Pebbling in dense graphs

ANDRZEJ CZYGRINOW   GLENN HURLBERT*

Department of Mathematics and Statistics
Arizona State University
Tempe
Arizona 85287-1804
U.S.A.
andrzej@math.la.asu.edu   hurlbert@asu.edu

Abstract

A configuration of pebbles on the vertices of a graph is solvable if one can place a pebble on any given root vertex via a sequence of pebbling steps. The pebbling number of a graph G is the minimum number $\pi(G)$ so that every configuration of $\pi(G)$ pebbles is solvable. A graph is Class 0 if its pebbling number equals its number of vertices. A function is a pebbling threshold for a sequence of graphs if a randomly chosen configuration of asymptotically more pebbles is almost surely solvable, while one of asymptotically fewer pebbles is almost surely not. Here we prove that graphs on $n \geq 9$ vertices having minimum degree at least $\lfloor n/2 \rfloor$ are Class 0, as are bipartite graphs with $m \geq 336$ vertices in each part having minimum degree at least $\lfloor m/2 \rfloor + 1$. Both bounds are best possible. In addition, we prove that the pebbling threshold of graphs with minimum degree δ, with $\sqrt{n} \ll \delta$, is $O(n^{3/2}/\delta)$, which is tight when δ is proportional to n.

1   Introduction

1.1   Pebbling Numbers

Let $G = (V, E)$ be a connected graph with $n = n(G)$ vertices $V = \{v_1, \ldots, v_n\}$ and having edge set E. A *configuration* C of t pebbles on G is an assignment of t indistinguishable pebbles to the vertices of G. The notation $C(v) = x$ means that x pebbles have been assigned to the vertex v. (Notation such as $C(a, b, c) = x$ means that x pebbles have been assigned to each of the vertices a, b, c.) The parameter t is known as the *size* of C, which is also denoted by $|C|$. A *pebbling step* from vertex u to vertex v involves the removal of two pebbles from u and the placement of one of them onto v. A configuration is *r-solvable* if it is possible to place a pebble on

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the *root* vertex r via a (possibly empty) sequence of pebbling steps. A configuration is *solvable* if it is r-solvable for all choices of a root r. The *pebbling number* of a graph G, denoted $\pi(G)$, is the least number t for which every configuration of size t is solvable. One should read [10] for the history and main results of the theory of pebbling in graphs.

Clearly $\pi(G) \geq n(G)$ for every G, for the configuration which places no pebble on r and one pebble on all other vertices is not r-solvable. The authors of [4] defined G to be *Class 0* when $\pi(G) = n(G)$. Examples of Class 0 graphs include cliques (via the Pigeonhole Principle) and cubes (see [2]), as well as the 5-cycle, the Petersen graph, and many others. They proved that all 3-connected, diameter 2 graphs are Class 0, and conjectured that fixed diameter graphs with large enough connectivity are also Class 0. This conjecture was proved in [6], where the result was used to prove that the random graph in which each edge appears independently with probability $p \gg (n \lg n)^{1/d}/n$ for some $d > 0$ is almost surely Class 0.

Let $\mathbf{B}(m)$ be the set of all connected bipartite graphs with m vertices in each part. It was proven in [9] that every regular graph in $\mathbf{B}(m)$ having degree at least $\lceil 2m/3 \rceil + 1$ is Class 0. In this paper we derive a more general and stronger result for large m. Let $b = b(m)$ be the minimum number so that every $G \in \mathbf{B}(m)$ having minimum degree at least b is Class 0.

Theorem 1 *For all $m \geq 336$, $b(m) = \lfloor m/2 \rfloor + 1$.*

We prove this in Section 2.1. We also derive an analogous result for connected graphs. Let $\mathbf{G}(n)$ be the set of all connected graphs on n vertices, and let $g(n)$ be the minimum number g so that every $G \in \mathbf{G}(n)$ having minimum degree at least g is Class 0. We prove the following in Section 2.2.

