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Note on summation formulas derived from an identity of F. H. Jackson

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Abstract

An identity of F. H. Jackson is used to derive general bilateral summation formulae for q-series which unifies several generalizations of Gasper's bibasic summation formula with different independent bases, and we also obtain the dual formula for the different generalization related to Gasper's formula.

1 Introduction

Gasper [4] showed that some bibasic summation formulae derived by Carlitz [2], Al-Salam and Verma [1], and Wm. Gosper could be extended to the following indefinite bibasic summation formula:

Theorem 1. *If n is a non-negative integer, then*

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^k/q^k)}{(1-a)(1-b)} \frac{(a,b;p)_k(c,a/bc;q)_k}{(q,aq/b;q)_k(ap/c,bcp;p)_k} q^k = \frac{(ap,bp;p)_n(cq,aq/bc;q)_n}{(q,aq/b;q)_n(ap/c,bcp;p)_n}, \tag{1}$$

where p and q are independent bases and a, b, c are arbitrary parameters.

He used it for obtaining quadratic and cubic summation and transformations formulae for q-hypergeometric series. A little later Gasper and Rahman [5] obtained the following bilateral extension of Gasper's bibasic summation formula (1) by using a difference operator.

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Theorem 2. *If m, n are non-negative integers, then*

$$(2)\qquad \sum_{k=-m}^{n} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(c,ad^2/bc;q)_k}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k} q^k$$
$$= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)}$$
$$\cdot\{\frac{(ap,bp;p)_n(cq,ad^2q/bc;q)_n}{(dq,adq/b;q)_n(adp/c,bcp/d;p)_n} q^n$$
$$-\frac{(c/ad,d/bc;p)_{m+1}(1/d,b/ad;q)_{m+1}}{(1/c,bc/ad^2;q)_{m+1}(1/a,1/b;p)_{m+1}}\}$$

where $n, m = 0, \pm 1, \pm 2, \ldots$.

They use this formula to derive some rather general summation and transformation formulae. It should be noted that in (2) and elsewhere we employ the standard convention of defining

$$(3)\qquad \sum_{k=m}^{n} a_k = \begin{cases} a_m + a_{m+1} + \cdots + a_n, & m \leq n, \\ 0, & m = n+1, \\ -(a_{n+1} + a_{n+2} + \cdots + a_{m-1}), & m \geq n+2. \end{cases}$$

Jain and Verma [8] also used the difference operator to obtain a summation formula involving three independent bases:

Theorem 3. *If m, n are non-negative integers, then*

$$(4)\qquad \sum_{k=-m}^{n} \frac{(c-d)(1-cyp^kP^k\beta/d)(1-yP^kq^{-k}/d)(1-p^kq^{-k}\beta/d)}{(1-\beta)(1-c)(1-y)(1-cy\beta/d^2)}$$
$$\frac{(\beta;p)_k(c;q)_k(y;P)_k(cy\beta/d^2;pP/q)_kq^k}{(dq;q)_k(cp\beta/d;p)_k(y\beta/d\cdot pP/q;pP/q)_k(cyP/d;P)_k}$$
$$= \left\{\frac{(1/d;q)_{m+1}(d/c\beta;p)_{m+1}(d/y\beta;pP/q)_{m+1}(d/cy;P)_{m+1}}{(1/\beta;p)_{m+1}(1/c;q)_{m+1}(1/y;P)_{m+1}(d^2/cy\beta;pP/q)_{m+1}}\right.$$
$$\left.-\frac{(p\beta;p)_n(cq;q)_n(yP;P)_n(cy\beta/d^2\cdot pP/q;pP/q)_n}{(dq;q)_n(cp\beta/d;p)_n(y\beta/d\cdot pP/q)_n(cyP/d;P)_n}\right\}.$$

When $P = q$, (4) reduces to the summation formula (2).

Chu [3] obtained a generalization of Gasper-Rahman's formula (2) (after suitably renaming the sequences so as to remove redundant sequences).

