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On connected resolvability of graphs

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Abstract

For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices and a vertex v in a connected graph G, the representation of v with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$, where $d(x, y)$ represents the distance between the vertices x and y. The set W is a connected resolving set for G if distinct vertices of G have distinct representations with respect to W and the subgraph $\langle W \rangle$ induced by W is a nontrivial connected subgraph of G. The minimum cardinality of a connected resolving set in a graph G is its connected resolving number $cr(G)$. A connected resolving set in G of cardinality $cr(G)$ is called a cr-set of G. An upper bound for the connected resolving number of a connected graph that is not a path is presented. We study how the connected resolving number of a connected graph is affected by adding a vertex to the graph. It is shown that for every integer $k \geq 2$, there exists a connected graph with a unique cr-set. Moreover, for every pair k, r of integers with $k \geq 2$ and $0 \leq r \leq k$, there exists a connected graph G with connected resolving number k such that there are exactly r vertices in G that belong to every cr-set of G.

1 Introduction

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. For an ordered set $W = \{w_1, w_2, \cdots, w_k\} \subseteq V(G)$ and a vertex v of G, we refer to the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$$

as the (*metric*) *representation of* v *with respect to* W. The set W is called a *resolving set* for G if distinct vertices have distinct representations with respect to W. A

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resolving set for G containing a minimum number of vertices is a *minimum resolving set* or a *basis* for G. The (*metric*) *dimension* $\dim(G)$ is the number of vertices in a basis for G. For a nontrivial connected graph G, its vertex set $V(G)$ is always a resolving set. Moreover, $\langle V(G)\rangle = G$ is a nontrivial connected graph. In [11] a resolving set W of G is defined to be *connected* if the subgraph $\langle W\rangle$ induced by W is a nontrivial connected subgraph of G. The minimum cardinality of a connected resolving set W in a graph G is the *connected resolving number* $cr(G)$. A connected resolving set of cardinality $cr(G)$ is called a *cr-set* of G. To illustrate this concept, consider the graph G of Figure 1.

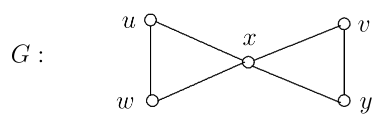

Figure 1: A graph G with $\dim(G) = 2$ and $cr(G) = 3$

The set $W = \{u, v\}$ is a basis for G and so $\dim(G) = 2$. Since $\langle\{u, v\}\rangle$ is disconnected, W is not a connected resolving set. On the other hand, the set $W' = \{u, v, x\}$ is a connected resolving set. Since G contains no 2-element connected resolving set, it follows that $cr(G) = 3$.

The concepts of resolving set and minimum resolving set have previously appeared in the literature. In [9] and later in [10], Slater introduced these ideas and used *locating set* for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph G as its *location number*. Slater described the usefulness of these ideas when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations. Harary and Melter [6] discovered these concepts independently as well but used the term *metric dimension* rather than location number, the terminology that we have adopted. These concepts were rediscovered by Johnson [7] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. The structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [1, 2, 4, 8]. It was noted in [5, p.204] that determining the dimension of a graph is an NP-complete problem.

In many instances, the vertices in a minimum resolving set in a graph are located at significant distances from one another. For graphs representing networks, a resolving set represents a set of detecting devices in a network so that for every station in the network, there are two detecting devices whose distances from the station are distinct. Since it is important that the devices be properly maintained and have easy access to one another, it is convenient if these devices are located in close proximity to one another. For this reason, we are led to investigate resolving sets W whose induced subgraph $\langle W \rangle$ is connected and to the connected resolving number, first introduced and studied in [11]. We refer to [3] for graph theory notation and terminology not described here.

Certainly, every connected resolving set is a resolving set. Thus it was noted in [11] that if G is a connected graph of order $n \geq 3$, then

$$1 \leq \dim(G) \leq cr(G) \leq n - 1.$$

Also, $\dim(G) = cr(G)$ if and only if G contains a connected basis. Two vertices u and v of a connected graph G is defined in [11] to be *distance similar* if $d(u,x) = d(v,x)$ for all $x \in V(G) - \{u, v\}$. Certainly, distance similarity in a graph G is an equivalence relation in $V(G)$. The following observation [11] is useful.