Theorem 2 *For all $n \geq 9$, $g(n) = \lfloor n/2 \rfloor$.*

1.2 Thresholds

We next consider a randomized version of pebbling, introduced in [3], in which we consider the probability space of all configurations of t pebbles, each equally likely. The pebbling number is the minimum t for which the probability that a configuration is solvable equals 1. Now we wish to find t so that this probability is nearly 1. To be more precise, let us introduce some notation.

For two functions $f = f(n)$ and $g = g(n)$ we say that $f \ll g$ $(g \gg f)$ if $f/g \to 0$ as $n \to \infty$. We set $o(g) = \{f \mid f \ll g\}$ and $\omega(f) = \{g \mid g \gg f\}$. We also write $f \sim g$ whenever $f/g \to 1$ as $n \to \infty$. Further, we set $O(f) = \{g \mid \text{for some } c, k > 0, g < cf \text{ for all } n > k\}$, and similarly $\Omega(g) = \{f \mid \text{for some } c, k > 0, f > cg \text{ for all } n > k\}$. Finally we define $\Theta(f) = O(f) \cap \Omega(f)$.

We consider sequences $\mathcal{G} = (G_1, \ldots, G_n, \ldots)$ of graphs for which the number of vertices increases with n (e.g. G_n has n vertices). For a function $t = t(n)$ we denote by $\mathrm{Pr}_t(n)$ the probability that a randomly chosen configuration of t pebbles on G_n is solvable. A function $\tau = \tau(n)$ is a *pebbling threshold* for $\mathcal{G}$ if $\mathrm{Pr}_t(n) \to 1$ for all $t \gg \tau$ and $\mathrm{Pr}_t(n) \to 0$ for all $t \ll \tau$. We denote by $\tau(\mathcal{G})$ the set of all pebbling

thresholds for $\mathcal{G}$. It is not immediately evident that every graph sequence has such a threshold, but it is proven so in [1]. The first threshold result, from [3], established that $\tau(\mathcal{K}) = \Theta(n^{1/2})$, where $\mathcal{K}$ is the sequence of complete graphs. More results on the pebbling thresholds of paths, cubes, and other sequences appear in [1, 5, 7, 8]. For instance, it is known that if $t \in \tau(\mathcal{G})$ for some graph sequence $\mathcal{G}$, then $t \in \Omega(n^{1/2})$.

For our purposes let us define $\mathbf{G}(n, \delta)$ to be the set of all connected graphs on n vertices having minimum degree at least $\delta = \delta(n)$. Let $\mathcal{G}_\delta = (G_1, \ldots, G_n, \ldots)$ denote any sequence of graphs with each $G_n \in \mathbf{G}(n, \delta)$. In Section 2.3 we prove the following theorem.

Theorem 3 *For every function $\sqrt{n} \ll \delta = \delta(n) \leq n - 1$, $\tau(\mathcal{G}_\delta) \subseteq O(n^{3/2}/\delta)$. In particular, if in addition $\delta \in \Omega(n)$ then $\tau(\mathcal{G}_\delta) = \Theta(n^{1/2})$.*

2 Proofs

2.1 Theorem 1

Lower bound. First we give a proof of the lower bound, that $b(m) \geq \lfloor m/2 \rfloor + 1$ for all $m \geq 7$.

We define, for each m, the bipartite graph B_m as follows. Let $|L| = |R| = m$ with $L = L_1 \cup L_2$ and $R = R_1 \cup R_2$ so that $|L_1| = |R_1| = \lceil m/2 \rceil$ and $|L_2| = |R_2| = \lfloor m/2 \rfloor$. Let the induced subgraphs on $L_1 \cup R_1$ and on $L_2 \cup R_2$ each be complete bipartite with one missing edge. Let the two missing edges be xy, with $x \in L_1$ and $y \in R_1$, and wz, with $w \in L_2$ and $z \in R_2$. Finally include the two edges wy and xz. Note that for $\delta = \lfloor m/2 \rfloor$ the graph B_m has minimum degree δ, and is δ-regular when m is even.

Now we define a configuration C of size $n = n(B_m) = 2m$, and show that it is unsolvable when $m \geq 7$. We choose the root $r \in L_2 - \{w\}$ and define $C(r, w, x, y, z) = 0$. We find $a, b, c \in L_1 - \{x\}$ and define $C(a, b) = 3$ and $C(c) = 2$. Finally we define $C(v) = 1$ for all other vertices v. Clearly, $|C| = n$.