Theorem 4. *If m, n are non-negative integers, then*

$$(5)\qquad \sum_{k=-m}^{n} \frac{(1-\alpha a_kb_k)(b_k-\frac{a_k}{\alpha d})}{(1-\alpha a_0b_0)(b_0-\frac{a_0}{\alpha d})} \frac{\prod_{j=0}^{k-1}[(1-a_j)(1-\frac{a_j}{d})(1-cb_j)(1-\frac{\alpha^2 d}{c}b_j)]}{\prod_{j=1}^{k}[(1-\alpha b_j)(1-\frac{\alpha a_j}{c})(1-\alpha db_j)(1-\frac{c}{d\alpha}a_j)]}$$

$$= \frac{(1-a_0)(1-\frac{a_0}{d})(1-b_0c)(1-\frac{\alpha^2 d}{c}b_0)}{\alpha(1-\alpha a_0 b_0)(1-\frac{c}{\alpha})(b_0-\frac{a_0}{d\alpha})(1-\frac{\alpha d}{c})}$$
$$\cdot\left\{\prod_{j=1}^{n}\left[\frac{(1-a_j)(1-\frac{a_j}{d})(1-cb_j)(1-\frac{\alpha^2 d}{c}b_j)}{(1-\alpha b_j)(1-\frac{\alpha a_j}{c})(1-\alpha d b_j)(1-\frac{c}{d\alpha}a_j)}\right]\right.$$
$$\left.-\prod_{j=-m}^{0}\left[\frac{(1-\alpha b_j)(1-\frac{\alpha a_j}{c})(1-\alpha d b_j)(1-\frac{c}{d\alpha}a_j)}{(1-a_j)(1-\frac{a_j}{d})(1-cb_j)(1-\frac{\alpha^2 d}{c}b_j)}\right]\right\}$$

where $\langle a_j\rangle$ and $\langle b_j\rangle$ are arbitrary sequences such that none of the terms in the denominators vanish.

This reduces to the formula (2) on setting $a_k = ap^k, b_k = q^k$ and replacing α and d by d and a/b, respectively.

Recently, Subbarao and Verma [9] obtained a generalization of Chu's summation formula (5) involving four arbitrary sequences which is also obtained by using difference operator.

Theorem 5. *If m, n are non-negative integers, then*

$$\sum_{k=-m}^{n}\frac{(1-u_kv_kw_kz_k)(1-\frac{w_kz_k}{u_kv_k})(1-\frac{v_kz_k}{u_kw_k})(1-\frac{u_kz_k}{w_kv_k})}{(1-u_0v_0w_0z_0)(1-\frac{w_0z_0}{u_0v_0})(1-\frac{v_0z_0}{u_0w_0})(1-\frac{u_0z_0}{w_0v_0})} \tag{6}$$
$$\times\frac{\prod_{j=0}^{k-1}[(1-u_j^2)(1-v_j^2)(1-w_j^2)(1-z_j^2)]\frac{u_kv_kw_k}{z_k}}{\prod_{j=1}^{k}[(1-\frac{v_jw_jz_j}{u_j})(1-\frac{u_jw_jz_j}{v_j})(1-\frac{u_jv_jz_j}{w_j})(1-\frac{u_jv_jw_j}{z_j})]}$$
$$=\frac{(1-u_0^2)(1-v_0^2)(1-w_0^2)(1-z_0^2)}{(1-u_0v_0w_0z_0)(1-\frac{w_0z_0}{u_0v_0})(1-\frac{v_0z_0}{u_0w_0})(1-\frac{u_0z_0}{w_0v_0})}$$
$$\times\left\{\prod_{j=1}^{n}\left[\frac{(1-u_j^2)(1-v_j^2)(1-w_j^2)(1-z_j^2)]}{(1-\frac{v_jw_jz_j}{u_j})(1-\frac{u_jw_jz_j}{v_j})(1-\frac{u_jv_jz_j}{w_j})(1-\frac{u_jv_jw_j}{z_j})}\right]\right.$$
$$\left.-\prod_{j=-m}^{0}\left[\frac{(1-\frac{v_jw_jz_j}{u_j})(1-\frac{u_jw_jz_j}{v_j})(1-\frac{u_jv_jz_j}{w_j})(1-\frac{u_jv_jw_j}{z_j})}{(1-u_j^2)(1-v_j^2)(1-w_j^2)(1-z_j^2)}\right]\right\}$$

where $\langle u_k\rangle$, $\langle v_k\rangle$, $\langle w_k\rangle$, $\langle z_k\rangle$ are arbitrary sequences such that none of the terms in denominators vanish and m, n are non-negative integers.

