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Strong edge-magic labelling of a cycle with a chord

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Abstract

Suppose G is a finite graph with vertex-set $V(G)$ and edge-set $E(G)$. An *edge-magic total labelling* on G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \ldots, |V(G) \cup E(G)|$ with the property that, given any edge (x, y),

$$\lambda(x) + \lambda(x, y) + \lambda(y) = k$$

for some constant k. Such a labelling is called *strong* if the smallest labels appear on the vertices. In this paper, we investigate the existence of strong edge-magic total labellings of graphs derived from cycles by adding one chord.

1 Preliminaries

All graphs are finite, simple and undirected. The graph G has vertex-set $V(G)$ and edge-set $E(G)$. Unless otherwise noted, $V(G) = v$ and $E(G) = e$.

A *labelling* of a graph is any map that carries some set of graph elements to numbers (usually to the positive or non-negative integers). *Magic* labellings are one-to-one maps onto the appropriate set of consecutive integers starting from 1, with some kind of "constant-sum" property. An *edge-magic total labelling* on G is a one-to-one map λ from $V(G) \cup E(G)$ onto the integers $1, 2, \ldots, |V(G) \cup E(G)|$ with the property that, given any edge (x, y),

$$\lambda(x) + \lambda(x, y) + \lambda(y) = k$$

for some constant k. Edge-magic total labellings were first discussed by Kotzig and Rosa [4]. A recent discussion of edge-magic total labellings can be found in [5].

2 Strong labellings

An edge-magic total labelling will be called *strong* if it has the property that the vertex-labels are the integers $1, 2, \ldots, v$, the *smallest* possible labels. A graph with a strong edge-magic total labelling will be called *strongly edge-magic*. (Note: the phrase "super-edge-magic" has been used in the literature. The change in terminology is suggested because "super" has been used with other meanings in the study of magic labellings.)

If vertex x_i has degree d_i and is to receive label a_i, it is necessary to find an arrangement $\{a_i\}$ of the first v integers such that

$$e \text{ divides } \sum_{i=1}^{v+e} i + \sum_{i=1}^{v} (d_i - 1)a_i. \tag{1}$$

It follows that the even cycles are not strongly edge-magic. However, every odd cycle has a strong labelling [2]. The following labelling of the odd cycle with vertices $x_1, x_2, \ldots, x_{2n+1}$ is strong:

$$\lambda(x_i) \equiv 1 + ni (\bmod 2n + 1) \tag{2}$$

(the members of $\mathbb{Z}_{2n+1}$ being taken as $1, 2, \ldots, 2n + 1$). This labelling has been rediscovered several times.

2.1 Some facts about strong labellings

Enomoto *et al.* proved the following useful result.

Lemma 1 [2] *Any strongly edge-magic graph other than K_1 satisfies $e \leq 2v - 3$.*

It was noticed in [3] that a graph G is strongly edge-magic if and only if there is a map λ from $V(G)$ onto $\{1, 2, \ldots, v\}$ such that

$$S = \{\lambda(x) + \lambda(y) \mid xy \in E(G)\} \tag{3}$$

is a set of consecutive integers. (This fact is also implicit in [1].) The strong edge-magic total labelling is constructed from λ as follows: if s is the largest member of S, the constant k is $v+1+s$, and $\lambda(xy) = k - \lambda(x) - \lambda(y)$ for each edge xy. We call this the edge-magic total labelling *induced by* the vertex labelling λ; the use of the same symbol λ for both labellings should cause no confusion. The vertex labelling λ is in fact an edge-antimagic vertex labelling, so we could say that G is strongly edge-magic if and only if it has an edge-antimagic vertex labelling such that the edge weights are consecutive. It is convenient to call $\lambda(x)+\lambda(y)$ the *vertex-weight* of the chord xy.

There is a natural duality for strong edge-magic total labellings. If λ is a vertex labelling for which (3) is consecutive, then the map λ^*, defined by

$$\lambda^*(x) = v + 1 - \lambda(x),$$

also has this property. We call λ^* the *strong dual* of λ. Strong duality will be useful in complete listings of small strong labellings. Sometimes the strong dual is identical to the original labelling (an example is the labelling in Figure 2); in this case the labelling is *strongly self-dual.*

3 Cycle plus a chord

The graph constructed by adding a chord to a cycle always satisfies Lemma 1. We wondered whether all such graphs are strongly edge-magic. The short answer is "no"; an exception is the graph formed from a 6-cycle by joining two vertices of distance 2. But maybe this is the only exception. We shall provide solutions for all cases of an odd cycle with a chord, and the majority of cases of an even cycle with a chord.

