmod 2$. Then $G$ is said to be $n$-critical if, for any subset $S \subseteq V(G)$ with $|S| = n$, $G - S$ has a perfect matching.

There has been some research on this topic (see [4–9]). In [12], Yu gives a Tutte style necessary and sufficient condition for $n$-critical graphs. However, it does not help to design an efficient algorithm to determine $n$-critical graphs. In [8], Lou and Zhong give a necessary and sufficient condition which characterizes all bicritical graphs and develop an algorithm to determine whether a graph is bicritical using the condition. In this paper, we give a necessary and sufficient condition to characterize all factor critical graphs. Using our necessary and sufficient condition, we can design a linear time algorithm to determine all factor critical graphs.

## 2 A necessary and sufficient condition

In this section, we give a necessary and sufficient condition for the factor critical graphs. It serves as a basis for the algorithm in Section 3. First, we give a lemma.

**Lemma 1:** (Lou and Zhong [8]) *Let $G$ be a graph with a perfect matching $M_0$. Then the following propositions are equivalent:*

*1. $G$ is bicritical;*

*2. For any perfect matching $M$ and any two different vertices $x$ and $y$ in $G$, there is an $M$-alternating path between $x$ and $y$, which starts and ends with edges in $E(G) \setminus M$;*

*3. For every pair of vertices $x$ and $y$ in $G$, there is an $M_0$-alternating path between $x$ and $y$, which starts and ends with edges in $E(G) \setminus M_0$.*

Now we give a theorem which shows the relation between $n$-critical graphs and $(n+1)$-critical graphs.

**Theorem 2:** *Let $G$ be a connected graph and $w$ be a vertex not in $V(G)$. Then $G$ is n-critical if and only if $G + w$ is $(n+1)$-critical.*

**Proof.** We prove necessity first. Suppose $G$ is $n$-critical. Then, for any subset $S \subseteq V(G)$ such that $|S| = n$, $G - S$ has a perfect matching. Let $G' = G + w$. Let $S' \subseteq V(G')$ such that $|S'| = n + 1$. If $w \in S'$, then $S_1 = S' \setminus \{w\} \subseteq V(G)$ such that $|S_1| = n$, so $G' - S' = G - S_1$ has a perfect matching. If $w \notin S'$ , let $x \in S'$ and let $S_2 = S' \setminus \{x\}$. Then $S_2 \subseteq V(G)$ and $|S_2| = n$, and $G - S_2$ has a perfect matching $M_1$. Assume $xy \in M_1$. Then $G - S' = (G - S_2) - \{x\}$ has a near perfect matching $M_2 = M_1 \setminus \{xy\}$. Since $w$ is adjacent to every vertex of $G$, $wy \in E(G')$. So $G' - S'$ has a perfect matching $M_2 \cup \{yw\}$. Hence $G'$ is $(n + 1)$-critical.

Now we prove sufficiency. Suppose $G' = G + w$ is $(n + 1)$-critical. Then, for any $S' \subseteq V(G')$ with $|S'| = n + 1$, $G' - S'$ has a perfect matching. Let $S$ be any subset of $V(G)$ such that $|S| = n$ and let $S' = S \cup \{w\}$. Then $S' \subseteq V(G')$ and $|S'| = n+1$. Since $G'$ is $(n + 1)$-critical, $G - S = G' - S'$ has a perfect matching. Hence $G$ is $n$-critical. □

**Theorem 3:** *Let $G$ be a graph with odd order, $M$ be a near perfect matching in $G$, and $u$ be the $M$-unsaturated vertex of $G$. Then $G$ is factor critical if and only if, for any vertex $v \neq u$ in $G$, there is a $(u, v)$ $M$-alternating path $Q$ such that $Q$ starts and ends with edges in $E(G) \setminus M$.*

**Proof.** First, we prove sufficiency. Suppose that, for any vertex $v \neq u$ in $G$, there is a $(u, v)$ $M$-alternating path $Q$ such that $Q$ starts and ends with edges in $E(G) \setminus M$. We are going to prove that $G$ is factor critical.

Let $w \in V(G)$ and $G' = G - w$. If $w = u$ then the original matching $M$ is a perfect matching of $G'$. Otherwise, if $w \neq u$, since $G$ has the near perfect matching

$M$ and $u$ is the only $M$-unsaturated vertex, then there must exists a vertex $v$ such that $wv \in M$. By the hypothesis of this theorem, there is a $(u, v)$ $M$-alternating path $Q$ such that $Q$ starts and ends with edges in $E(G) \setminus M$. Notice that $w$ does not belong to $Q$. Then $M' = M \triangle E(Q)$ is a perfect matching of $G'$, where $M \triangle E(Q)$ denotes the symmetric difference of $M$ and $E(Q)$.

Then we prove necessity. Let $G' = G + w$ such that $w \notin V(G)$. Obviously, $G'$ has a perfect matching $M_1 = M \cup \{wu\}$. Since $G$ is factor critical, by Theorem 2, $G'$ is bicritical.

By Lemma 1, we know that, in particular, there is a $(u, v)$ $M_1$-alternating path $Q$ for each $v \neq u \in V(G)$ such that $Q$ starts and ends with edges in $E(G') \setminus M_1$.

Since $uw \in M_1$ and $Q$ is an $M_1$-alternating path starting and ending with edges in $E(G') \setminus M_1$, $w \notin V(Q)$.

Hence $Q$ is a $(u, v)$ $M$-alternating path in $G$ such that $Q$ starts and ends with edges in $E(G) \setminus M$. The necessity is then proved. □

## 3 Description of the algorithm

In this section, we give a linear time algorithm to determine whether a connected graph $G$ is factor-critical if one of its maximum matching is given.

