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Exponents of primitive graphs*

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Abstract

Suppose G is a primitive graph with $n > 1$ vertices. Let $L(u)$ be the length of a shortest closed walk of odd length containing vertex $u \in V(G)$, and let $M = \max\{L(u) \mid u \in V(G)\}$. We prove that the exponent of G is equal to $M - 1$ if $M \geq n - g + 1$ and less than or equal to $n - g$ if $M \leq n - g + 1$, where g is the length of a shortest cycle in G of odd length. We then determine the exponent set of primitive graphs with given n and g.

Let $G = (V, E)$ be a digraph with vertex set V and arc set E. All of our digraphs are finite, and loops are permitted but no multiple arcs. A digraph G is symmetric if for all $u, v \in V(G)$, (u, v) is an arc if and only if (v, u) is.

A walk, W, from u to v is a sequence of not necessarily distinct vertices $u, u_1, \ldots,$ $u_p = v$ and a sequence of arcs (u, u_1), $(u_1, u_2), \ldots, (u_{p-1}, v)$. A closed walk is a walk where $u = v$. A path is a walk with distinct vertices. A cycle is a closed walk where all the vertices except the first and last are distinct. The length of a walk W, denoted $|W|$, is the number of arcs in it. The walk $W = A + B$ is obtained by identifying the final vertex of A with the initial vertex of B. If u and v are vertices on a walk W, then $W(u, v)$ denotes the portion of W from u to v.

If G is symmetric and $W(u, v)$ is a portion of a walk W in G, then $W'(v, u)$ denotes the walk from v to u whose vertex sequence consists of the vertices of $W(u, v)$ listed in reverse order. If u and v are the initial and terminal vertices of W, then we write $W' = W'(v, u)$.

A digraph G is said to be primitive if there exists a positive integer k such that there is a walk of length k from u to v for all $u, v \in V(G)$. The minimum such k is called the exponent of G, denoted $\exp(G)$. Define $\exp(G; u, v)$ to be the minimum integer k such that there is a walk of length m from u to v for all $m \geq k$. Clearly, $\exp(G) = \max_{u,v \in V(G)} \exp(G; u, v)$. Primitive digraphs and the corresponding exponents have been extensively studied.

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We will be concerned with symmetric digraphs. Note that a symmetric digraph G can naturally correspond to an (undirected) graph $\tilde{G}$ by replacing each pair of arcs (u, v) and (v, u) by an edge (u, v) and that G is primitive if and only if $\tilde{G}$ is connected and G contains at least one odd cycle, where an odd cycle is a cycle of odd length. The odd girth of a primitive digraph G is the length of a shortest odd cycle in G. In this paper, we refer to symmetric digraphs simply as graphs. Shao [5] and Liu et al. [3] determined respectively the exponent set of primitive graphs and the exponent set of primitive graphs with no loops. These are proved more expediently by Neufeld [4] recently. We consider the concept of odd girth in conjunction with primitive graphs. Using the results and techniques of Neufeld [4], we first propose an upper bound for the exponents of primitive graphs with given order and odd girth. Then we prove a general property (Theorem 2) for estimating the exponents, which is the main result of this paper and is finally used to determine the exponent set of this class of primitive graphs.

The following lemmas are due to Neufeld [4].

Lemma 1 *Let G be a primitive graph and let $u, v \in V(G)$. If there are walks P and Q of opposite parity from u to v, then*

$$\exp(G; u, v) \leq \max\{|P|, |Q|\} - 1.$$

The proof of the following lemma can be found in [4, pp. 135–136]. To be more self-contained in this paper, a proof is reproduced here.

Lemma 2 *Let G be a primitive graph. Let $u \in V(G)$ and let W be a shortest closed walk of odd length containing u. Then no vertex of G can occur more than twice in W.*

Proof. Suppose a vertex $v \in V(G)$ occurs three times in W. Let v_1, v_2, and v_3 be the first, second, and third occurrences of v in W. Then $W = W_1 + W_2 + W_3 + W_4$ where $W_1 = W(u, v_1)$, $W_2 = W(v_1, v_2)$, $W_3 = W(v_2, v_3)$ and $W_4 = W(v_3, u)$. The closed walk $W_1 + W_4$ containing u is shorter than W and hence must be of even length. Then exactly one of W_2 or W_3, say, W_2 is of odd length. But now the closed walk $W_1 + W_2 + W_4$ containing u is of odd length and shorter than W, a contradiction. This proves this lemma. □