In order that C is r-solvable one must be able to move 2 pebbles onto either w or z, and consequently 4 pebbles onto either x or y. It is not difficult to see that both cases are impossible, since at most three pebbles can be put in motion via pebbling steps from a, b, and c. Hence, for all $m \geq 7$, B_m is not Class 0 and so $b(m) \geq \lfloor m/2 \rfloor + 1$. $\qquad\square$

Upper bound. Second we give a proof of the upper bound, that $b(m) \leq \lfloor m/2 \rfloor + 1$ for all $m \geq 336$.

Let $B \in \mathbf{B}(m)$ have bipartition L, R and minimum degree at least $\lfloor m/2 \rfloor + 1$, where $|L| = |R| = m$. Choose any configuration C of size $n = n(B) = 2m$ and let r be any chosen root (which we may assume lies in L). We assume that C is r-unsolvable and derive a contradiction.

We will make use of the following two observations about B. First, every pair of vertices in the same part has a common neighbor. Second, from this it is clear that the diameter of B is at most 3. We will derive a contradiction by accumulating 8 pebbles on some vertex, from which we can obviously pebble to r.

Denote the neighborhood of a vertex v by $N(v)$, and the union of neighborhoods of a set S of vertices by $N(S)$. Then we must have $C(r) = 0$ and $C(v) \leq 1$ for all $v \in N(r)$. We also know that $C(v) \leq 3$ for all $v \in N(N(r)) = L$. Let $Z = \{v \mid C(v) = 0\}$, $U = \{u \mid C(u) = 1\}$, and $H = V - Z - U$. We let $Z_L = Z \cap L$, with Z_R, U_L, U_R, H_L, and H_R defined analogously.

Claim. $|Z| > m/2$.

Proof. The claim is trivial if $N(r) \subseteq Z$ so we assume otherwise and pick some $r' \in N(r) \cap U$. Now we know that we cannot move another pebble to r'. We note that H is nonempty because Z is nonempty. Also we note that if there is some vertex v with $C(v) \geq 4$, then we can move a pebble to either r or r'. Hence we assume $C(v) \leq 3$ for all v. Moreover, we note that it must be impossible to ever put 4 pebbles on any vertex.

Suppose that there is a vertex t having $C(t) = 3$; without loss of generality we assume that $t \in L$ (if $t \in R$, we think of r' as our new root and argue similarly). Since $N(r) \cap N(t) \subseteq Z$, we know that $|Z| \geq 2$. Moreover, $N(t) \subseteq Z$ implies $|Z| > m/2$, so we assume otherwise and pick $t' \in N(t) \cap (U \cup H)$. Then $N(t') \cap N(r') \subseteq Z$, and so $|Z| \geq 3$, which implies that $|H| \geq 2$.

If there is a vertex $s \in H_L - \{t\}$ then we can argue as follows. Let $X = N(r)$, $T = N(t)$, and $S = N(s)$. Of course, $(X \cap S) \cup (X \cap T) \cup (S \cap T) \subseteq Z$ ($S \cap T \subseteq Z$ since otherwise we could place 4 pebbles on t), and $X \cap S \cap T = \emptyset$. Therefore we have that

$$
\begin{aligned}
m &\geq |X \cup S \cup T| \\
&= |X| + |S| + |T| - |X \cap S| - |X \cap T| - |S \cap T| \\
&> 3m/2 - |Z|,
\end{aligned}
$$

which implies that $|Z| > m/2$. If instead there is a vertex $s' \in H_R$ then we know that, either $|Z| > m/2$ because $N(s') \subseteq Z$, or there is some $s \in N(s') \cap (U \cup H) - \{t\}$. In the latter case we move a pebble from s' to s and argue as above. Henceforth we may assume that $C(v) \leq 2$ for all v.