Observation 1.1 *Let G be a connected graph and let $V_1, V_2, \cdots, V_k$ be the k ($k \geq 1$) distinct distance-similar equivalence classes of $V(G)$. If W be a resolving set of G, then W contains at least $|V_i| - 1$ vertices from each equivalence class V_i for all i with $1 \leq i \leq k$ and so $cr(G) \geq \dim(G) \geq n - k$.*

2 An upper bound for the connected resolving number of a graph

Observe that if W is a resolving set of a connected graph G and $W \subseteq W'$, then W' is also a resolving set of G. Therefore, if W is a basis of G such that $\langle W \rangle$ is disconnected, then surely there is a smallest superset W' of W for which $\langle W' \rangle$ is connected. In fact, if H is a nontrivial connected subgraph of G such that $W \subseteq V(H)$, then $V(H)$ is a connected resolving set of G. This observation suggests an upper bound for $cr(G)$. First, we need some additional definitions. For a set S of vertices in a connected graph G, the *Steiner distance* $d(S)$ of S is the minimum size of a connected subgraph in G containing all vertices of S. Necessarily, each such subgraph is a tree and is called a *Steiner tree with respect to S* or a *Steiner S-tree.* A basis W of G is called a *Steiner basis* of G if

$$d(W) = \min\{d(W') : \ W' \text{ is a basis of } G\}.$$

It was shown in [2] that the path of order $n \geq 2$ is the only connected graph of order n with dimension 1. Thus for a connected graph G that is not a path, if W is a Steiner basis of G and T is a Steiner W-tree, then $V(T)$ is a connected resolving set for G. These observations yield an upper bound for the connected resolving number of a nontrivial connected graph that is not a path in terms of the Steiner distances of its bases.

Proposition 2.1 *Let G be a nontrivial connected graph that is not a path. If W is a Steiner basis of G, then*

$$cr(G) \leq d(W) + 1.$$

The upper bound in Proposition 2.1 is sharp. For example, it was shown in [11] that $cr(K_{1,n-1}) = n-1$, where $K_{1,n-1}$ is a star of order $n \geq 4$. On the other hand, every basis W of $K_{1,n-1}$ contains exactly $n-2$ end-vertices of $K_{1,n-1}$ and so $d(W) = n-2$. Therefore, $cr(K_{1,n-1}) = d(W)+1$. Next, we show that $cr(G)$ can be strictly less than $d(W)+1$ for some connected graphs G.

Theorem 2.2 *For every pair k, N of integers with $k \geq 5$ and $N \geq 0$, there exists an infinite class of connected graphs G such that $cr(G) = k$ and*

$$cr(G) \leq d(W) + (1-N),$$

where W is a Steiner basis of G.

Proof. For integers $p, q \geq 3$, let G be that graph obtained from two odd cycles C_{2p+1} and C_{2q+1} by (1) identifying a vertex of C_{2p+1} with a vertex of C_{2q+1} and denoting the identified vertex by x and (2) adding the $k-4$ (≥ 1) new vertices $y_1, y_2, \cdots, y_{k-4}$ and joining each y_i ($1 \leq i \leq k-4$) to x. The graph G is shown in Figure 2.

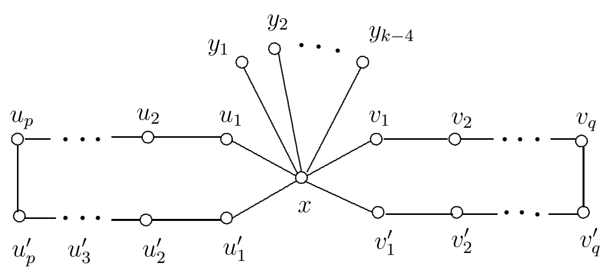

Figure 2: The graph G

First we make an observation. Let $U = \{u_1, u_2, \cdots, u_p\}$, $U' = \{u'_1, u'_2, \cdots, u'_p\}$, $V = \{v_1, v_2, \cdots, v_q\}$, $V' = \{v'_1, v'_2, \cdots, v'_q\}$, and $Y = \{y_1, y_2, \cdots, y_{k-4}\}$. If S is a resolving set of G, then S contains at one vertex from each of $U \cup U'$ and $V \cup V'$. For otherwise, if $S \subseteq U \cup U' \cup Y \cup \{x\}$, then $r(v_1 \mid S) = r(v'_1 \mid S)$. If $S \subseteq V \cup V' \cup Y \cup \{x\}$, then $r(u_1 \mid S) = r(u'_1 \mid S)$. In either case, S is not a resolving set, which is a contradiction. Moreover, by Observation 1.1, every resolving set of G contains at least $k-5$ (≥ 0) vertices from Y.

