AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **28** (2003), Pages 305–315

Distance magic labelings of graphs

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Abstract

As a natural extension of previously defined graph labelings, we introduce in this paper a new magic labeling whose evaluation is based on the neighbourhood of a vertex.

We define a 1-vertex-magic vertex labeling of a graph with v vertices as a bijection f taking the vertices to the integers $1, 2, \ldots, v$ with the property that there is a constant k such that at any vertex x, $\sum_{y \in N(x)} f(y) = \mathsf{k}$, where $N(x)$ is the set of vertices adjacent to x.

We completely solve the existence problem of 1-vertex-magic vertex labelings for all complete bipartite, tripartite and regular multipartite graphs, and obtain some non-existence results for other natural families of graphs.

1 Introduction

We assume that G=$G(V, E)$ is a finite, simple, and undirected graph with v vertices and e edges. We use $\overline{G}$ to denote *the complement* of a graph G (for other graph-theoretic notions see [6]). By a *labeling* we mean a one-to-one mapping that carries a set of graph elements onto a set of numbers, called *labels*.

Historically, the notion of a magic labeling was first introduced in 1963 by Sedláček [4]. He defined a *magic labeling* of a graph $G = G(V, E)$ as a bijection f from E to a set of positive integers such that

(i) $f(e_i) \neq f(e_j)$ for all distinct $e_i, e_j \in E$, and

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(ii) $\sum_{e \in N_E(x)} f(e)$ is the same for every $x \in V$, where $N_E(x)$ is the set of edges incident to x.

Using similar magic conditions, Kotzig and Rosa [2] and MacDougall *et al.* [3] defined two other magic labelings, namely, the *edge-magic total labeling* and the *vertex-magic total labeling*, respectively. The *edge-magic edge labeling* and the *vertex-magic vertex labeling* could be defined analogously. However, these turn out to be trivial. Note that all the above labelings are defined by considering the incidence of a particular vertex or edge.

More generally, graph labelings can be defined based on distances between vertices. For example, motivated by applications in radio transmissions, Griggs and Yeh [1] defined $L_d(2,1)$*-labeling* of G where the labels of vertices distance 1 and 2 have to differ by $2d$ and d, respectively.

In this paper we consider a graph labeling that is both magic and distance based. The domain of the labeling will be the set of all vertices and the codomain will be $\{1, 2, \ldots, v\}$. We call this labeling a *vertex labeling.* We define the *1-vertex-weight* of each vertex x in G under a vertex labeling to be the sum of vertex labels of the vertices adjacent to x (that is, distance 1 from x). If all vertices in G have the same weight k, we call the labeling a 1*-vertex-magic vertex labeling* (for some examples, see Figures 1, 2, 3 and 4). More formally, we have the following definition.

Definition 1 *A* 1*-vertex-magic vertex labeling is a bijection* $f : V \rightarrow \{1, 2, \ldots, v\}$ *with the property that there is a constant* k *such that at any vertex* x

$$\sum_{y \in N(x)} f(y) = \mathsf{k}$$

where $N(x)$ *is the set of vertices adjacent to* x*.*

Thus the following holds.

Lemma 1 *A necessary condition for the existence of a* 1*-vertex-magic vertex labeling* f *of a graph* G *is*

$$\mathsf{k}v = \sum_{x \in V} d(x) f(x)$$

where $d(x)$ *is the degree of vertex* x*.*

Proof. Clearly the sum of all 1-vertex-weights of all vertices in G is $\mathsf{k}v$. On the other hand, for each vertex $x \in V$, this sum counts the label of x exactly $d(x)$ times. Thus, the equation holds. $\square$

Unlike the vertex-magic vertex labeling, the 1-vertex-magic vertex labeling is not trivial. In this paper we provide 1-vertex-magic vertex labelings for various complete multipartite graphs (see Section 2), and prove that several families of graphs do not have 1-vertex-magic vertex labelings (see Section 3). It will cause no confusion if from now on we refer to the 1-vertex-weight of a vertex x simply as the *weight* $w(x)$ and if we refer to the 1-vertex-magic vertex labeling as the *labeling.*

2 Complete multipartite graphs

We begin with a useful result that can be repeatedly used when dealing with complete multipartite graphs.