It will be convenient to have a notation for a cycle with a chord added. We shall write C_v^t to mean the graph constructed from a C_v by joining two vertices whose distance in the cycle is t. In this notation, we have just remarked that C_6^2 is not strongly edge-magic.

Suppose C_v^t has a strong edge-magic total labelling, and suppose the endpoints of the chord receive labels a and b. Then (1) is

$$(v+1)k = 2(1+2+\ldots+v) + a + b + ((v+1)+(v+2)+\ldots+(2v+1)) \qquad (4)$$

so $v+1$ must divide $a+b+\delta(v+1)$, where δ is 0 when v is even and $\frac{1}{2}$ when v is odd.

When v is even, say $v = 2n$, we have $a+b \equiv 0 (\bmod 2n+1)$, so $a+b = 0$ or $2n+1$ or $4n+2\ldots$ and the condition $0 \le a, b \le 2n$ implies $a+b = 2n+1$. From (4), $k = 2n+1$ and the set of sums (3) is

$$\{n+1, n+2, \ldots, 2n+1, \ldots, 3n+1\}.$$

For odd $v = 2n+1$ we get $a+b = n+1$ or $3n+3$ yielding sums

$$\{n+1, n+2, \ldots, 3n+2\} \text{ or } \{n+2, n+2, \ldots, 3n+3\}$$

respectively. In this case the vertex labels that induce a strong edge-magic total labelling of any C_v^t will also induce such a labelling of C_v.

It may well happen that one vertex labelling of C_v provides strong edge-magic total labellings of C_v^t for all possible chord lengths t. A labelling with this desirable property will be called *universal.*

3.1 Odd cycles

A chord in C_v of length t is of course also a chord of length $v-t$. This means that we can restrict our attention to chords of length at most $\frac{1}{2}v$, or, in the case of odd cycles, to chords of odd length.

It is easy to see that the labelling (2) of C_{2n+1} provides chords of all lengths t where $t \equiv 3 (\bmod 4)$, so C_v^t is strongly magic for every $t \equiv 3 (\bmod 4)$, when v is odd. However, we have constructed better labellings.

Theorem 1 *The following vertex labelling λ of C_{4m+3} is universal:*

i *values*	$\lambda(x_i)$
i *even*, $2 \le i \le 2m$	$2m+3+\frac{1}{2}i$
i *even*, $2m+2 \le i \le 4m+2$	$\frac{1}{2}(i+2)$
i *odd*, $1 \le i \le 2m+1$	$\frac{1}{2}(i+1)$
i *odd*, $2m+3 \le i \le 4m+1$	$2m+\frac{1}{2}(i+5)$
$i = 4m+3$	$2m+3$

Proof. It is easy to verify that λ yields edges with vertex-weights $\{2m+3, 2m+4, \ldots, 6m+5\}$. So it induces a strongly edge-magic labelling of C_{4m+3}. To prove that it is a strongly edge-magic labelling of C_{4m+3}^t it suffices to exhibit a chord of length t with vertex-weight $2m+2$ or $6m+6$.

For $1 \le i \le m$, $\lambda(x_{2i-1}) = i$ and $\lambda(x_{4m+2-2i}) = 2m+2-i$. So the chord $x_{2i-1}x_{4m+2-2i}$ of length $4(m-i)+3$ has vertex-weight $2m+2$. This covers chords of lengths $\{3, 7, \ldots, 4m-1\}$.

For $1 \le i \le m-1$, $\lambda(x_{2i}) = 2m+3+i$ and $\lambda(x_{4m+1-2i}) = 4m+3-i$. So the chord $x_{2i}x_{4m+1-2i}$ of length $4(m-i)+1$ has vertex-weight $6m+6$. This covers chords of lengths $\{5, 9, \ldots, 4m-3\}$.