Lemma 3 *Let G be a primitive graph with odd girth g and let $u \in V(G)$. Let W be a shortest closed walk of odd length containing u and let $l(W)$ be the number of distinct vertices in W. Then $l(W) \geq (|W| + g)/2$. Also, if equality holds, then W contains a cycle of length g and is unique (up to isomorphism).*

Proof. Observe that W must contain an odd cycle, say, C. Let v be the first vertex of W which is also a vertex of C. Suppose $x \neq v$ is on C and occurs twice in W. Let $W = W_1 + C + W_2 + W_3$ where $W_1 = W(u, v)$, $W_2 = W(v, x)$, $W_3 = W(x, u)$. Then one of the closed walks $W_1 + C(v, x) + W_3$ and $W_1 + C'(v, x) + W_3$ containing u

is of odd length and shorter than W, which is a contradiction. Hence every vertex of C except v occurs in W exactly once. By Lemma 2, no vertex of G can occur more than twice in W. Hence $l(W) \geq (|W| - |C|)/2 + |C| = (|W| + |C|)/2 \geq (|W| + g)/2$.

If $l(W) = (|W| + g)/2$, then $|C| = g$, and every vertex of $(W \backslash C) \cup \{v\}$ occurs exactly twice in W. Let w_a and w_b be the first and second occurrences of a vertex $w \in (W \backslash C) \cup \{v\}$ in W. Since $W(u, w_a) + W(w_b, u)$ is a closed walk and shorter than W, $W(w_a, w_b)$ is of odd length.

Let (u, x) and (y, u) be arcs of W. Suppose $x \neq y$. Note that both $|W(x_a, x_b)|$ and $|W(y_a, y_b)|$ are odd and hence $W'(y_b, y_a)$ is odd. Since $(u, x_a) + W(x_b, y_b) + (y_b, u)$ is a closed walk and shorter than W, $|W(x_b, y_b)|$ is odd. Now $(u, x_a) + W(x_b, y_b) + W'(y_b, y_a) + (y_b, u)$ is a closed walk and shorter than W, a contradiction. So we have $x = y$.

Observe that x is contained in a closed walk $W_1 = W(x_a, x_b)$ of odd length $|W_1| = |W| - 2$, $l(W_1) = (|W_1| + g)/2$ and $W_1 \subseteq W$. By repeating this observation we obtain a sequence of walks $W_1, W_2, \ldots, W_k$ where $l(W_i) = (|W_i| + g)/2$ $(1 \leq i \leq k)$, $W_1 \supseteq W_2 \supseteq \ldots \supseteq W_k$ and $|W_1| = |W| - 2$, $|W_2| = |W_1| - 2, \ldots, |W_k| = |W_{k-1}| - 2 = g$. Thus $W_k = W(v, v) = C$, and the decomposition of W shows that it is unique (up to isomorphism). □

We first establish the following.

Theorem 1 [7] *Let G be a primitive graph on $n > 1$ vertices with odd girth g. Then* $\exp(G) \leq 2n - g - 1$. *Moreover, if $G_{n,g} = (V, E)$ where $V = \{1, \ldots, n\}$, $E = \{(i, i+1) : 1 \leq i \leq n-1\} \cup \{(n, n-g+1)\}$, then $G_{n,g}$ is the unique (up to isomorphism) graph with exponent* $2n - g - 1$.

Proof. Let d be the diameter of G. Then [4] $\exp(G) \leq 2d$, and equality hold if and only if there is a vertex u such that a shortest closed walk of odd length containing u has length $2d + 1$.

Note that $d \leq n - (g+1)/2$. We have $\exp(G) \leq 2n - g - 1$. If $\exp(G) = 2n - g - 1$, then $d = n - (g+1)/2$ and there is a vertex u in G such that a shortest closed walk W of odd length containing u is of length $2(n - (g+1)/2) + 1 = 2n - g$. By Lemma 3, the number of distinct vertices in W is at least $(|W| + g)/2 = n$. Thus, also by Lemma 3, the extremal graph $G_{n,g}$ is the unique (up to isomorphism) graph on n vertices and with odd girth g and exponent $2n - g - 1$. □

Theorem 1 gives the maximum value of exponents of primitive graphs with given order and odd girth and the corresponding extremal graphs. Now we prove the main result.