Consequently the equality

$$|Z| + |U| + |H| = n(B) = |C| = |U| + 2|H|$$

tells us that $|H| = |Z|$, and so $|H| \geq 3$. Therefore, again without loss of generality, $|H_L| \geq 2$, say $\{s, t\} \subseteq H_L$. If $S \cap T \subseteq Z$ then we may copy the above argument that $m \geq |X \cup S \cup T|$ implies $|Z| > m/2$. Otherwise we may move a pebble from s through $S \cap T$ to t, find $p \in H - \{s, t\}$ and use the original argument for the case that $C(t) = 3$. This completes the proof of the Claim.   ◇

Now we can use the relations

$$|Z| + |U| + |H| = n(B) = |C| = |U| + \sum_{v \in H} C(v)$$

to see that

$$\frac{m}{2} < |Z| = \sum_{v \in H} C(v) - |H| \leq 6|H| \ ,$$

so that $|H| > m/12$. From this we can assume, without loss of generality, that $|H_L| > m/24$, so that the number of edges with one end in H_L is more than $(m/24)(m/2)$. Since $m \geq 336$ there must be some $x \in R$ having at least 8 neighbors in H_L, so that we can put 8 pebbles on x, a contradiction.

This contradiction proves that C is r-solvable.   □

2.2   Theorem 2

Lower bound. First we give a proof of the lower bound, that $g(n) \geq \lfloor n/2 \rfloor$ for all $n \geq 9$.

We define, for each n, the graph G_n as follows. Let the vertex set $V = L \cup R$, with $|L| = \lceil n/2 \rceil$ and $|R| = \lfloor n/2 \rfloor$. Let the induced subgraphs on L and on R each be complete with one missing edge. Suppose the edge xy is missing from the subgraph on L, and the edge wz is missing from the subgraph on R. Finally include the two edges wy and xz. Note that for $\delta = \lfloor n/2 \rfloor - 1$ the graph G_n has minimum degree δ, and is δ-regular when n is even.

Next we define a configuration C of size $n = n(G_n)$, and show that it is unsolvable when $n \geq 9$. We choose the root $r \in R - \{w, z\}$ and define $C(r, w, x, y, z) = 0$. We find $a, b, c \in L - \{x, y\}$ and define $C(a, b) = 3$ and $C(c) = 2$. Finally we define $C(v) = 1$ for all other vertices v. Clearly, $|C| = n$.

In order that C is r-solvable one must be able to move 2 pebbles onto either w or z, and consequently 4 pebbles onto either x or y. It is not difficult to see that both cases are impossible. Hence for all $n \geq 9$, G_n is not Class 0, and so $b(n) \geq \lfloor n/2 \rfloor$. □

Upper bound. Second we give a proof of the upper bound, that $g(n) \leq \lfloor n/2 \rfloor$ for all $n \geq 6$.

Let G be graph with minimum degree $\lfloor n/2 \rfloor$. We suppose that G is not Class 0 and derive a contradiction. Because complete graphs are Class 0, G has diameter at least 2, and because every pair of vertices of G has a common neighbor, the diameter of G is exactly 2. It is proven in [11] that every graph G of diameter two has pebbling number $n(G)$ or $n(G) + 1$ (*Class 1*). In [4] we find the following characterization of Class 1 graphs of diameter two (see Figure 1).

In the figure, F_0 is any (possibly empty) graph, F_1 is any nonempty graph, and F_2 and F_3 are any nonempty connected graphs. The solid lines indicate an edge from the given vertex to every vertex in the corresponding set. At least two of the three dashed lines must be present, and the arrows indicate that every vertex in F_0 must have at least one edge to $\{a, b\}$.
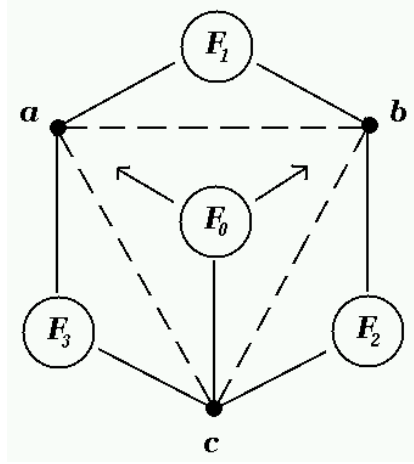
Figure 1: A schematic diagram of Class 1 graphs of diameter two