Finally, $x_{4m+1}x_{4m+3}$ is a chord of length $4m+1$ whose vertex-weight is $\lambda(x_{4m+1}) + \lambda(x_{4m+3}) = (4m+3)+(2m+3) = 6m+6$. So the construction covers all lengths. $\square$

We have not found a universal labelling for all cycles C_{4m+1}. We present two theorems, one of which has chords with the appropriate weight of all odd lengths except 5 and 9 (and, consequently, all even lengths except $4m-4$ and $4m-8$), and another for chords of all lengths congruent to 1 (or 0) modulo 4 other than $4m-3$ (or 4). So all odd cycles with chords are strongly edge-magic.

It should be noted that the constructions fail when $m < 3$. However, universal labellings of C_5 and C_9 exist; see Figure 1.

Theorem 2 *The following vertex labelling λ induces a strongly edge-magic labelling of C_{4m+1}^t for every t other than $t = 5, 9, 4m-4, 4m-8$, provided $m \ge 3$:*

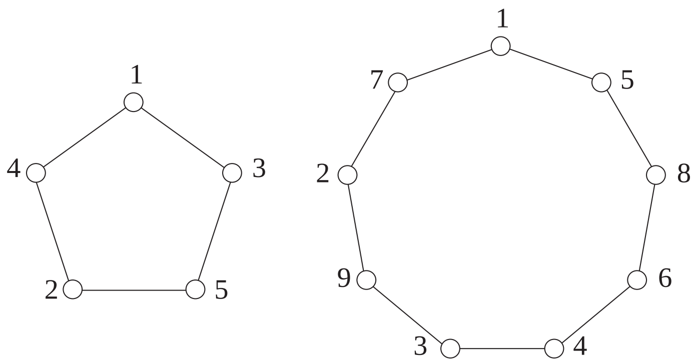

Figure 1: Universal labellings of C_5 and C_9.

i *values*	$\lambda(x_i)$
i *even,* $2 \leq i \leq 2m-2$	$2m+4+\frac{1}{2}i$
i *even,* $2m \leq i \leq 4m$	$\frac{1}{2}i+2$
i *odd,* $1 \leq i \leq 2m+1$	$\frac{1}{2}(i+1)$
i *odd,* $2m+3 \leq i \leq 4m-3$	$2m+\frac{1}{2}(i+5)$
$i = 4m \pm 1$	$\frac{1}{2}(i+7)$

Proof. Again it is easy to verify that λ induces a strongly edge-magic labelling of C_{4m+1}. To prove that it is a strongly edge-magic labelling of C_{4m+1}^t it suffices to exhibit a chord of length t with vertex-weight $2m+1$ or $6m+3$.

For $1 \leq i \leq m-1$, $\lambda(x_{2i-1}) = i$ and $\lambda(x_{4m-2-2i}) = 2m+2-i$. So the chord $x_{2i-1}x_{4m-2-2i}$ of length $4(m-i)-1$ has vertex-weight $2m+1$. This covers chords of lengths $\{3, 7, \ldots, 4m-5\}$.

For $1 \leq i \leq m-5$, $\lambda(x_{2m+2i+1}) = 3m+3+i$ and $\lambda(x_{2m-8-2i}) = 3m-i$. So the chord $x_{2m+2i+1}x_{2m-8-2i}$ of length $4i+9$ has vertex-weight $6m+3$. This covers chords of lengths $\{13, 17, \ldots, 4m-11\}$.

Finally, we have the following chords: $x_{2m-1}x_{2m+1}$, length $4m-1$, vertex-weight $m+(m+1) = 2m+1$; $x_{4m-7}x_{4m+1}$, length $4m-7$, vertex-weight $(2m+4)+(4m-1) = 6m+3$; and $x_{4m-1}x_{4m-5}$, length $4m-3$, vertex-weight $(2m+3)+4m = 6m+3$.

So the construction covers all required lengths. □

Notice that the construction fails when $m < 3$. When $m = 1$ two values are assigned to $\lambda(x_3)$ and $\lambda(x_5) > 5$; when $m = 2$ the value assigned to $\lambda(x_{4m-5})$ comes from the third line of the table, not the fourth line, so chord length $4m-3$ does not arise.

