AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **28** (2003), Pages 131–136

Composition of regular coverings of graphs and voltage assignments

Jana Šiagiová

Department of Mathematics
SvF, Slovak University of Technology
Radlinského 11, 813 68 Bratislava, Slovakia
siagiova@vox.svf.stuba.sk

Abstract

Consider a composition of two regular coverings $\pi_1 : \Gamma_0 \to \Gamma_1$ and $\pi_2 : \Gamma_1 \to \Gamma_2$ of graphs, given by voltage assignments α_1, α_2 on Γ_1, Γ_2 in groups G_1 and G_2, respectively. In the case when $\pi_2 \circ \pi_1$ is regular we present an explicit voltage assignment description of the composition in terms of G_1, G_2, α_1, α_2, and walks in Γ_1.

1 Introduction

Lifts of graphs have become a widely adopted means in algebraic and topological graph theory. The corresponding coverings are in many cases regular and hence admit a description in terms of voltage assignments.

It is well known that a composition of two regular coverings need not be regular in general. Various necessary and sufficient conditions for regularity of such a composition can be found in [7]. In a situation when a composition of graph coverings is regular and both coverings are given by voltage assignments, it is natural to ask about a voltage assignment description of the composition of the two coverings. The purpose of this note is to present such a description (Section 3). In conclusion we apply our results to a composition of regular covering of graphs that has arisen in constructions of currently largest vertex-transitive graphs of diameter two and given valence.

For a general treatment of graph coverings we refer to [4, 5]. Our preference for regular coverings and for their description in the language of voltage assignments is motivated by their wide use, ease of understanding, and the potential of being applied by researchers outside algebraic graph theory. The few necessary prerequisites regarding graph coverings and voltage assignments are presented in Section 2.

2 Preliminaries

Let Γ be a graph, possibly with loops or parallel edges. An edge of Γ with a preassigned orientation will be called an *arc*; the set of all arcs of Γ will be denoted by $D(\Gamma)$.

If x is an arc of $D(\Gamma)$ then x^{-1} is its reverse. Given two graphs Γ and Γ', a mapping $f : D(\Gamma) \to D(\Gamma')$ is a *graph homomorphism* from Γ into Γ' if $f(x^{-1}) = (f(x))^{-1}$ for each arc $x \in D(\Gamma)$ and if for any arcs $x, y \in D(\Gamma)$ emanating from the same vertex of Γ the arcs $f(x), f(y) \in D(\Gamma')$ (possibly $f(x) = f(y)$) also emanate from a common vertex of Γ'. A bijective homomorphism $A : \Gamma \to \Gamma$ is a *graph automorphism.* The group of all automorphisms of Γ will be denoted by $Aut(\Gamma)$.

A graph homomorphism $\pi : \Gamma \to \Gamma'$ is a *covering* if for every vertex v of Γ, the arcs emanating from v are mapped bijectively by π onto the arcs emanating from $\pi(v)$. A *covering transformation* of π is an automorphism $A : \Gamma \to \Gamma$ such that $\pi(A(x)) = \pi(x)$ for each arc $x \in D(\Gamma)$. The covering π is *regular* if the group of its covering transformations acts regularly on each fibre $\pi^{-1}(v)$ for any vertex v of Γ'.

A *walk* $W = x_1x_2 \dots x_m$ of length m in Γ is a sequence of arcs x_i of Γ, such that the arc x_{i-1} terminates in the initial vertex of x_i, $2 \le i \le m$. The walk W is *closed* if x_1 emanates from the terminal vertex of x_m.

Let Γ be a connected graph and let G be a group. We say that a mapping $\alpha : D(\Gamma) \to G$ is a *voltage assignment* on Γ if $\alpha(x^{-1}) = (\alpha(x))^{-1}$ for each arc $x \in D(\Gamma)$. To specify a voltage assignment we usually fix in advance an orientation of the undirected graph Γ and assign voltages to arcs obtained this way; the reverse arcs will automatically receive the corresponding inverse voltages. The voltage assignment α can be extended to walks in the obvious way: If $W = x_1x_2 \dots x_m$ is a walk in Γ then $\alpha(W) = \alpha(x_1)\alpha(x_2)\dots\alpha(x_m)$.