Lemma 2 *If $G = H \times \overline{K_{2k}}$, where H is an r-regular graph, then G has a labeling.*

Proof. Consider an r-regular graph H on n vertices $\{x_1, \ldots, x_n\}$. By replacing every vertex in H with $2k$ pairs of vertices, each joined to all vertices corresponding to the neighbours of the original vertex of H, we obtain the $2kr$-regular graph $G = H \times \overline{K_{2k}}$. For $1 \leq i \leq 2k$ and $1 \leq j \leq n$, let x_{ij} be the vertices of G that replace $x_j, 1 \leq j \leq n$ in H. Label the vertices in the following way

$$f(x_{ij}) = \begin{cases} j + (i-1)n, & 1 \leq j \leq n \text{ and } i \text{ odd}, \\ n - j + 1 + (i-1)n, & 1 \leq j \leq n \text{ and } i \text{ even}. \end{cases}$$

Notice that for every i and j,

$$f(x_{2i-1,j}) + f(x_{2i,j}) = n + 1 + (4i-3)n.$$

So the sum of the labels in the jth part is

$$\begin{aligned} &= k(n+1) + (1 + 5 + \ldots + 4k - 3)n \\ &= k(n+1) + n\frac{k}{2}(4k-2) \\ &= k(2nk+1), \end{aligned}$$

which is independent of j. Therefore, for every $x \in G$,

$$w(x) = rk(2nk+1).$$

□

Let $H_{n,p}, n \geq 1$ and $p \geq 1$, denote the complete symmetric multipartite graph with p parts, each of which contains n vertices. Obviously, $H_{n,1}$ has a labeling. On the other hand, it is easy to see that $H_{1,p}$ does not have a labeling (see Theorem 3(c)). Next we will settle the labeling problem for all other graphs in this family.

Theorem 1 *Let $n > 1$ and $p > 1$. $H_{n,p}$ has a labeling if and only if either n is even or both n and p are odd.*

Proof. Suppose $H_{n,p}$ has a labeling. By Lemma 1,

$$\begin{aligned} \mathsf{k}(np) &= n(p-1)(1 + \ldots + np) \\ \mathsf{k}p &= n(p-1)\frac{(np+1)(np-1)}{2}. \end{aligned}$$

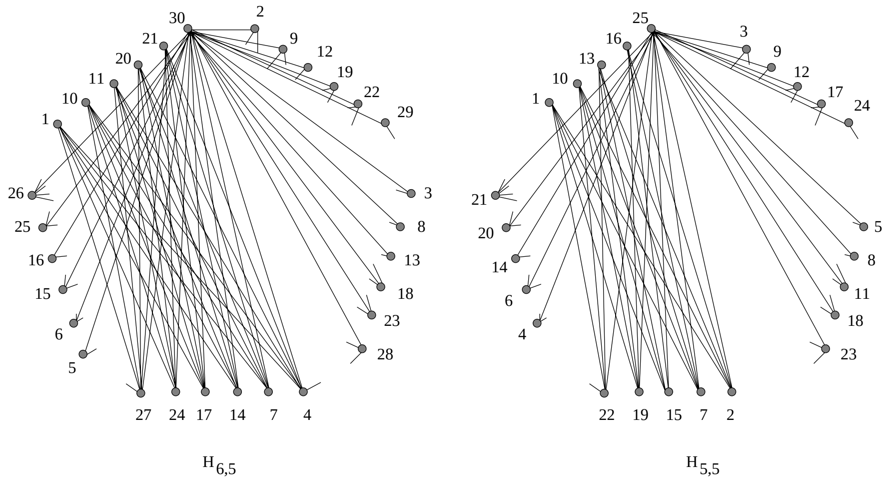

Figure 1: Labelings for multipartite graphs

To guarantee that both sides are integers, either n has to be even or both n and p must be odd.