Theorem 2 *Let G be a primitive graph on $n > 1$ vertices with odd girth g. Let $u \in V(G)$ and let W_u be a shortest closed walk of odd length containing u. Let $M = \max_{u \in V(G)} |W_u|$. If $M \geq n - g + 1$, then* $\exp(G) = M - 1$ *and if $M \leq n - g + 1$, then* $\exp(G) \leq n - g$.

Proof. Let $u, v \in V(G)$. We wish to show that $\exp(G; u, v) \le \max\{M, n-g+1\} - 1$.

Case 1: W_u and W_v intersect. Let x be a vertex in both W_u and W_v. Assume without loss of generality $|W_u(u,x)| < |W_u(x,u)|$ and $|W_v(x,v)| < |W_v(v,x)|$. Let $U = W_u(u,x) + W_v(x,v)$. Then $|U| \le \max\{|W_u|, |W_v|\}$. If $|W_u(u,x)| < |W_v(x,v)|$, then the walk $W_u(u,x) + W'(x,v)$ is u to v, have parity opposite U, and is of length less than $|W_v|$. If $|W_u(u,x)| \ge |W_v(x,v)|$, then the walk $W'(u,x) + W_v(x,v)$ is u to v, have parity opposite U, and is of length less than $|W_u|$. By Lemma 1, we have $\exp(G; u, v) \le \max\{|W_u|, |W_v|\} - 1 \le M - 1$.

Case 2: W_u and W_v do not intersect. Then G has at least two vertex disjoint odd cycles C_1 and C_2 and they are contained respectively in W_u and W_v. Let P be a shortest path from u to v. Note that at least $(|C_1|-1)/2 + (|C_2|-1)/2 \ge g-1$ vertices on C_1 and C_2 do not lie on P. Then P has at most $n-g+1$ vertices, and hence $|P| \le n-g$.

Let x be the final vertex of P which is also a vertex of W_u and let y be the first vertex of P after x which is also a vertex of W_v. Suppose without loss of generality that $|W_u(u,x)| < |W_u(x,u)|$ and $|W_v(y,v)| < |W_v(v,y)|$. Clearly $|P(u,x)| \le |W_u(u,x)|$. In fact, $|P(u,x)| = |W_u(u,x)|$, for otherwise, one of the closed walks $W_u(u,x) + P'(x,u)$ or $W'_u(u,x) + P'(x,u)$ containing u would be shorter than W_u and of odd length since $W_u(u,x)$ and $W'_u(u,x)$ have different parity. Similarly, $|P(y,v)| = |W_v(y,v)|$. Let $L_1 = W_u(u,x) + P(x,y) + W_v(y,v)$. Then $|L_1| = |P| \le n-g$.

Let $L_2 = W'_u(u,x) + P(x,y) + W_v(y,v)$ and $L_3 = W_u(u,x) + P(x,y) + W'_v(y,v)$. Suppose $\min\{|L_2|, |L_3|\} > \max\{M, n-g+1\}$, i.e., $\min\{|W'_u(u,x)| + |P(x,y)| + |W_v(y,v)|, |W_u(u,x)| + |P(x,y)| + |W'_v(y,v)|\} > \max\{M, n-g+1\}$. Then $|W_u| + 2|P(x,y)| + |W_v| > 2\max\{M, n-g+1\}$. By Lemma 3, the number of distinct vertices in W_u, P and W_v total at least $(|W_u|+g)/2 + |P(x,y)| - 1 + (|W_v|+g)/2 = (|W_u| + 2|P(x,y)| + |W_v|)/2 + g - 1 > \max\{M, n-g+1\} + g - 1 \ge n$, which is a contradiction. Hence we have $\min\{|L_2|, |L_3|\} \le \max\{M, n-g+1\}$. Suppose without loss of generality that $|L_2| \le \max\{M, n-g+1\}$.

Note that the walks L_1 and L_2 are both from u to v and have opposite parity. By Lemma 1, $\exp(G; u, v) \le \max\{|L_1|, |L_2|\} - 1 \le \max\{M, n-g+1\} - 1$.