We now determine the dimension of G and all its bases. Let Y' be any subset of Y with $|Y'| = k-5$ and let

$$W_1 = \{u_p, v_q\} \cup Y', \quad W_2 = \{u'_p, v_q\} \cup Y',$$
$$W_3 = \{u_p, v'_q\} \cup Y', \quad W_4 = \{u'_p, v'_q\} \cup Y'.$$

Since each W_i ($1 \leq i \leq 4$) is a basis for G, it follows that $\dim(G) = k-3$. In what follows, we consider two bases of G are equal if they contain exactly same vertices from $V(G) - Y$. We show next that every basis of G is one of W_i for $1 \leq i \leq 4$

Assume, to the contrary, that G contains a basis W distinct from all W_i $(1 \le i \le 4)$. By the observation above, W contains exactly one vertex from each of $U \cup U'$ and $V \cup V'$ and exactly $k-5$ vertices from Y. Thus we may assume that $W = \{s,t\} \cup Y'$, where $s \in U \cup U'$, $t \in V \cup V'$, and $Y' = Y - \{y_1\}$. We consider two cases.

Case 1. *$s = u_p$ or $s = u'_p$.* Assume, without loss of generality, that $s = u_p$. Since $W \ne W_i$ for $1 \le i \le 4$, it follows that $t = v_j$ or $t = v'_j$ for $1 \le j \le q-1$. If $t = v_j$ for $1 \le j \le q-1$, then $d(v'_1, t) = d(y_1, t)$. Since $d(v'_1, s) = d(y_1, s)$, it follows that $r(v'_1 \mid W) = r(y_1 \mid W)$, a contradiction. If $t = v'_j$ for $1 \le j \le q-1$, then $d(v_1, t) = d(y_1, t)$. Since $d(v_1, s) = d(y_1, s)$, it follows that $r(v_1 \mid W) = r(y_1 \mid W)$, a contradiction.

Case 2. *$s = u_i$ or $s = u'_i$, where* $1 \le i \le p-1$. Assume, without loss of generality, that $s = u_i$ for some i with $1 \le i \le p-1$. Then $t = v_j$ or $t = v'_j$ for $1 \le j \le q$. However, in either case, $r(u'_1 \mid W) = r(y_1 \mid W)$, which is a contradiction.

Therefore, W is not a basis of G and so every basis of G is one of W_i $(1 \le i \le 4)$.

Next we show that $cr(G) = k$. Since $S_0 = \{u_1, u'_1, v_1, v'_1, x\} \cup (Y - \{y_1\})$ is a connected resolving set, $cr(G) \le |S_0| = k$. Assume, to the contrary, that $cr(G) \le k-1$. Since W_i $(1 \le i \le 4)$ are the only bases of G and none of W_i $(1 \le i \le 4)$ are connected, G contains no connected basis. Thus $cr(G) \ge \dim(G) + 1 = k-2$. Let S be a cr-set of G. Then $|S| = k-2$ or $|S| = k-1$. Since S is a resolving set, S contains at least $k-5$ vertices from Y. So we may assume that $Y' = Y - \{y_1\}$ is a subset of S. We consider two cases.

Case 1. $|S| = k-2$. By the observation above, S contains at least one vertex from each of $V \cup V'$ and $U \cup U'$. This implies that $x \in S$ and so $S \subseteq N[x]$, where $N[x] = \{u_1, u'_1, v_1, v'_1, x\} \cup Y$ be the closed neighborhood of x. Since $\langle N[x] \rangle = K_{1,k}$ and $cr(K_{1,k}) = k$, it follows that S is not a connected resolving set of G, a contradiction.