The authors observed that the proofs of (1)–(6) are mainly obtained by means of difference operators and the sticking point is how to obtain the closed form of difference of two consecutive terms of a sequence. We find that we can use F. H. Jackson's identity [7] as our starting point for getting the summation formula. The identity is:

$$1-\frac{a(1-b)(1-c)(1-d)(1-a^2bcd)}{(1-ab)(1-ac)(1-ad)(1-abcd)} = \frac{(1-a)(1-abc)(1-abd)(1-acd)}{(1-ab)(1-ac)(1-ad)(1-abcd)}. \tag{7}$$

In fact, as Jackson pointed out, this identity is the instance $n = q = 1$ of his q-analogue of Dougall's theorem [7]. Jackson's identity also can be written in another form:

$$1-\frac{(1-a)(1-abc)(1-abd)(1-acd)}{(1-ab)(1-ac)(1-ad)(1-abcd)}=\frac{a(1-b)(1-c)(1-d)(1-a^2bcd)}{(1-ab)(1-ac)(1-ad)(1-abcd)}. \tag{8}$$

If one summation formula is derived from (7), the summation formula derived from (8) is called its dual summation formula, and vice versa.

In this paper we will use (7) or (8) to derive a generalization of Subbarao and Verma's summation formula (6) which generalizes Chu's summation formula (5) involving four arbitrary sequences and other general bilateral summations in Section 2. Subsequently, in Section 3, we shall exhibit the dual formulae for the above mentioned identities.

2 Unified formulae

We begin this section by giving two applications of Jackson's identity (7).

Theorem 6. *If $\langle a_k\rangle$, $\langle b_k\rangle$,$\langle c_k\rangle$, $\langle d_k\rangle$ are arbitrary sequences such that none of the terms in denominators vanish and m, n are non-negative integers, then*

$$\begin{aligned}
&\sum_{k=-m}^{n}\frac{a_k(1-b_k)(1-c_k)(1-d_k)(1-a_k^2b_kc_kd_k)}{(1-b_0)(1-c_0)(1-d_0)(1-a_0^2b_0c_0d_0)}\\
&\times\frac{\prod_{j=0}^{k-1}[(1-a_j)(1-a_jb_jc_j)(1-a_jb_jd_j)(1-a_jc_jd_j)]}{\prod_{j=1}^{k}[(1-a_jb_j)(1-a_jc_j)(1-a_jd_j)(1-a_jb_jc_jd_j)]}\\
&=\frac{(1-a_0)(1-a_0b_0c_0)(1-a_0b_0d_0)(1-a_0c_0d_0)}{(1-b_0)(1-c_0)(1-d_0)(1-a_0^2b_0c_0d_0)}\\
&\times\left\{\prod_{j=-m}^{0}\left[\frac{(1-a_jb_j)(1-a_jc_j)(1-a_jd_j)(1-a_jb_jc_jd_j)}{(1-a_j)(1-a_jb_jc_j)(1-a_jb_jd_j)(1-a_jc_jd_j)}\right]\right.\\
&\left.-\prod_{j=1}^{n}\left[\frac{(1-a_j)(1-a_jb_jc_j)(1-a_jb_jd_j)(1-a_jc_jd_j)}{(1-a_jb_j)(1-a_jc_j)(1-a_jd_j)(1-a_jb_jc_jd_j)}\right]\right\}.
\end{aligned} \tag{9}$$

Proof. Let

$$s_n=\frac{\prod_{j=1}^{n}(1-a_j)(1-a_jb_jc_j)(1-a_jb_jd_j)(1-a_jc_jd_j)}{\prod_{j=1}^{n}(1-a_jb_j)(1-a_jc_j)(1-a_jd_j)(1-a_jb_jc_jd_j)}$$

for $n = 0, \pm1, \pm2, \ldots$, and define the difference operator Δ by

$$\Delta s_n = s_n - s_{n-1}.$$

Then

$$\begin{aligned}\Delta s_k &= s_k - s_{k-1}\\ &= \frac{\prod_{j=1}^{k-1}(1-a_j)(1-a_jb_jc_j)(1-a_jb_jd_j)(1-a_jc_jd_j)}{\prod_{j=1}^{k-1}(1-a_jb_j)(1-a_jc_j)(1-a_jd_j)(1-a_jb_jc_jd_j)}\\ &\times \left\{\frac{(1-a_k)(1-a_kb_kc_k)(1-a_kb_kd_k)(1-a_kc_kd_k)}{(1-a_kb_k)(1-a_kc_k)(1-a_kd_k)(1-a_kb_kc_kd_k)} - 1\right\}.\end{aligned}$$