To prove the sufficiency, if n is even then the theorem follows from Lemma 2 since $H_{2k,p} = K_p \times \overline{K_{2k}}$.

If n is odd, consider the graph $H_{2k+1,2m+1}$ and the labeling f defined bellow.

$$f(x_{ij}) = \begin{cases} 2j-1, & 1 \le j \le m+1 \text{ and } i=1, \\ 2(j-m-1), & m+2 \le j \le 2m+1 \text{ and } i=1, \\ 4m+3-j, & 1 \le j \le 2m+1 \text{ and } i=2, \\ 5m+4-j & 1 \le j \le m+1 \text{ and } i=3, \\ 7m+5-j & m+2 \le j \le 2m+1 \text{ and } i=3, \\ j+(i-1)(2m+1), & 1 \le j \le 2m+1 \text{ and } i>3,\ i \text{ even}, \\ 2m+2-j+(i-1)(2m+1), & 1 \le j \le 2m+1 \text{ and } i>3,\ i \text{ odd}. \end{cases}$$

Clearly, for every j

$$f(x_{1,j}) + f(x_{2,j}) + f(x_{3,j}) = 9m+6$$

and for $i > 1$,

$$f(x_{2i,j}) + f(x_{2i+1,j}) = 2m+2+(4i-1)(2m+1).$$

Thus, the sum of the labels in the jth part is

$$\begin{aligned} &= f(x_{1,j}) + f(x_{2,j}) + f(x_{3,j}) + \sum_{i=2}^{k} 2m+2+(4i-1)(2m+1) \\ &= (9m+6) + (k-1)(2m+2) + 2(k-1)(k+2)(2m+1) - (k-1)(2m+1) \\ &= (9m+6) + (k-1)(2m+2) + (k-1)(2k+3)(2m+1). \end{aligned}$$

Therefore, under the labeling f, every vertex x obtains the same weight,

$$w(x) = 2m\{(9m+6) + (k-1)(2m+2) + (k-1)(2k+3)(2m+1)\}.$$

□

Of course, there are many multipartite graphs in which the number of vertices is not the same for all parts. The following result completely settles this labeling problem for complete bipartite and tripartite graphs. (It is likely that the proof could be extended to more parts, so notation is used in the proof with this in mind. However, our efforts to make use of such an extension became quite technical and messy.)

Theorem 2 *Let $1 \le a_1 \le \ldots \le a_p$ where $2 \le p \le 3$. Let $s_i = \sum_{j=1}^{i} a_j$. There exists a labeling of the complete multipartite graph $H_{a_1,\ldots,a_p}$ if and only if the following conditions hold.*

(a) $a_2 \ge 2$,

(b) $n(n+1) \equiv 0 \bmod 2p$, *where* $n = s_p = |V(H_{a_1,\ldots,a_p})|$, *and*

(c) $\sum_{j=1}^{s_i}(n+1-j) \ge \frac{in(n+1)}{2p}$ *for* $1 \le i \le p$.

Proof. First suppose that such a labeling exists. The necessity of (a) follows directly from Lemma 3. To prove (b), let $A_1, \ldots, A_p$ be the partite sets of $H_{a_1,\ldots,a_p}$. Then by the magic property, if $x \in A_i$ and $y \in A_j$,

$$\sum_{v \in V \setminus A_i} w(v) = w(x) = w(y) = \sum_{u \in V \setminus A_j} w(u),$$

so $\sum_{v \in A_i} w(v) = \sum_{u \in A_j} w(u)$. Therefore, since the vertices are labeled with $1, \ldots, n$, the magic constant of the labeling is equal to

$$\sum_{v \in A_i} w(v) = \frac{n(n+1)}{2p} \text{ for } 1 \le i \le p,$$

so condition (b) holds. Since for $1 \le i \le p$ the largest that the sum of the labels in $\cup_{j=1}^{i} A_j$ could be is $\sum_{j=1}^{s_i}(n+1-j)$, and since (as we just observed) the sum of the labels of the vertices in A_j is $\frac{n(n+1)}{2p}$, the necessity of (c) follows.