Case 2. $|S| = k-1$. Again, S contains at least one vertex from each of $V \cup V'$ and $U \cup U'$. Thus an argument similar to the one used in Case 1 shows that if $S \subseteq N[x]$, then S is not a connected resolving set for G. So $S \not\subseteq N[x]$. Since $|S| = k-1$ and S is connected, S must contain x and exactly one vertex from each of $\{u_1, u'_1\}$, $\{v_1, v'_1\}$, and $\{u_2, u'_2, v_2, v'_2\}$. Thus we may assume, without loss of generality, that $S = \{u_2, u_1, x, v_1\} \cup Y'$ for some subset Y' of Y with $|Y'| = k-5$. However, then $r(u'_1 \mid S) = r(v'_1 \mid S)$, which is a contradiction.

Therefore, $cr(G) = k$. Since each basis of G is one of W_i $(1 \le i \le 4)$ and $d(W_i) = (k-5)+p+q = k+p+q-5$, it follows that every basis W of G is a Steiner basis of G with $d(W) = k+p+q-5$. For each positive integer N, choose integers p and q such that $p, q \ge 3$ and $p+q \ge 4+N$. Therefore, $cr(G) \le d(W) + (1-N)$ for every basis W of G. ■

3 On connected resolving numbers of graphs with an added vertex

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. In this section, we consider how the connected resolving number of a connected graph G is affected by the addition of a single vertex (and, of course, at least one edge incident with this vertex). It was shown in [1] that if G' is a graph obtained by adding a pendant edge to a nontrivial connected graph G, then

$$\dim(G) \le \dim(G') \le \dim(G) + 1.$$

Thus if a pendant edge is added to a graph G, then the dimension of the resulting graph either stays the same or increases by at most one. However, if a pendant edge is added to a graph G, then the connected resolving number of the resulting graph can increase significantly. To show this, we need some additional definitions. For a set W of vertices of a graph G and a vertex v of G, the *distance between v and W* is defined as

$$d(v, W) = \min\{d(v, u) :\ u \in W\}.$$

Thus $d(v, W) = 0$ if and only if $v \in W$. The *distance between v and the cr-sets* of G is defined as

$$d_{cr}(v) = \min\{d(v, W) :\ \ W \text{ is a } cr\text{-set of } G\}.$$

Certainly, $d_{cr}(v) = 0$ if and only if v belongs to some cr-set of G.

Theorem 3.1 *If G' is the graph obtained by adding a pendant edge to a nontrivial connected graph G at a vertex v, then*

$$cr(G) \le cr(G') \le cr(G) + 1 + d_{cr}(v).$$

Proof. Suppose that G' is obtained from G by adding a pendant edge vx, where $v \in V(G)$ and $x \notin V(G)$. We first show that $cr(G) \le cr(G')$. Let W be a cr-set of G'. We consider two cases.

Case 1. $x \notin W$. Then $W \subseteq V(G)$ and so $\langle W \rangle$ is a connected subgraph in G. Since $d_G(u, w) = d_{G'}(u, w)$ for all $u \in V(G)$ and $w \in W$, it follows that W is a resolving set of G and so W is a connected resolving set of G. Thus $cr(G) \le |W| = cr(G')$.

Case 2. $x \in W$. Since $\langle W \rangle$ is a connected subgraph in G', it follows that $v \in W$. Let $W_1 = W - \{x\}$. Certainly, $\langle W_1 \rangle$ is a connected subgraph in G since x is an end-vertex of G'. Next we show that W_1 is a resolving set of G. Assume, to the contrary, that $r(s|W_1) = r(t|W_1)$ for some $s, t \in V(G)$. Then $d(s, w) = d(t, w)$ for all $w \in W_1$ and so $d(s, v) = d(t, v)$. Since $d(s, x) = d(s, v) + 1$ and $d(t, x) = d(t, v) + 1$, it follows that $d(s, x) = d(t, x)$. This implies that $r(s|W) = r(t|W)$ in G', which is a contradiction. Therefore, W_1 is a resolving set of G and so