Now summing with respect to k from $-m$ to n, and using the fact that $\sum_{k=-m}^{n} \Delta s_k = s_n - s_{-m-1}$ and keeping in mind (8), we get (9) on simplification.

It should be noticed that (9) is (6) when a_j, b_j, c_j, d_j are replaced by $z_j^2, w_jv_j/u_jz_j, w_ju_j/v_jz_j, u_jv_j/w_jz_j$, respectively.

Similarly, if we let

$$s_n = \frac{\prod_{j=1}^{n} a_j(1-b_j)(1-c_j)(1-d_j)(1-a_j^2b_jc_jd_j)}{\prod_{j=1}^{n}(1-a_jb_j)(1-a_jc_j)(1-a_jd_j)(1-a_jb_jc_jd_j)}$$

where $n = 0, 1, \ldots,$ and use (7), we can get the dual summation formula of (9), that is:

Theorem 7. *If $\langle a_k\rangle$, $\langle b_k\rangle$, $\langle c_k\rangle$, $\langle d_k\rangle$ are arbitrary sequences such that none of the terms in denominators vanish and m, n are non-negative integers, then*

$$\begin{aligned}&\sum_{k=-m}^{n} \frac{(1-a_k)(1-a_kb_kc_k)(1-a_kb_kd_k)(1-a_kc_kd_k)}{(1-a_0)(1-a_0b_0c_0)(1-a_0b_0d_0)(1-a_0c_0d_0)}\\ &\times \frac{\prod_{j=0}^{k-1}[a_j(1-b_j)(1-c_j)(1-d_j)(1-a_j^2b_jc_jd_j)]}{\prod_{j=1}^{k}[(1-a_jb_j)(1-a_jc_j)(1-a_jd_j)(1-a_jb_jc_jd_j)]}\\ &= \frac{a_0(1-b_0)(1-c_0)(1-d_0)(1-a_0^2b_0c_0d_0)}{(1-a_0)(1-a_0b_0c_0)(1-a_0b_0d_0)(1-a_0c_0d_0)}\\ &\times \left\{\prod_{j=-m}^{0}\left[\frac{(1-a_jb_j)(1-a_jc_j)(1-a_jd_j)(1-a_jb_jc_jd_j)}{a_j(1-b_j)(1-c_j)(1-d_j)(1-a_j^2b_jc_jd_j)}\right]\right.\\ &\left. - \prod_{j=1}^{n}\left[\frac{a_j(1-b_j)(1-c_j)(1-d_j)(1-a_j^2b_jc_jd_j)}{(1-a_jb_j)(1-a_jc_j)(1-a_jd_j)(1-a_jb_jc_jd_j)}\right]\right\}.\end{aligned} \tag{10}$$

By setting

$$a_j = abcQ^{2j}, b_j = \frac{d}{abc}\frac{P^jq^j}{p^jQ^j}, c_j = \frac{P^jp^j}{bq^jQ^j}, d_j = \frac{p^jq^j}{cP^jQ^j}$$

in (10), we get a summation formula:

$$
\begin{aligned}
(11)\qquad & \sum_{k=-m}^{n} \frac{(1-abcQ^{2k})(1-ap^{2k})(\frac{b}{d}-P^{2k})(\frac{c}{d}-q^{2k})}{(1-abc)(1-a)(\frac{b}{d}-1)(\frac{c}{d}-1)} \\
\times\ & \frac{(\frac{abc}{d};\frac{pQ}{Pq})_k(b;\frac{qQ}{pP})_k(c;\frac{PQ}{pq})_k(ad;pqPQ)_k(\frac{Q}{pqP})^k}{(d\frac{qPQ}{p};\frac{qPQ}{p})_k(ac\frac{pPQ}{q};\frac{pPQ}{q})_k(ab\frac{pqQ}{P};\frac{pqQ}{P})_k(\frac{bc}{d}\frac{Q}{pqP};\frac{Q}{pqP})_k} \\
=\ & \frac{(1-\frac{abc}{d})(1-ad)(1-b)(1-c)}{(1-abc)(d-ad)(1-\frac{b}{d})(1-\frac{c}{d})} \\
\times\ & \left\{\frac{(\frac{abc}{d}\frac{pQ}{Pq};\frac{pQ}{Pq})_n(b\frac{qQ}{pP};\frac{qQ}{pP})_n(c\frac{PQ}{pq};\frac{PQ}{pq})_n(adpqPQ;pqPQ)_n}{(d\frac{qPQ}{p};\frac{qPQ}{p})_n(ac\frac{pPQ}{q};\frac{pPQ}{q})_n(ab\frac{pqQ}{P};\frac{pqQ}{P})_n(\frac{bc}{d}\frac{Q}{pqP};\frac{Q}{pqP})_n}\right. \\
-\ & \left.\frac{(\frac{1}{d};\frac{qPQ}{p})_{m+1}(\frac{1}{ac};\frac{pPQ}{q})_{m+1}(\frac{1}{ab};\frac{pqQ}{P})_{m+1}(\frac{d}{bc};\frac{Q}{pqP})_{m+1}}{(\frac{d}{abc};\frac{pQ}{Pq})_{m+1}(\frac{1}{b};\frac{qQ}{pP})_{m+1}(\frac{1}{c};\frac{PQ}{pq})_{m+1}(\frac{1}{ad};pqPQ)_{m+1}}\right\}.
\end{aligned}
$$

Letting $m=0, c=(\frac{pq}{PQ})^n$ in (11), we have:

Corollary 1. *If n is a non-negative integer, then*

$$
\begin{aligned}
(12)\qquad & \sum_{k=0}^{n} \frac{(1-ab(\frac{pq}{PQ})^nQ^{2k})(1-ap^{2k})(b-dP^{2k})((\frac{pq}{PQ})^n-dq^{2k})}{(1-ab(\frac{pq}{PQ})^n)(1-a)(b-d)((\frac{pq}{PQ})^n-d)} \\
\times\ & \frac{(\frac{ab}{d}(\frac{pq}{PQ})^n;\frac{pQ}{Pq})_k(b;\frac{qQ}{pP})_k((\frac{pq}{PQ})^n;\frac{PQ}{pq})_k(ad;pqPQ)_k(\frac{Q}{pqP})^k}{(d\frac{qPQ}{p};\frac{qPQ}{p})_k(a(\frac{pq}{PQ})^n\frac{pPQ}{q};\frac{pPQ}{q})_k(ab\frac{pqQ}{P};\frac{pqQ}{P})_k(\frac{b}{d}(\frac{pq}{PQ})^n\frac{Q}{pqP};\frac{Q}{pqP})_k} \\
=\ & \frac{(1-ad)(1-d)(a-(\frac{PQ}{pq})^n)(b-d(\frac{PQ}{pq})^n)}{(1-a)(b-d)(ab-(\frac{PQ}{pq})^n)(1-d(\frac{PQ}{pq})^n)},
\end{aligned}
$$

which is the generalization of (2.8) in [5].

By setting $m=0, d=1$ in (11), then we obtain

$$
\begin{aligned}
(13)\qquad & \sum_{k=0}^{n} \frac{(1-abcQ^{2k})(bP^{-k}-P^k)(p^{-k}-ap^k)(cq^{-k}-q^k)}{(1-abc)(1-b)(1-a)(1-c)} \\
\times\ & \frac{(abc;\frac{pQ}{Pq})_k(b;\frac{qQ}{pP})_k(c;\frac{PQ}{pq})_k(a;pqPQ)_kQ^k}{(\frac{qPQ}{p};\frac{qPQ}{p})_k(ac\frac{pPQ}{q};\frac{pPQ}{q})_k(ab\frac{pqQ}{P};\frac{pqQ}{P})_k(bc\frac{Q}{pqP};\frac{Q}{pqP})_k} \\
=\ & \frac{(abc\frac{pQ}{Pq};\frac{pQ}{Pq})_n(b\frac{qQ}{pP};\frac{qQ}{pP})_n(c\frac{PQ}{pq};\frac{PQ}{pq})_n(apqPQ;pqPQ)_n}{(\frac{qPQ}{p};\frac{qPQ}{p})_n(ac\frac{pPQ}{q};\frac{pPQ}{q})_n(ab\frac{pqQ}{P};\frac{pqQ}{P})_n(bc\frac{Q}{pqP};\frac{Q}{pqP})_n}.
\end{aligned}
$$