So now suppose the conditions (a) to (c) hold. We define the labels on the vertices in $A_1, \ldots, A_p$ in turn. Informally, at each step, the labels on the vertices in A_i will be made up from the α_i smallest labels available, the β_i largest available, and at most i other labels, where the precise values of α_i and β_i depend upon $a_1, \ldots, a_p$. Let $t(i) = \sum_{j=1}^{i} j$. Note that for $1 \le i \le p$, $s_i \le \frac{in}{p}$ because $a_1 \le \ldots \le a_p$, so

$$t(s_i) = \frac{s_i(s_i+1)}{2} \le \frac{in(s_i+1)}{2p} \le \frac{in(n+1)}{2p}. \quad (*)$$

We shall consider two cases:

Case 1 Suppose that either $p = 2$ or $\sum_{j=a_1+1}^{a_1+a_2}(n+1-j) \geq \frac{n(n+1)}{2p}$. We begin by showing that for each t satisfying $t(a_1) \leq t \leq t(n) - t(n-a_1)$, there exists a subset $L_1(t)$ of $\mathcal{L}_1 = \{1, 2, \ldots, n\}$ of size a_1 such that $\sum_{l \in L_1(t)} l = t$. As the following argument shows, this can be achieved so that

$$L_1(t) = \{1, \ldots, \alpha_1(t), \gamma_1(t), n - \beta_1(t) + 1, \ldots, n\} \backslash O_1(t)$$

for some $O_1(t) \subseteq \{\gamma_1(t)\}$ and some integers $\alpha_1(t), \beta_1(t)$ and possibly $\gamma_1(t)$, with $\alpha_1(t) + 1 < \gamma_1(t) < n - \beta_1(t)$ (we allow $\alpha_1(t) = 0$ and $\beta_1(t) = 0$ to describe empty ranges). Clearly $L_1(t(n) - t(n-a_1)) = \{n - a_1 + 1, \ldots, n\}$; so $\alpha_1(t(n) - t(n-a_1)) = 0, \beta_1(t(n) - t(n-a_1)) = a_1$ and $O_1(t(n) - t(n-a_1)) = \{\gamma_1(t(n) - t(n-a_1))\}$ (in such a case we may also say that $\gamma_1(t(n) - t(n-a_1))$ is undefined). If $t < t(t(n) - t(n-a_1))$ then $L_1(t)$ can be defined inductively as follows: if $\gamma_1(t+1)$ is defined in $L_1(t+1)$ then let

$$L_1(t) = (L_1(t+1) \cup \{\gamma_1(t+1) - 1\}) \backslash \{\gamma_1(t+1)\};$$

otherwise let

$$L_1(t) = (L_1(t+1) \cup \{n - \beta_1(t+1)\}) \backslash \{n - \beta_1(t+1) + 1\}.$$

Recall that by the inequality $(*)$ we know that $t(a_1) = t(s_1) \leq \frac{n(n+1)}{2p}$. Also, applying condition (c), with $i = 1$, we see that $\frac{n(n+1)}{2p} \leq t(n) - t(n-a_1)$. Thus there exists a label set for A_1, namely $L_1 = L_1(\frac{n(n+1)}{2p})$ with $\alpha_1 = \alpha_1(\frac{n(n+1)}{2p}), \beta_1 = \beta_1(\frac{n(n+1)}{2p})$ and if defined $\gamma_1 = \gamma_1(\frac{n(n+1)}{2p})$. If $p = 2$, this concludes the proof, since $L_2 = \{1, 2, \ldots, n\} \backslash L_1$ must be a label set for A_2.