Theorem 3 *The following vertex labelling λ induces a strongly edge-magic labelling of C^t_{4m+1} for every $t \equiv 1 (\mathrm{mod}\, 4)$ except $4m-3$:*

i *values*	$\lambda(x_i)$
i *even,* $2 \le i \le 2m-2$	$2m+3+\frac{1}{2}i$
$i = 2m$	$m+2$
i *even,* $2m+2 \le i \le 4m-2$	$2m+2+\frac{1}{2}i$
$i = 4m$	$2m+2$
i *odd,* $1 \le i \le 2m+1$	$\frac{1}{2}(i+1)$
i *odd,* $2m+3 \le i \le 4m-1$	$\frac{1}{2}(i+3)$
$i = 4m+1$	$\frac{1}{2}(i+5)$

Proof. It is easy to check that λ is an edge-magic labelling with vertex labels $\{1, 2, \ldots, 4m+1\}$ whose vertex-weights are $\{2m+2, 2m+3, \ldots, 6m+2\}$. For odd i, $1 \le i \le 2m-3$, the chord $x_i x_{4m-2-i}$ has vertex-weight $2m+1$ and length $4m-4-2i$, so the labelling induces a strongly magic labelling of $C^{4m-4-2i}_{4m+1}$. As i ranges from 1 to $2m-3$, this produces labellings for chords of lengths $5, 9, \ldots, 4m-7$. $\square$

3.2 Even cycles

Suppose there is a strongly magic labelling of C^t_{2n}. Suppose the sums of adjacent labels are $k, k+1, \ldots, k+2n$, where the sum of the labels on the chord is $k+i$. If we add the sums for all edges of the cycle, we add the label on each vertex twice, so

$$\sum_{j=0}^{2n}(k+j) = 2\sum_{j=1}^{2n} j + (k+i),$$

or

$$\begin{aligned}(2n+1)k + \frac{2n(2n+1)}{2} &= 2n(2n+1) + k + i,\\ i &= 2n\left(k - \frac{2n+1}{2}\right).\end{aligned}$$

As $1 \le i \le 2n$, and as k is an integer, the only possibility is for $k - \frac{2n+1}{2}$ to equal $\frac{1}{2}$, so that $i = n$ and $k = n+1$. If a suitable labelling of the vertices is found, we may take the chord to be any pair of vertices whose labels add to $2n+1$.

For example, consider the C_8 whose vertices are labeled (sequentially) $1, 4, 3, 8, 2, 6, 7, 5$. A chord could be added joining the vertices labeled $1, 8$, giving a strongly magic labelling of C^3_8, or $2, 7$, giving a strongly magic labelling of C^2_8. This example does not provide a strongly magic labelling of C^4_8, but such labellings are available (an example is $1, 5, 3, 2, 8, 4, 7, 6$).

A complete search shows that the only examples for C_4 and C_6 are $1, 3, 4, 2$ and $1, 3, 2, 6, 4, 5$. These give strongly magic labellings of C^2_4 and C^3_6. There is no such labelling of C^2_6.

3.3 Some general constructions

Theorem 4 *The following vertex labelling λ induces a strongly edge-magic labelling of C^t_{4m} for all $t \equiv 2 (\operatorname{mod} 4)$:*

i *values*	$\lambda(x_i)$
i *even*, $2 \le i \le 2m-2$	$2m + \frac{1}{2}(i+2)$
i *even*, $2m \le i \le 4m$	$\frac{1}{2}(i+2)$
i *odd*, $1 \le i \le 2m-1$	$\frac{1}{2}(i+1)$
i *odd*, $2m+1 \le i \le 4m-1$	$2m + \frac{1}{2}(i+1)$

Proof. There is no problem in seeing that the vertex labels run from 1 to $4m$ and the vertex-weights run from $2m+1$ to $6m+1$, omitting $4m+1$.

When i is odd and $1 \le i \le 2m-1$, the chord $x_i x_{4m-i}$, of length $4m-2$, has vertex-weight $4m+1$. So the construction induces strong edge-magic labellings of C^t_{4m} in all the required cases. □

This construction gives no other cases: the other chords with the appropriate vertex-weight duplicate these lengths.