Letting $c=(\frac{pq}{PQ})^n$, then we have:

Corollary 2. *If n is a non-negative integer, then*

$$\sum_{k=0}^{n}\frac{(1-ab(\frac{pq}{PQ})^nQ^{2k})(bP^{-k}-P^k)(p^{-k}-ap^k)((\frac{pq}{PQ})^nq^{-k}-q^k)}{(1-ab(\frac{pq}{PQ})^n)(1-b)(1-a)(1-(\frac{pq}{PQ})^n)} \times \frac{(ab(\frac{pq}{PQ})^n;\frac{pQ}{Pq})_k(b;\frac{qQ}{pP})_k((\frac{pq}{PQ})^n;\frac{PQ}{pq})_k(a;pqPQ)_kQ^k}{(\frac{qPQ}{p};\frac{qPQ}{p})_k(a(\frac{pq}{PQ})^n\frac{pPQ}{q};\frac{pPQ}{q})_k(ab\frac{pqQ}{P};\frac{pqQ}{P})_k(b(\frac{pq}{PQ})^n\frac{Q}{pqP};\frac{Q}{pqP})_k}=\delta_{n,0} \tag{14}$$

which is the generalization of (2.3) in [4].

If we let

$$s_n=\frac{\prod_{j=1}^{n}(1-a_j)(1-b_j)(1-c_j)(1-E^2\frac{a_jc_j}{b_j})}{\prod_{j=1}^{n}(1-Ea_j)(1-\frac{b_j}{E})(1-Ec_j)(1-E\frac{a_jc_j}{b_j})}$$

where $n=0,1,\ldots,$ and E is a complex parameter, by using (8), then we obtain the generalization of Jain-Verma summation formula (4).

Theorem 8. *If m,n are non-negative integers, then*

$$\begin{aligned}&\sum_{k=-m}^{n}\frac{a_k(1-E)(1-\frac{b_k}{a_kE})(1-\frac{c_kE}{b_k})(1-a_kc_kE)}{(1-a_0)(1-b_0)(1-c_0)(1-\frac{E^2a_0c_0}{b_0})}\\&\times\frac{\prod_{j=0}^{k-1}[(1-a_j)(1-b_j)(1-c_j)(1-\frac{E^2a_jc_j}{b_j})]}{\prod_{j=1}^{k}[(1-Ea_j)(1-\frac{b_j}{E})(1-Ec_j)(1-\frac{Ea_jc_j}{b_j})]}\\&=\left\{\prod_{j=-m}^{0}\left[\frac{(1-Ea_j)(1-\frac{b_j}{E})(1-Ec_j)(1-\frac{Ea_jc_j}{b_j})}{(1-a_j)(1-b_j)(1-c_j)(1-\frac{E^2a_jc_j}{b_j})}\right]\right.\\&\left.-\prod_{j=1}^{n}\left[\frac{(1-a_j)(1-b_j)(1-c_j)(1-\frac{E^2a_jc_j}{b_j})}{(1-Ea_j)(1-\frac{b_j}{E})(1-Ec_j)(1-\frac{Ea_jc_j}{b_j})}\right]\right\}.\end{aligned} \tag{15}$$

If in (15) we replace E,a_j,b_j,c_j by $c/d,\beta p^j,cq^j,yP^j$, respectively, we obtain Jain-Verma's summation formula (4), which in turn incorporates (2) and (1) as special cases.

3 Dual formulae

Let us now give the dual formula of (15).