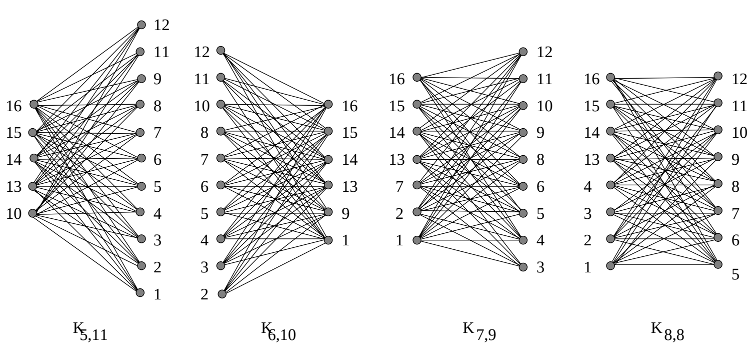

Figure 2: Labelings for complete bipartite graphs on 16 vertices

If $p = 3$ then we proceed to define L_2 in exactly the same way as L_1, except that we need some care to avoid placing γ_1, if defined, in L_2. Let $\mathcal{L}_2 = \mathcal{L}_1 \backslash L_1 = \{\alpha_1 + 1, \ldots, n - \beta_1\} \backslash \{\gamma_1\}$ be the set of labels still available for vertices in the remaining parts. Let $S(\mathcal{L}_2)$ be the set of a_2 smallest numbers in $\mathcal{L}_2$; let $s(\mathcal{L}_2) = \sum_{a \in S(\mathcal{L}_2)} a$; let

$L(\mathcal{L}_2)$ be the set of a_2 largest numbers in $\mathcal{L}_2$, and let $l(\mathcal{L}_2) = \sum_{a\in L(\mathcal{L}_2)} a$. For every $t, s(\mathcal{L}_2) \le t \le l(\mathcal{L}_2)$, we want to show that there exists a subset $L_2(t)$ of $\mathcal{L}_2$ of size a_2 with $\sum_{l\in L_2(t)} l = t$. Similar to the definition of $L_1(t)$, $L_2(t)$ can be defined to be a subset of $\mathcal{L}_2$ of size a_2 so that

$$L_2(t) = \{\alpha_1+1,\dots,\alpha_2(t),\gamma_2(t),\delta_2(t),n-\beta_1-\beta_2(t)+1,\dots,n-\beta_1\}\backslash O_2(t)$$

for some $O_2(t) \subseteq \{\gamma_1,\gamma_2(t),\delta_2(t)\}$ and for some integers $\alpha_2(t),\beta_2(t)$ and possibly $\gamma_2(t),\delta_2(t)$, with $\alpha_2(t)+1 < \gamma_2(t) < \delta_2(t) < n-\beta_1-\beta_2(t)$ (we allow $\alpha_2(t) = 0$ and $\beta_2(t) = 0$ to describe empty ranges). For each t, we assume that if $O_2(t)$ has one element then it is $\gamma_2(t)$. Clearly, $L_2(l(\mathcal{L}_2)) = L(\mathcal{L}_2)$ (where $\alpha_2(l(\mathcal{L}_2)) = 0$). If $t < l(\mathcal{L}_2)$ then $L_2(t)$ can be defined inductively as follows. If neither of $\gamma_2(t+1)$ nor $\delta_2(t+1)$ exists and $n-\beta_1-\beta_2(t+1) \ne \gamma_1$ then

$$L_2(t) = (L_2(t+1)\cup\{n-\beta_1-\beta_2(t+1)\})\backslash\{n-\beta_1-\beta_2(t+1)+1\}.$$

If neither of $\gamma_2(t+1)$ nor $\delta_2(t+1)$ exists and $n-\beta_1-\beta_2(t+1) = \gamma_1$ then

$$L_2(t) = (L_2(t+1)\cup\{\alpha_2(t+1)+1, n-\beta_1-\beta_2(t+1)-1\})\backslash\{\alpha_2(t+1), n-\beta_1-\beta_2(t+1)+1\}.$$