With the help of a voltage assignment α on Γ in a nontrivial group G we can construct a new larger graph Γ^α, called a *lift* of Γ. The vertex and arc sets of the lift are $V(\Gamma^\alpha) = V(\Gamma) \times G$ and $D(\Gamma^\alpha) = D(\Gamma) \times G$, respectively. An arc x_g in Γ^α emanates from a vertex u_g and terminates at a vertex v_h if and only if x is an arc in Γ from u to v and $h = g\alpha(x)$. The reverse of the arc x_g is the arc $(x^{-1})_{g\alpha(x)}$. Such a pair of mutually reverse arcs forms an edge in the undirected graph Γ^α.

A voltage assignment characterization of regular coverings can be given in the following form [2].

Lemma 1 *Let $\pi : \Gamma_1 \to \Gamma_2$ be a covering of connected graphs. Then π is regular if and only if there exists a voltage assignment α on Γ_2 such that π is isomorphic to the covering $\Gamma_2^\alpha \to \Gamma_2$.* □

Using walks and their voltages it is possible to describe lifts of automorphisms. We say that an automorphism $A \in Aut(\Gamma)$ *lifts* to an automorphism $A' \in Aut(\Gamma^\alpha)$ if $\pi \circ A' = A \circ \pi$. Further, a voltage assignment α is said to be *A-compatible* if $\alpha W = 1_G \iff \alpha AW = 1_G$ for each closed walk W in Γ. The characterization of lifts we give here appeared first (in this form) in [3]; for a restricted version see e.g. [1], and for later development and generalizations see [4, 5].

Lemma 2 *Let α be a voltage assignment on a connected graph Γ in a group G and let A be an automorphism of Γ. Then A lifts to an automorphism of Γ^α if and only if α is A-compatible.* □

3 Main result

To present our main result we need to introduce more terminology and notation. Let Γ be a connected graph, and let α be a voltage assignment on Γ in a group H such that the lift Γ^α is connected; let $\pi_\alpha : \Gamma^\alpha \to \Gamma$ be the corresponding (regular) covering. Let β be a voltage assignment on Γ^α in a group K such that the lift $(\Gamma^\alpha)^\beta$ is connected, with the associated covering $\pi_\beta : (\Gamma^\alpha)^\beta \to \Gamma^\alpha$. We know that in regular coverings, the voltage group is isomorphic to the covering transformation group. Keeping the notation from the previous section, all covering transformations of π_α have the form $I_h : x_a \mapsto x_{ha}$, where $h \in H$ and x_a is an arbitrary arc of Γ^α; similarly, all covering transformations of the covering π_β are given by $J_k : (x_a)_b \mapsto (x_a)_{kb}$ where $k \in K$. For a walk W in Γ^α we will often use the abbreviated notation hW instead of $I_h(W)$.

By Lemma 1, the composition $\pi_\alpha \circ \pi_\beta : (\Gamma^\alpha)^\beta \to \Gamma$ is a regular covering if and only if there exists a voltage assignment γ on the graph $(\Gamma^\alpha)^\beta$ in some group G such that the coverings $\Gamma^\gamma \to \Gamma$ and $(\Gamma^\alpha)^\beta \to \Gamma$ are isomorphic (which means that there exists an isomorphism $f : \Gamma^\gamma \to (\Gamma^\alpha)^\beta$ such that $\pi_\gamma = \pi_\alpha \circ \pi_\beta \circ f$).

We shall be more specific now and describe the structure of the group G; moreover, we give an explicit description of a corresponding voltage assignment γ. (It is well known that such an assignment need not be unique; see [2] for basic information about equivalence of voltage assignments.)