Suppose that G is labeled as in Figure 1. Choose any vertices $p \in F_3, q \in F_2, r \in F_1$, and recall that $N(x)$ denotes the neighborhood of a vertex x. Since $\{p, q\}$ is not an edge, $|N(p) \cap N(q)| = 1$ and $r \notin N(p) \cup N(q)$, we have $deg(p) + deg(q) \le n - 2$, and so at least one of p or q has degree smaller than $\lfloor n/2 \rfloor$, a contradiction. Hence G is Class 0.   □

In fact it is easy to show by induction that every diameter two Class 1 graph has minimum degree at most $\lfloor n/3 \rfloor$.

2.3   Theorem 3

In this section we prove Theorem 3. The proof is divided into two steps. First, we will show that it is possible to partition the vertices of $G_n = (V, E)$ into $O(\frac{n}{\delta})$ subgraphs of diameter at most two. Second, we prove that if a distribution has enough pebbles then there will be in every subgraph "many" vertices with two pebbles each.

We call a partition $V_1, \ldots, V_l, W$ a q-*star partition* of V if

1. for every $1 \le i \le l$, V_i contains a star on at least q vertices and

2. every vertex of W has a neighbor in V_i for some $1 \le i \le l$.

The following procedure constructs a $(\delta + 1)$-star partition $V_1, \ldots, V_l, W$ of V with $l = O(n/\delta)$. Select $v \in V$ arbitrarily and let $V_1 = N(v) \cup \{v\}$. For a general step, suppose $V_1, \ldots, V_k$ have been selected and let $U = V \setminus \bigcup_{i=1}^{k} V_i$. Either every vertex from U has a neighbor in V_i for some $1 \le i \le k$, in which case we stop the process with $l = k$ and $W = U$, or there is a vertex $u \in U$ such that $N(u) \subseteq U$. In the latter case we set $V_{i+1} = N(u) \cup \{u\}$ and continue the process.

Assume that $V_i = \{v_{i0}, v_{i1}, \ldots, v_{ik_i}\}$ with $k_i \ge \delta$ and let C be a pebbling distribution with $t = \omega n^{3/2}/\delta$ pebbles where $\omega = \omega(n) \to \infty$ is such that $t \le n-1$. Consider

the random variable X_{ij}, which is equal to one if $C(v_{ij}) = 2$ and zero otherwise, and let $X_i = \sum_{j=0}^{k_i} X_{ij}$. Then

$$\mathbf{E}[X_{ij}] = \frac{\binom{t+n-4}{t-2}}{\binom{t+n-1}{t}} \; ;$$

using the assumptions about t and n it is easy to check that $\mathbf{E}[X_{ij}X_{ik}] \le \mathbf{E}[X_{ij}]\mathbf{E}[X_{ik}]$ for $j \ne k$. Consequently

$$var[X_i] = \mathbf{E}[X_i^2] - \mathbf{E}[X_i]^2 \le \mathbf{E}[X_i] \; ,$$

and by Chebyshev's Inequality

$$\Pr\left[X_i < \frac{\mathbf{E}[X_i]}{2}\right] \le \Pr\left[|X_i - \mathbf{E}[X_i]| \ge \frac{\mathbf{E}[X_i]}{2}\right] \le \frac{4}{\mathbf{E}[X_i]} \; .$$

Thus the probability that there exists an $1 \le i \le l$ such that $X_i < \mathbf{E}[X_i]/2$ is $O(n/\delta\mathbf{E}[X_i])$. But, with our choice of t and n,

$$\mathbf{E}[X_i] = \Omega\left(\frac{\delta t^2}{n^2}\right) = \Omega\left(\frac{\omega^2 n}{\delta}\right) \; .$$

Thus, with probability tending to one, for all i, we have $X_i \ge \mathbf{E}[X_i]/2$, which is at least 8 for large enough n. Therefore, with probability tending to one, 4 pebbles can be accumulated on the center of every star, and since every vertex is within distance two of some center, it is possible to move a pebble to any given root vertex.

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