If $\gamma_2(t+1)$ exists, $\delta_2(t+1)$ does not exist, and $\gamma_2(t+1) \ne \gamma_1+1$ then

$$L_2(t) = (L_2(t+1)\cup\{\gamma_2(t+1)-1\})\backslash\{\gamma_2(t+1)\}.$$

If $\gamma_2(t+1)$ exists, $\delta_2(t+1)$ does not exist, and $\gamma_2(t+1) = \gamma_1+1$ then

$$L_2(t) = (L_2(t+1)\cup\{n-\beta_1-\beta_2(t+1)\})\backslash\{n-\beta_1-\beta_2(t+1)+1\}.$$

If both of $\gamma_2(t+1)$ and $\delta_2(t+1)$ exist then: if $\delta_2(t+1) = \gamma_2(t+1)+1 = \gamma_1+2$ then

$$L_2(t) = (L_2(t+1)\cup\{\gamma_2(t+1)-2, \delta_2(t+1)+1\})\backslash\{\gamma_2(t+1),\delta_2(t+1)\}$$

(notice that in this case $\gamma_1 - 1 \ne \alpha_2(t+1)$, for if it was an equality instead then $L_2(t+1)$ would contain $\{\alpha_1+1,\dots,\gamma_1-1,\gamma_1+1,\gamma_1+2\}$, and so $\alpha_2(t+1)$ would in fact be γ_1+2); and otherwise

$$L_2(t) = (L_2(t+1)\cup\{X-1\})\backslash\{X\},\ \text{where } X\in\{\gamma_2(t+1),\delta_2(t+1) | X \ne \gamma_1+1\}.$$

Next we have to show that $s(\mathcal{L}_2) \le \frac{n(n+1)}{2p} \le l(\mathcal{L}_2)$. Since $\sum_{l\in L_1} l = \frac{n(n+1)}{2p}$, and since $\sum_{l\in\mathcal{L}_1} l = \frac{n(n+1)}{2}$, we have $\sum_{l\in\mathcal{L}_2} l = \frac{n(n+1)}{2} - \frac{n(n+1)}{2p} = \frac{p-1}{p}\frac{n(n+1)}{2}$. But $a_2 \le a_3 \le \ldots \le a_p$, so $\frac{a_2}{\sum_{i=2}^{p} a_i} \le \frac{1}{p-1}$. Thus the sum of the a_2 smallest elements of $\mathcal{L}_2$ is at most $\frac{a_2}{\sum_{i=2}^{p} a_i}\sum_{l\in\mathcal{L}_2} l \le \frac{1}{p-1}\frac{p-1}{p}\frac{n(n+1)}{2} = \frac{n(n+1)}{2p}$. By the assumption of Case 1, $\sum_{j=a_1+1}^{a_1+a_2}(n+1-j) \ge \frac{n(n+1)}{2p}$, and so $s(\mathcal{L}_2) \le \frac{n(n+1)}{2p} \le l(\mathcal{L}_2)$.

Thus, there exists a subset $L_2 = L_2(\frac{n(n+1)}{2p})$ of $\mathcal{L}_2$ of size a_2 such that $\sum_{l\in L_2} l = \frac{n(n+1)}{2p}$. We conclude the proof for Case 1 by labeling the vertices in A_2 with elements of L_2, and the vertices in A_3 with the elements of $L_3 = \mathcal{L}_2\backslash L_2$.

Case 2 Suppose that $p = 3$ and $\sum_{j=a_1+1}^{a_1+a_2}(n+1-j) < \frac{n(n+1)}{2p}$. In this case, begin with $L_1 = \{n-a_1+1, \ldots, n\}$ and $L_2 = \{n-a_1-a_2+1, \ldots, n-a_1\}$. By condition **(c)** and the assumption of Case 2, $\sum_{l\in L_1} l > \frac{n(n+1)}{2p}$. Recursively swap the smallest element ξ in L_1, for which $\xi - 1 \in L_2$, with $\xi - 1$ (so the sum of the elements in L_1 decreases by 1 and the sum of the elements in L_2 increases by 1) until $\sum_{l\in L_2} l = \frac{n(n+1)}{2p}$. Then, since the process does not alter $\sum_{l\in L_1\cup L_2} l$, and since by **(c)** we have the sum is at least $\frac{2n(n+1)}{2p}$, it follows that in these modified sets we now have $\sum_{l\in L_1} l \geq \frac{n(n+1)}{2p}$ and that $\sum_{l\in L_2} l = \frac{n(n+1)}{2p}$. Furthermore, this process ensures that L_1 is of the form