Since u and v are artitrary vertices of G, we have from Cases 1 and 2 that $\exp(G) \le \max\{M, n-g+1\} - 1$. Hence, if $M \le n-g+1$, then $\exp(G) \le n-g$. Observe that u is contained in no closed walk of length $|W_u| - 2$ which implies $\exp(G) \ge \max_{u \in V(G)} \exp(G; u, u) \ge \max_{u \in V(G)} |W_u| - 1 = M - 1$. Hence, if $M \ge n-g+1$, then $\exp(G) = M - 1$. □

Remark Let G be a primitive digraph on $n > 1$ vertices with odd girth g. With notation as in Theorem 2, it is easy to see that $M \le 2n-g$ and hence $\exp(A) \le 2n-g-1$; equality here implies $M = 2n-g$ and hence G is isomorphic to $G_{n,g}$. So Theorem 1 follows from Theorem 2.

Finally we apply Theorem 2 to determine the exponent set for primitive graphs with given order and odd girth.

Theorem 3 [6, 7] *The exponent set of primitive graphs on $n > 1$ vertices with odd* *girth g, $E(n, g)$, is the set* $\{g-1, \ldots, 2n-g-1\}\backslash S$ *where S is the set of odd integers in* $\{n-g+1, \ldots, 2n-g-2\}$ *and* 0.

Proof. The case $g = 1$ has been proved in [4]. In the following we suppose $g \geq 3$.

By Theorem 2, $E(n, g) \subseteq \{g-1, \ldots, 2n-g-1\}\backslash S$ since for a primitive graph G with odd girth g on n vertices, $\exp(G) \leq 2n-g-1$, and if $\exp(G) \geq n-g+1$, then $\exp(G)$ is even. We need to show that the reverse inclusion holds. Note that the largest odd integer less than or equal to $n-g$ is $2\lfloor (n-g-1)/2 \rfloor + 1$.

For any even $k \in \{g-1, g+1, \ldots, 2n-g-1\}$, let $r = (k+g+1)/2$ and consider the graph $G = (V, E)$ where $V = V(G_{r,g}) \cup \{r+1, \ldots, n\}$ and $E = E(G_{r,g}) \cup \{(r-1, i), (i, r-g+1) : r+1 \leq i \leq n\}$. Clearly $\exp(G) = \exp(G_{r,g}) = 2r-g-1 = k$. Hence $\{g-1, g+1, \ldots, 2n-g-1\} \subseteq E(n, g)$.

For any odd $k \in \{g, g+2, \ldots, 2\lfloor (n-g-1)/2 \rfloor + 1\}$, consider the graph $G = (V, E)$ where $V = \{1, \ldots, n\}$ and $E = \{(i, i+1) : 1 \leq i \leq k+g-2\} \cup \{(k+g-1, k)\} \cup \{(g-1, i), (i, 1) : k+g \leq i \leq n\}$. Let H be obtained by deleting vertices $k+g+1, \ldots, n$ from G. Clearly $\exp(G) = \exp(H)$. Note that the diameter of H is k. So $\exp(H) \geq k$. Let $u, v \in V(H)$ and let P be a shortest path from u to v in H. If either u or v is on a cycle of length g, then there is a walk from u to v of length at most $k+1$ and parity opposite P, and so $\exp(H; u, v) \leq k$ by Lemma 1.

Suppose $u, v \in \{g, g+1, \ldots, k-1\}$. Further, suppose without loss of generality that $u \leq v$ and $k-v \leq u-(g-1)$. Then the walk $u, u+1, \ldots, k, k+1, \ldots, k+g-1, k, k-1, \ldots, v$ from u to v has length $2k+g-(u+v) \leq k+1$ and parity opposite P. By Lemma 1, $\exp(H; u, v) \leq k$. Thus for all u, v we have $\exp(H; u, v) \leq k$ and so $\exp(G) = \exp(H) = k$. It follows that $\{g, g+2, \ldots, 2\lfloor (n-g-1)/2 \rfloor + 1\} \subseteq E(n, g)$.

Hence we have $\{g-1, \ldots, 2n-g-1\}\backslash S \subseteq E(n, g)$. The proof is now completed. □

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