Theorem 1 *Let α be a voltage assignment on a connected graph Γ in a group H and let β be a voltage assignment on the lift Γ^α in a group K such that the lift $(\Gamma^\alpha)^\beta$ is connected. Let π_α and π_β be the corresponding regular covering projections, and assume that their composition $\pi_\alpha \circ \pi_\beta$ is regular. Then there is a group G, a voltage assignment γ on the graph Γ in G, and an isomorphism $f : \Gamma^\gamma \to (\Gamma^\alpha)^\beta$ such that $\pi_\gamma = \pi_\alpha \circ \pi_\beta \circ f$. The group G is a product of H and K. Fixing an arbitrary vertex t of Γ, multiplication in G is given by*

$$(h, k) * (h', k') = (hh', k\beta(hV)), \tag{1}$$

where $h, h' \in H$, $k, k' \in K$, and V is a walk in Γ^α from the vertex t_{1_H} to the vertex $t_{h'}$ such that $\beta(V) = k'$.

The voltage assignment γ on arcs of Γ can be described as follows. Let W be a walk in Γ^α from t_{1_H} to w_{1_H} such that $\beta(W) = 1_k$. Then,

$$\gamma(x) = (\alpha(x), \beta(x_{1_H})\beta(\alpha(x)W)^{-1}). \tag{2}$$

With this choice of γ, the isomorphism f is given by

$$f(x_{(h,k)}) = (I_h)_k(x_{1_H})_{1_K}.$$

Proof. Assume that $\pi_\alpha \circ \pi_\beta$ is regular. Using the above notation this means that all covering transformations I_h have lifts (proved in [7]). According to Lemma 2, each I_h has a lift if and only if the voltage assignment β is I_h-compatible for each $h \in H$. In other words for each closed walk U in the graph Γ^α, based at a fixed vertex (e.g. t_{1_H}), we have $\beta(U) = 1_K$ if and only if $\beta(I_h U) = 1_K$ for any $h \in H$ (see [3]). Moreover, in such a case I_h lifts to $|K|$ covering transformations $(I_h)_k$. By [9], for each $k \in K$ the action of $(I_h)_k$ on arcs $(x_a)_b$ (and similarly on vertices $(u_a)_b$) of the lift $(\Gamma^\alpha)^\beta$ is given as follows. Let W be a walk in Γ^α from our fixed vertex t_{1_H} to the initial vertex v_a of the arc x_a such that $\beta(W) = b$. (The existence of such a walk for each $b \in K$ is equivalent to our connectivity assumption for the lift $(\Gamma^\alpha)^\beta$.) Then we have

$$(I_h)_k((x_a)_b) = (I_h)_k((x_a)_{\beta(W)}) = (x_{ha})_{k\beta(hW)} \ , \tag{3}$$

where hW stands for $I_h(W)$.

A formula for composition of two such covering transformations $(I_h)_k$ and $(I_{h'})_{k'}$ may also be derived from [3] or from a more detailed account in [9]. Let V be a walk from t_{1_H} to $t_{h'}$ in Γ^α such that $\beta(V) = k'$. Then,

$$(I_h)_k \circ (I_{h'})_{k'} = (I_{hh'})_{k\beta(hV)} \ . \tag{4}$$

The fact that the definitions (3) and (4) do not depend on a particular choice of walks W and V is a consequence of compatibility of the voltage assignment β with the covering transformations of π_α. This way the information contained in [9] helped us to obtain an explicit description of the group $Cov(\pi_\alpha \circ \pi_\beta)$ of all transformations of the covering $\pi_\alpha \circ \pi_\beta$.

As the sought voltage group G is isomorphic to $Cov(\pi_\alpha \circ \pi_\beta)$, we define the elements of G to be ordered pairs (h, k) where $h \in H$ and $k \in K$, with the binary operation $*$ that, by (4), has the form

$$(h, k) * (h', k') = (hh', k\beta(hV)) \tag{5}$$

where V is any walk from t_{1_H} to $t_{h'}$ in Γ^α such that $\beta V = k'$. The mapping $\varphi : \ G \to Cov(\pi_\alpha \circ \pi_\beta)$ given by $\varphi(h, k) = (I_h)_k$ is then a group isomorphism.