Theorem 5 *The following vertex labelling λ induces a strongly edge-magic labelling of $C^t_{4m+2}, m > 1$, for all odd t other than 5, and for $t = 2, 6$:*

i *values*	$\lambda(x_i)$
$i = 2$	$2m+4$
$i = 4$	$4m+2$
$i = 2m$	$m+2$
$i = 2m+2$	$m+3$
i *even*, $6 \le i \le 2m-2$	$2m+2+\frac{1}{2}i$
i *even*, $2m+4 \le i \le 4m+2$	$2m+\frac{1}{2}i$
i *odd*, $1 \le i \le 2m+1$	$\frac{1}{2}(i+1)$
i *odd*, $2m+3 \le i \le 4m+1$	$\frac{1}{2}(i+5)$

Proof. Again, the vertex labels have the required values, and the vertex-weights cover all possibilities once each, with $4m+3$ missing. We see that the chords with vertex-weight $4m+3$ are:

$x_{2j+1}x_{4m+4-2j}$, for $j = 1, 2, \ldots, m$, producing chords of all lengths $\equiv 3 (\operatorname{mod} 4)$;

$x_{2m+2j+1}x_{4m+2j}$, for $j = 1, 2, \ldots, m-5$, giving chords of all lengths $\equiv 1 (\operatorname{mod} 4)$ other than 5;

$x_{2m-2}x_{2m}$ and $x_{2m-4}x_{2m+2}$, giving chords of lengths 2 and 6. □

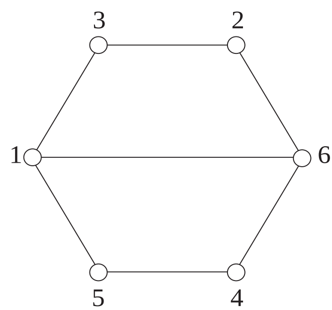

Figure 2: A strong labelling of C_6^3.

3.4 Small examples

Complete searches have been made for some small cases. A solution is denoted by the sequence of vertex labels around the cycle, followed by the labels on the endpoints of the chord. In the odd order cases, these vertex labels yield a strong labelling of the cycle.

Up to C_6, all the solutions of a given size have the same vertex labels, but there is more than one choice for the chord. Each is strongly self-dual.

C_4

The self-dual labelling $1, 2, 4, 3$ provides two solutions for C_4^2, using chords 1,4 and 2,3.

C_5

The self-dual labelling of C_5 shown in Figure 1 has chords 1,2 and 4,5 of length 2. There are no other examples.

C_6

The self-dual labelling $1, 3, 2, 6, 4, 5$ of C_6 can be interpreted as a strong labelling of C_6^3 in three ways, using chord namely 1,6 or 2,5 or 3,4. There is no solution for C_6^2.

The pattern in these small orders does not continue. In particular, strongly self-dual examples seem to be relatively rare. In the examples of orders 4 and 6, all longest chords have endpoints whose labels sum to the required total, but this promising pattern does not continue — there is no such example at order 8.

In the higher cases, it is common for several chord lengths to be provided by the same vertex-labelling. We represent these in an obvious notation.

C_7

There are five vertex labellings for C_7, up to duality, all but one of which provide solutions for both C_7^2 and C_7^3:

Vertex labels	Chord	
	length 2	length 3
$1,4,7,3,6,2,5$		$1,3\ ;\ 5,7$
$1,4,2,7,3,5,6$	$5,7$	$1,3$
$1,4,2,5,6,3,7$	$1,3$	$5,7$
$1,5,6,4,3,2,7$	$5,7$	$1,3$
$1,4,3,7,2,6,5$	$1,3$	$5,7$

The first solution is self-dual.

C_8

The labellings for C_8 form four dual pairs and provide all chords:

Vertex labels	Chords		
	length 2	length 3	length 4
$1,4,3,8,2,6,7,5$	$2,7;\ 4,5$	$1,8;\ 3,6$	
$1,4,8,3,7,6,2,5$	$1,8;\ 2,7;$ $3,6;\ 4,5$		
$1,4,2,8,3,5,7,6$		$1,8;\ 3,6$	$2,7;\ 4,5$
$1,5,3,2,8,4,7,6$		$2,7;\ 3,6$	$1,8;\ 4,5$

However, observe that none is universal.

C_9

We have constructed all labellings of C_9 that induce strongly edge-magic labellings of C_9^t. There are thirteen dual pairs and five self-dual labellings. We have tabulated one member of each dual pair (the first column and the first four rows of the second column) followed by the self-dual labellings, showing the number of chords of various lengths. It will be seen that many are universal.