$$\{n - a_1 - a_2 + 1, \ldots, n - a_1 - a_2 + \alpha_1\} \cup O_1 \cup \{n - \beta_1 + 1, \ldots, n\}$$

for some $\alpha_1, \beta_1 \geq 0$ and $O_1 \subseteq \{\gamma_1\}$ for some γ_1 satisfying $n - a_1 - a_2 + \alpha_1 + 1 < \gamma_1 < n - \beta_1$.

If

$$\sum_{l\in L_1} l - \frac{n(n+1)}{2p} \leq \alpha_1(n - a_1 - a_2) \qquad (**)$$

then it is clearly possible to replace the numbers in $\{n-a_1-a_2+1, \ldots, n-a_1-a_2+\alpha_1\}$ in L_1 with smaller numbers so that $\sum_{l\in L_1} l = \frac{n(n+1)}{2p}$. For example, if equality holds in $(**)$ then these numbers are replaced with $\{1, 2, \ldots, \alpha_1\}$.

If $\sum_{l\in L_1} l - \frac{n(n+1)}{2p} > \alpha_1(n-a_1-a_2)$ then replace $\{n-a_1-a_2+1, \ldots, n-a_1-a_2+\alpha_1\}$ in L_1 with $\{1, 2, \ldots, \alpha_1\}$. Then we still have the resulting set L_1 with $\sum_{l\in L_1} l > \frac{n(n+1)}{2p}$. Now L_1 is of the form used in Case 1, and most importantly, it is now the case that $\sum_{l\in L_2} l = \frac{n(n+1)}{2p}$. Therefore, we can apply the method used in Case 1. □

We believe that a similar method can be used to label complete multipartite graphs with more than three parts but the current proof technique becomes technically too messy. So we hesitantly propose the following conjecture, looking for a neater, more manageable, proof. We suspect that condition **(c)** may need strengthening when considering $p \geq 4$.

Conjecture 1 *Let* $1 \leq a_1 \leq \ldots \leq a_p, p > 1$. *Let* $s_i = \sum_{j=1}^{i} a_j$ *and* $n = s_p$. *There exists a labeling of the complete multipartite graph* $H_{a_1,\ldots,a_p}$ *if and only if the following conditions hold.*

(a) $a_2 \geq 2$,

(b) $n(n + 1) \equiv 0 \bmod 2p$, *and*

(c) $\sum_{j=1}^{s_i}(n + 1 - j) \geq \frac{in(n+1)}{2p}$ *for* $1 \leq i \leq p$.

3 Graphs without labelings

We conclude by listing various families of graphs, all of which have no labelings. The results in this section contrast our 1-vertex magic vertex labelings with other types of

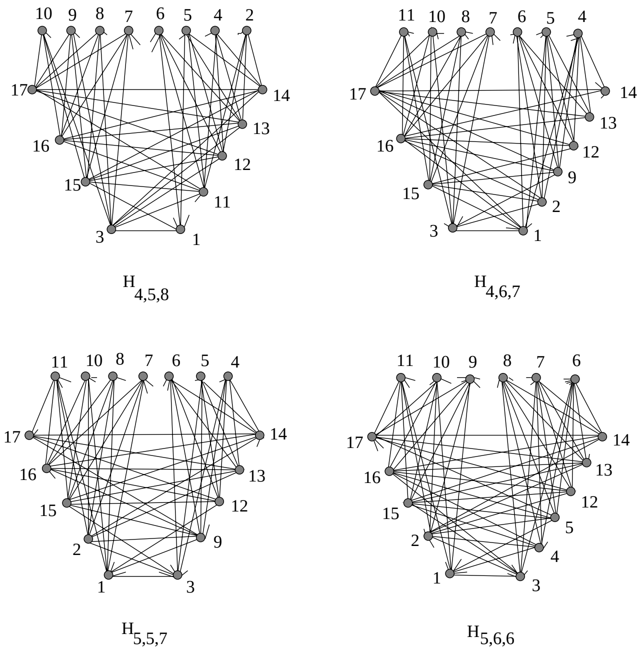

Figure 3: Labelings for complete tripartite graphs on 17 vertices

magic labelings where often the graphs under consideration do have magic labelings.