Finally, we need to find a voltage assignment γ on the base graph Γ in the group G and an isomorphism f of the coverings $\pi_\gamma : \ \Gamma^\gamma \to \Gamma$ and $\pi_\alpha \circ \pi_\beta : \ (\Gamma^\alpha)^\beta \to \Gamma$. Arcs of the lift Γ^γ have the form $x_{(h,k)}$ where $x \in \Gamma$ and $(h, k) \in G$. Since fibres in the lift are determined by the action of the covering transformation group, we define $f : \ \Gamma^\gamma \to (\Gamma^\alpha)^\beta$ as follows:

$$f(x_{(h,k)}) = \varphi(h, k)((x_{1_H})_{1_K}) = (I_h)_k(x_{1_H})_{1_K}, \tag{6}$$

with an analogous formula for the f-images of vertices $u_{h,k}$:

$$f(u_{(h,k)}) = \varphi(h, k)((u_{1_H})_{1_K}) = (I_h)_k(u_{1_H})_{1_K}.$$

Note that with such a definition we have

$$f(x_{(h,k)*(a,b)}) = \varphi((h, k) * (a, b))(x_{1_H})_{1_K} = (I_h)_k \circ (I_a)_b(x_{1_H})_{1_K}. \tag{7}$$

Next we check whether f maps arcs onto arcs (the incidence at vertices is automatic). Let x be an arbitrary arc of the base graph Γ from a vertex v to a vertex w. Take an arbitrary element $(h, k) \in G$ and let $x_{(h,k)}$ be the arc of Γ^γ that emanates from the vertex $v_{(h,k)}$. The terminal vertex of $x_{(h,k)}$ depends on the voltage assignment γ; if $\gamma(x) = (a,b) \in G$ then $x_{(h,k)}$ terminates at $w_{(h,k)*(a,b)}$. The mapping f will send the arc $x_{(h,k)}$ of the graph Γ^γ (from $v_{(h,k)}$ to $w_{(h,k)*(a,b)}$) onto the arc $(I_h)_k(x_{1_H})_{1_K}$ of $(\Gamma^\alpha)^\beta$, starting at the vertex $(I_h)_k(v_{1_H})_{1_K}$ and ending at the vertex $(I_h)_k \circ (I_a)_b(w_{1_H})_{1_K}$. But applying the inverse covering transformation to $(I_h)_k$ we see that, in the graph $(\Gamma^\alpha)^\beta$, the arc $(I_h)_k(x_{1_H})_{1_K}$ starts at $(I_h)_k(v_{1_H})_{1_K}$ and ends at $(I_h)_k \circ (I_a)_b(w_{1_H})_{1_K}$ if and only if the arc $(x_{1_H})_{1_K}$ emanates from $(v_{1_H})_{1_K}$ and terminates at $(I_a)_b(w_{1_H})_{1_K}$. On the other hand, we know that the terminal vertex of $(x_{1_H})_{1_K}$ is $(w_{\alpha(x)})_{\beta(x_{1_H})}$. Therefore we must have

$$(I_a)_b(w_{1_H})_{1_K} = (w_{\alpha(x)})_{\beta(x_{1_H})}. \tag{8}$$

Let W be a walk in the intermediate graph Γ^α from our fixed vertex t_{1_H} to the vertex w_{1_H} (recall that we are working with an arc x from v to w) such that $\beta(W) = 1_K$. Then by (3) we have $(I_a)_b(w_{1_H})_{1_K} = (w_a)_{b\beta(aW)}$. Combining this with (8) yields $a = \alpha(x)$ and $b = \beta(x_{1_H})\beta(\alpha(x)W)^{-1}$. It means that a suitable voltage assignment γ on the $v \to w$ arc x of the base graph Γ is given by

$$\gamma(x) = (\alpha(x), \beta(x_{1_H})\beta(\alpha(x)W)^{-1}), \tag{9}$$

where W is a $t_{1_H} \to w_{1_H}$ walk in Γ^α with $\beta(W) = 1_K$. (Again, by the compatibility assumption this definition is independent of the choice of W.)