Theorem 9. *If m,n are non-negative integers, then*

$$\sum_{k=-m}^{n}\frac{(1-a_k)(1-b_k)(1-c_k)(1-E^2\frac{a_kc_k}{b_k})}{a_0(1-\frac{b_0}{a_0E})(1-\frac{c_0E}{b_0})(1-a_0c_0E)}$$

$$\times\frac{(1-E)^{k-1}\prod_{j=0}^{k-1}[a_j(1-\frac{b_j}{a_jE})(1-\frac{c_jE}{b_j})(1-a_jc_jE)]}{\prod_{j=1}^{k}[(1-Ea_j)(1-\frac{b_j}{E})(1-Ec_j)(1-\frac{Ea_jc_j}{b_j})]} \tag{16}$$

$$=\left\{\prod_{j=-m}^{0}\left[\frac{(1-Ea_j)(1-\frac{b_j}{E})(1-Ec_j)(1-\frac{Ea_jc_j}{b_j})}{a_j(1-E)(1-\frac{b_j}{a_jE})(1-\frac{c_jE}{b_j})(1-a_jc_jE)}\right]-\prod_{j=1}^{n}\left[\frac{a_j(1-E)(1-\frac{b_j}{a_jE})(1-\frac{c_jE}{b_j})(1-a_jc_jE))}{(1-Ea_j)(1-\frac{b_j}{E})(1-Ec_j)(1-\frac{Ea_jc_j}{b_j})}\right]\right\}.$$

Proof. Set

$$s_n=\frac{\prod_{j=1}^{n}a_j(1-E)(1-\frac{b_j}{a_jE})(1-\frac{c_jE}{b_j})(1-a_jc_jE)}{\prod_{j=1}^{n}(1-Ea_j)(1-\frac{b_j}{E})(1-Ec_j)(1-E\frac{a_jc_j}{b_j})};$$

by a similar method as in the proof of (9), we can obtain the result.

Summation formula (16), on setting $E=c/d$ and replacing a_j, b_j, c_j by $\beta p^j, cq^j, yP^j$, respectively, reduces to the dual summation formula of (4), which is:

Corollary 3. *If m, n are non-negative integers, then*

$$\sum_{k=-m}^{n}\frac{(1-\beta p^k)(1-cq^k)(1-yP^k)(1-\frac{\beta yc}{d^2}\frac{p^kP^k}{q^k})}{(1-\frac{d}{\beta})(1-\frac{y}{d})(1-\frac{\beta yc}{d})}$$

$$\times\frac{(\beta-\frac{\beta c}{d})^{k-1}p^{\binom{k}{2}}(\frac{d}{\beta};\frac{q}{p})_k(\frac{y}{d};\frac{P}{q})_k(\frac{\beta yc}{d};pP)_k}{(\frac{c\beta}{d}p;p)_k(dq;q)_k(\frac{cy}{d}P;P)_k(\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q})_k} \tag{17}$$

$$=\frac{(\frac{d}{c\beta};p)_{m+1}(\frac{1}{d};q)_{m+1}(\frac{d}{cy};P)_{m+1}(\frac{d}{\beta y};\frac{pP}{q})_{m+1}}{(\beta-\frac{\beta c}{d})^{m+1}p^{-\binom{m+1}{2}}(\frac{\beta}{d};\frac{q}{p})_{m+1}(\frac{d}{y};\frac{P}{q})_{m+1}(\frac{d}{\beta yc};pP)_{m+1}}$$

$$-\frac{(\beta-\frac{\beta c}{d})^np^{\binom{n+1}{2}}(\frac{d}{\beta}\frac{q}{p};\frac{q}{p})_n(\frac{y}{d}\frac{P}{q};\frac{P}{q})_n(\frac{\beta yc}{d}pP;pP)_n}{(\frac{c\beta}{d}p;p)_n(dq;q)_n(\frac{cy}{d}P;P)_n(\frac{\beta y}{d}\frac{pP}{q};\frac{pP}{q})_n}.$$

By setting $P=q, \beta=b, y=\frac{ad^2}{bc}$ in (17), we get the dual formula of the Gasper-Rahman's bibasic summation formula (2), which is:

Corollary 4. *If m, n are non-negative integers, then*

$$\sum_{k=-m}^{n}\frac{(1-bp^k)(1-cq^k)(1-\frac{ad^2}{bc}q^k)(1-ap^k)}{(1-\frac{d}{b})(1-\frac{ad}{bc})(1-ad)}$$

$$\times\frac{(b-\frac{bc}{d})^{k-1}p^{\binom{k}{2}}(\frac{d}{b};\frac{q}{p})_k(1-\frac{ad}{bc})^k(ad;pq)_k}{(\frac{cb}{d}p;p)_k(dq;q)_k(\frac{ad}{b}q;q)_k(\frac{ad}{c}p;p)_k} \tag{18}$$

$$=\frac{(\frac{d}{cb};p)_{m+1}(\frac{1}{d};q)_{m+1}(\frac{b}{ad};q)_{m+1}(\frac{c}{ad};p)_{m+1}}{(b-\frac{bc}{d})^{m+1}p^{-\binom{m+1}{2}}(\frac{b}{d};\frac{q}{p})_{m+1}(1-\frac{bc}{ad})^{m+1}(\frac{1}{ad};pq)_{m+1}}$$

$$-\frac{(b-\frac{bc}{d})^n p^{\binom{n+1}{2}}(\frac{d}{b}\frac{q}{p};\frac{q}{p})_n(1-\frac{ad}{bc})^n(adpq;pq)_n}{(\frac{cb}{d}p;p)_n(dq;q)_n(\frac{ad}{b}q;q)_n(\frac{ad}{c}p;p)_n}.$$

From the formula (18), we can see the dual formula of the bibasic summation formula (2) has four dependent bases. If we let $p=1$, then we can get the bilateral summation formula

$$\begin{aligned}
&\sum_{k=-m}^{n}\frac{(1-a)(1-b)(1-cq^k)(1-\frac{ad^2}{bc}q^k)}{(1-ad)(b-d)(1-\frac{c}{d})(1-\frac{ad}{bc})}\\
&\times\frac{(d-c)^k(cb-ad)^k(\frac{d}{b};q)_k(ad;q)_k}{(d-cb)^k(c-ad)^k(dq;q)_k(\frac{ad}{b}q;q)_k}\\
&=\frac{(bcd-d^2)^{m+1}(c-ad)^{m+1}(\frac{1}{d};q)_{m+1}(\frac{b}{ad};q)_{m+1}}{(b^2cd-b^2c^2)^{m+1}(bc-ad)^{m+1}(\frac{b}{d};q)_{m+1}(\frac{1}{ad};q)_{m+1}}\\
&-\frac{(d-c)^n(cb-ad)^n(\frac{d}{b}q;q)_n(adq;q)_n}{(d-cb)^n(c-ad)^n(dq;q)_n(\frac{ad}{b}q;q)_n}.
\end{aligned}\tag{19}$$

When $m=0$ and $d=1$, (18) reduces to the following dual formula of Gasper's formula (1).

Corollary 5. *If n is a non-negative integer, then*

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{(1-ap^k)(1-bp^k)(1-cq^k)(1-\frac{a}{bc}q^k)}{(1-a)(1-b)(1-\frac{a}{bc})}\\
&\times\frac{(1-c)^{k-1}(b-\frac{a}{c})^k p^{\binom{k}{2}}(\frac{1}{b};\frac{q}{p})_k(a;pq)_k}{(q;q)_k(\frac{a}{b}q;q)_k(cbp;p)_k(\frac{a}{c}p;p)_k}\\
&=\frac{(1-c)^n(b-\frac{a}{c})^n p^{\binom{n+1}{2}}(\frac{1}{b}\frac{q}{p};\frac{q}{p})_n(apq;pq)_n}{(q;q)_n(\frac{a}{b}q;q)_n(cbp;p)_n(\frac{a}{c}p;p)_n}.
\end{aligned}\tag{20}$$

By setting $p=1$ and replacing $1/b$ by b in (20), we can get an interesting formula:

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{(1-cq^k)(1-\frac{ab}{c}q^k)}{(1-c)(1-\frac{ab}{c})}\frac{(1-c)^k(c-ab)^k(b;q)_k(a;q)_k}{(b-c)^k(c-a)^k(q;q)_k(abq;q)_k}\\
&=\frac{(1-c)^n(c-ab)^n(bq;q)_n(aq;q)_n}{(b-c)^n(c-a)^n(q;q)_n(abq;q)_n}.
\end{aligned}\tag{21}$$

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