Lemma 3 *If G contains two vertices u and v such that $|N(u) \cap N(v)| = d(v) - 1 = d(u) - 1$ then G has no labeling.*

Proof. Suppose G has a labeling and let u' (v', respectively) be the one neighbour u and v, respectively that is not adjacent to v (u, respectively). Then $\sum_{x \in N(u)} f(x) = w(u) = w(v) = \sum_{x \in N(v)} f(x)$, so $f(u') = f(v')$, a contradiction. $\square$

Let P_n denote the *path* on n vertices, C_n the *cycle* of length n and let the *wheel* W_n ($n \geq 3$) denote the graph obtained by joining all vertices of cycle C_n to a further vertex called the *center*.

Theorem 3 **(a)** *There exists a labeling of P_n if and only if $n \in \{1, 3\}$.*

(b) *There exists a labeling of C_n if and only if $n = 4$.*

(c) *There exists a labeling of K_n if and only if $n = 1$.*

(d) *There exists a labeling of W_n if and only if $n = 4$.*

Proof. In each case, the non-existence of a labeling is a direct consequence of Lemma 3. A labeling for $P_1 = K_1$ is trivial, and labelings in each of the other cases are given in Figure 4. □

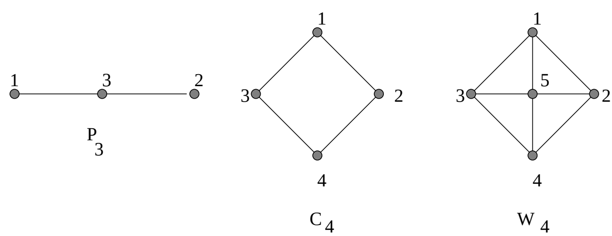

Figure 4: Labelings for small graphs

Lemma 4 *Let G be a graph on v vertices, with maximum degree Δ and minimum degree δ. If $\Delta(\Delta+1) > \delta(2v-\delta+1)$ then G does not have a labeling.*

Proof. Suppose G has a labeling. Let x_Δ be a vertex of degree Δ and x_δ a vertex of degree δ. Clearly,

$$1 + \ldots + \Delta \le w(x_\Delta) \le v + (v-1) + \ldots + (v - \Delta + 1)$$

and

$$1 + \ldots + \delta \le w(x_\delta) \le v + (v-1) + \ldots + (v - \delta + 1).$$

Since every weight has to be equal then $\frac{\Delta(\Delta+1)}{2} \le w(x_\Delta) = w(x_\delta) \le \frac{\delta(2v-\delta+1)}{2}$. This concludes the proof. □

Theorem 4 *There exists a labeling of a tree if and only if the tree is either P_1 or P_3.*

Proof. First we consider the star on $n+1$ vertices. Since $v = \Delta = n+1$ and $\delta = 1$, by Lemma 4, the labeling does not exist for $n > 2$. If the tree is not a star then it contains two vertices of degree one with no common neighbour. Thus, by Lemma 3, it has no labeling either. □

Theorem 5 *Every r-regular graph with odd r does not have a labeling.*

Proof. Let G be an $(2k+1)$-regular graph on v vertices. Thus, v has to be even. If the labeling exists then, by Lemma 1, $v\mathsf{k} = (2k+1)\sum_{i=1}^{v} i = \frac{(2k+1)(v+1)v}{2}$, which implies that the constant k is not an integer. □

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(Received 10 Aug 2002)