Vertex labels	Chord length		
	2	3	4
$1,5,3,7,2,9,4,8,6$	1	2	1
$1,5,3,9,4,7,2,8,6$	1		3
$1,5,2,8,6,3,9,4,7$	2	1	1
$1,5,2,9,4,8,6,3,7$		2	2
$1,5,4,3,9,2,8,6,7$	3	1	
$1,5,8,6,4,3,9,2,7$	1	2	1
$1,5,9,2,8,4,3,6,7$		1	3
$1,5,7,3,4,9,2,6,8$	1	2	1
$1,5,3,4,8,6,7,2,9$	1	2	1

Vertex labels	Chord length		
	2	3	4
$1,5,7,2,6,8,3,4,9$	1	2	1
$1,5,8,6,2,7,4,3,9$		3	1
$1,6,8,5,4,2,9,3,7$	1	1	2
$1,7,5,8,6,3,4,2,9$	2	1	1
$1,5,9,4,8,3,7,2,6$	2	2	
$1,5,9,2,6,7,3,4,8$	2	2	
$1,6,8,3,5,7,2,4,9$	2	2	
$1,7,4,2,5,8,6,3,9$	2		2
$1,8,6,7,5,3,4,2,9$	2	2	

C_{10}

The labellings for C_{10} come in 17 dual pairs. There are no self-dual labellings. We tabulate the number of chords of various lengths for one member of each pair:

Vertex labels	Chord length			
	2	3	4	5
$1, 5, 2, 6, 4, 10, 3, 9, 7, 8$	1	1	1	2
$1, 5, 3, 10, 4, 8, 2, 7, 9, 6$	2	3		
$1, 5, 4, 10, 3, 9, 7, 8, 2, 6$	1	3	1	
$1, 5, 2, 8, 4, 10, 3, 6, 9, 7$		1	2	2
$1, 5, 4, 3, 10, 2, 8, 6, 9, 7$		3	2	
$1, 5, 8, 2, 10, 4, 3, 6, 9, 7$			4	1
$1, 5, 9, 4, 3, 6, 10, 2, 8, 7$			4	1
$1, 5, 9, 6, 10, 3, 4, 8, 2, 7$	2	1	2	
$1, 5, 10, 2, 8, 6, 3, 4, 9, 7$	3		1	1

Vertex labels	Chord length			
	2	3	4	5
$1, 5, 10, 4, 3, 9, 7, 6, 2, 8$	1	2	1	1
$1, 5, 2, 6, 9, 7, 3, 10, 4, 8$	2	3		
$1, 5, 2, 6, 3, 10, 4, 8, 7, 9$	2	2		1
$1, 5, 4, 3, 10, 2, 6, 8, 7, 9$			4	1
$1, 5, 10, 3, 4, 8, 6, 2, 7, 9$	3		1	1
$1, 6, 8, 2, 4, 5, 10, 3, 9, 7$			2	3
$1, 6, 4, 2, 10, 3, 5, 9, 7, 8$			4	1
$1, 6, 9, 7, 5, 3, 10, 4, 2, 8$		1	4	

In this case there are two universal labellings, the first two listed.

C_{11}

There are 308 suitable labellings of C_{11}. Several are self-dual: one example is $(1, 7, 4, 10, 3, 6, 9, 8, 2, 5, 11)$.

C_{12}

Altogether there are 503 suitable labellings of C_{11}. Several are self-dual: one example is $(1, 6, 10, 4, 11, 7, 12, 5, 3, 9, 2, 8)$. One universal labelling is $(1, 6, 2, 12, 3, 8, 4, 5, 11, 7, 10, 9)$.

4 Two unsolved problems

1. For which values of t does C_{2n}^t have a strongly magic labelling?

2. For which n is there a universal vertex labelling of C_{2n}?

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References

[1] B. D. Acharya and S. M. Hegde, Strongly indexable graphs. *Discrete Math.* **93** (1991), 123–129.

[2] H. Enomoto, A. S. Llado, T. Nakamigawa and G. Ringel, Super edge-magic graphs. *SUT J. Math.* **2** (1998), 105–109.

[3] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, The place of super edge magic labellings among other classes of labellings. *Discrete Math.* **231** (2001), 153–168.

[4] A. Kotzig and A. Rosa, Magic valuations of finite graphs. *Canad. Math. Bull.* **13** (1970), 451–461.

[5] W. D. Wallis, E. T. Baskoro, M. Miller and Slamin, Edge-magic total labellings. *Australas. J. Combin.* **22** (2000), 177–190.

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