Working backwards, one may check that introducing a voltage assignment γ on Γ in the group G as in (9), the function f defined by (6) is indeed an isomorphism of the coverings π_γ and $\pi_\alpha \circ \pi_\beta$, as desired. □

4 An application

For a prime power q of the form $q = 4k + 1$ let Γ be the graph consisting of two vertices v and w joined by q parallel edges, and $(q-1)/4$ loops attached to each vertex. Let F^+ be the additive group of $F = GF(q)$. To define a voltage assignment on Γ we pick the orientation of all the parallel edges $x_i, i \in F$, from v to w. The orientation of the loops is arbitrary. The loops at the vertex v will be denoted by y_j, and at the vertex w by z_j, where $j \in \{0, 1, \dots (q-5)/4\}$. We define a voltage assignment α on Γ in the group F^+ by $\alpha(x_i) = i, \alpha(y_j) = 0, \alpha(z_j) = 0$. It is easy to see that the lift Γ^α is connected, isomorphic to a complete bipartite graph $K_{q,q}$ with $(q-1)/4$ loops at each vertex.

Let us describe a voltage assignment β on the graph Γ^α in the group F^+. Each edge of Γ^α, denoted by $(x_i)_k$, $i, k \in F$, will be oriented from the vertex v_k to the vertex w_l, where $l = k + \alpha(x_i)$. Again the orientation on the loops is chosen arbitrarily. The $(q-1)/4$ directed loops at the vertex v_k and $(q-1)/4$ loops at the vertex w_k will be denoted by $(y_j)_k$, and $(z_j)_k$, respectively. We fix a primitive element ξ of F and define a voltage assignment β by $\beta((x_i)_k) = kl$, $\beta((y_j)_k) = \xi^{2j}$, and $\beta((z_j)_k) = \xi^{2j+1}$.

It can be checked that the composition of the two coverings $\Gamma^\alpha \to \Gamma$ and $(\Gamma^\alpha)^\beta \to \Gamma^\alpha$ is regular. With the help of Theorem 1 one can work out the corresponding group G and a voltage assignment γ. Keeping the same orientation of edges of Γ as before and fixing the vertex v_0 of Γ^α we successively obtain $\gamma(y_j) = (0, \xi^{2j})$, $\gamma(z_j) = (0, \xi^{2j+1})$, and $\gamma(x_i) = (i, -i^2)$; the new voltage group G turns out to be the direct product $F^+ \times F^+$.

As was proved in [6], the lift $(\Gamma^\alpha)^\beta$ belongs to the family of the so-called McKay-Miller-Širáň graphs, the currently largest known vertex-transitive graphs of diameter two and valence $d = (3q-1)/2$, whose order is $\frac{8}{9}(d+\frac{1}{2})^2$. We thus have a new proof of an earlier fact [8] that the McKay-Miller-Širáň graphs are lifts of two-vertex graphs endowed with relatively simple voltage assignments.

Acknowledgement

This research was supported by the VEGA grant No. 1/9176/02 and by the Literary Fund of Slovakia.

References

[1] N. Biggs, *Algebraic graph theory* (2nd ed.), Cambridge University Press, Cambridge, 1993.

[2] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley, New York, 1987.

[3] P. Gvozdjak and J. Širáň, Regular maps from voltage assignments, in: *Graph Structure Theory* (Contemporary Mathematics, AMS Series) **147** (1993), 441–454.

[4] A. Malnić, Group actions, coverings and lifts of automorphisms, *Discrete Math.* 182 (1998), 203–218.

[5] A. Malnić, R. Nedela and M. Škoviera, Lifting graph automorphisms by voltage assignments, *European J. Combin.* 21 (2000), 927–947.

[6] B. D. McKay, M. Miller and J. Širáň, A note on large graphs of diameter two and given maximum degree, *J. Combin. Theory Ser. B* **74** (1998), 110–118.

[7] J. Šiagiová, Composition of regular coverings of graphs, *J. Electrical Engineering* **50** (1999), 75–77.

[8] J. Šiagiová, A note on the McKay-Miller-Širáň graphs, *J. Combin. Theory Ser. B* **81** (2001), 205–208.

[9] J. Širáň, The “walk calculus” of regular lifts of graph and map automorphisms, *Yokohama Math. Journal* **47** (1999), 113–128.

(Received 19 Apr 2002; revised 21 June 2002)