A near perfect matching $M$ is an input to this algorithm. Hence the existence of $M$ is tested prior to this algorithm. Moreover, $M$ should not necessarily be a near perfect matching. The algorithm can accept a maximum matching as an input, and test if it is a near perfect matching.

ALGORITHM:

1. Input a maximum matching $M$ of $G$; // $O(|E|)$

2. If $M$ is not a near perfect matching, then RETURN(false); ($G$ is not factor-critical) // $O(|V|)$

3. Else use Procedure 1 to construct an $M$-alternating tree from the $M$-unsaturated vertex $u$ to find an $M$-alternating path $P$ from $u$ to $v$ such that $P$ starts and ends with edges in $E(G) \setminus M$ for every $v \neq u$ in $V(G)$; // $O(|E|)$

4. If for some $v \neq u$ in $V(G)$, there is not such a path $P$ (Procedure 1 returns false), then RETURN(false); ($G$ is not factor-critical) // $O(1)$

5. Otherwise, RETURN(true); ($G$ is factor-critical). // $O(1)$

Procedure 1 uses the idea of [11] for finding an $M$-augmenting path to build an $M$-alternating tree starting from $u$. But it finds $M$-alternating paths from $u$ to every vertex $v \neq u$ such that the paths start and end with edges in $E(G) \setminus M$. We give the algorithm of Procedure 1 in the following.

Procedure 1:

1. Use BFS strategy to build an $M$-alternating tree $T$ rooted at $u$. First, $T := \emptyset$;

2. Put $u$ into the even vertex queue $Q$; Mark $u$ even;

3. Repeatedly take the first vertex $x$ from $Q$, do the following Steps 4−7 until $Q$ is empty;

4. For each edge $xy$ incident with $x$ do the following Steps 5−7;

5. Case 1: $y$ is not visited.

   Let $yy' \in M$;

   $T := T \cup \{xy, yy'\}$;

   Mark $y$ odd and $y'$ even;

   Put $y'$ into the queue $Q$;

   Set $Pre(y) := x$; $Pre(y') := y$;

End of Case 1;

6. Case 2: $y$ is marked odd.

   We do nothing in this case;

7. Case 3: $y$ is marked even.

   Track along the $Pre$ chains from $x$ and from $y$ respectively until we find the first common ancestor $t$ of $x$ and $y$. That is, we find vertex sequences $P = (x =) a_1, a_2, \ldots, a_m (= t)$ and $R = (y =) b_1, b_2, \ldots, b_n (= t)$ such that $Pre(a_i) = a_{i+1}$, $i = 1, 2, \ldots, m-1$, $Pre(b_j) = b_{j+1}$, $j = 1, 2, \ldots, n-1$, and $a_i \neq b_j$ unless $i = m$ and $j = n$;

   For each $a_i$ $(i = 1, 2, \ldots, m-1)$, set $Pre(a_i) := t$; if $a_i$ is marked odd, then mark $a_i$ even and put $a_i$ into the queue $Q$;

   For each $b_j$ $(j = 1, 2, \ldots, n-1)$, set $Pre(b_j) := t$; if $b_j$ is marked odd, then mark $b_j$ even and put $b_j$ into the queue $Q$;

End of Case 3;

8. If all vertices of $G$ are marked even, then RETURN(true);

   otherwise RETURN(false);

Notice that, for any even vertex $x$ in $T$, if $xy \in M$, then there is an $M$-alternating path from $u$ to $y$ in $T$ such that $P$ starts and ends with edges in $E(G) \setminus M$. So it suffices to check that every vertex (except $u$) is marked even after the execution of Steps 3−7 to determine that $G$ is factor critical. It is equivalent that every matching edge $xy$ in $M$ lies in a blossom of $T$.

In Procedure 1, $Pre$ is an array. For each vertex $v$ in $G$, $Pre$ has an element $Pre(v)$. If $v$ is not in any blossom of $T$, then $Pre(v)$ is the father of $v$ in $T$. If $v$ lies in a blossom of $T$, then $Pre(v)$ is the root of a blossom containing $v$ which has been processed. Here the terminology blossom and root of blossom comes from the classical paper of Edmonds [2].

Procedure 1 only takes $O(|E|)$ time since it uses BFS strategy to build the $M$-alternating tree $T$. Notice that, in Step 7, if the algorithm tracked the vertex sequences $P = (x =)\, a_1, a_2, \ldots, a_m (= t)$ and $R = (y =)\, b_1, b_2, \ldots, b_n (= t)$ once, then the algorithm will not track any subsequence of $P$ and $R$ of length at least 3 one more time because the algorithm has set $Pre(a_i) := t$ and $Pre(b_j) := t$, $i = 1, 2, \ldots, m-1$, $j = 1, 2, \ldots, n-1$. So it takes at most $O(|E| + |V|)$ time to process all blossoms. But we have assumed that $G$ is a connected graph. So $O(|E|) + O(|V|) = O(|E|)$

It is easy to see that the whole algorithm only takes $O(|E|)$ time. It is a linear time algorithm and hence is optimal. However, by [11], it takes $O(|V|^{1/2}|E|)$ time to find the near perfect matching $M$ in $G$. To determine whether a graph $G$ is factor critical spends the main time in finding a near perfect matching in $G$.

If we use the definition of factor critical graphs to design an algorithm, then we delete every vertex $v$ and try to find a perfect matching in $G - v$. In this case, the algorithm needs $O(|V|^{3/2}|E|)$ time. So our algorithm has higher efficiency.

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