$$cr(G) \le |W_1| = |W| - 1 = cr(G') - 1 < cr(G').$$

Next we show that $cr(G') \leq cr(G) + 1 + d_{cr}(v)$. Let W be a cr-set of G such that $d(v, W) = d_{cr}(v)$. So there exists $w_0 \in W$ with $d(w_0, v) = d_{cr}(v)$. Let P : $w_0 = v_0, v_1, \cdots, v_{d_{cr}(v)} = v$ be a $w_0 - v$ path of length $d_{cr}(v)$ in G'. Since $W' = W \cup \{x\} \cup V(P)$ is a connected resolving set of G', it follows that

$$cr(G') \leq |W'| = |W| + 1 + d_{cr}(v) = cr(G) + 1 + d_{cr}(v),$$

as desired. ■

The upper and lower bounds in Theorem 3.1 are both sharp. For example, for integers $k, n \geq 2$, let G be the graph obtained from the path $P_n : u_1, u_2, \cdots, u_n$ by adding the k new vertices $v_1, v_2, \cdots, v_k$ and joining each v_i $(1 \leq i \leq k)$ to v_1. Let G' be the graph obtained from G by adding a pendant edge $u_n x$ and let G'' be the graph obtained from G by adding a pendant edge $u_{n-1}x$. The graphs G, G' and G'' are shown in Figure 3. Let $V = \{v_1, v_2, \cdots, v_k\}$. There are $k+1$ cr-sets in G, that is, $W_0 = \{u_1, v_1, v_2, \cdots, v_k\}$ and $W_i = \{u_1, u_2\} \cup (V - \{v_i\})$ for $1 \leq i \leq k$. Thus $cr(G) = k+1$. Since each W_i is also a cr-set in G', it follows that $cr(G') = k+1$. Since $W_0, W_1, \cdots, W_k$ are all cr-sets in G', it follows that $d_{cr}(u_{n-1}) = d(u_{n-1}, u_2) = n-3$ and so $cr(G'') \leq cr(G) + 1 + d_{cr}(u_{n-1}) = k + n - 1$ by Theorem 3.1. On the other hand, the set $W = \{v_1, v_2, \cdots, v_{k-1}, u_1, u_2, \cdots, u_{n-1}, x\}$ is a cr-set of G'' and so $cr(G'') = |W| = k + n - 1 = cr(G) + 1 + d_{cr}(u_{n-1})$.

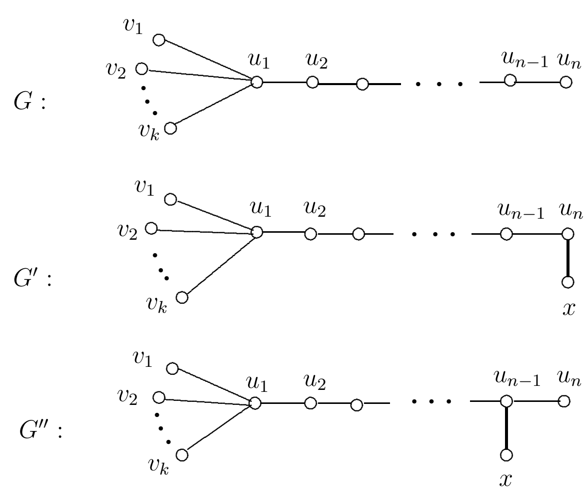

Figure 3: Graphs G, G', and G''

If a vertex v is added to a connected graph G such that more than one edge is incident with v, then the connected resolving number of the resulting graph can stay the same, decrease significantly, or increase significantly. We know that $cr(K_n) = n-1$. However, if we add a new vertex to K_n and join it to all vertices of K_n except one, the resulting graph still has connected resolving number $n-1$. Hence a new vertex may be added to a graph along with a large number of edges and not increase the connected resolving number.

If a vertex v is added to a connected graph G such that more than one edge is incident with v, then the dimension of the resulting graph can actually decrease by one. For example, consider the graphs G and G_1 in Figure 4. The dimension of G is 3, where $W = \{u, x, y\}$ is a basis for G. The graph G_1 is obtained from G by adding the vertex v and three edges uv, wv, and zv to G. Since $W_1 = \{u, v\}$ is a resolving set of G_1 and so $\dim(G_1) = 2$. On the other hand, we are unaware of any graph G with the property that if a vertex v is added to G such that more than one edge is incident with v, then the dimension of the resulting graph can decrease by more than one.

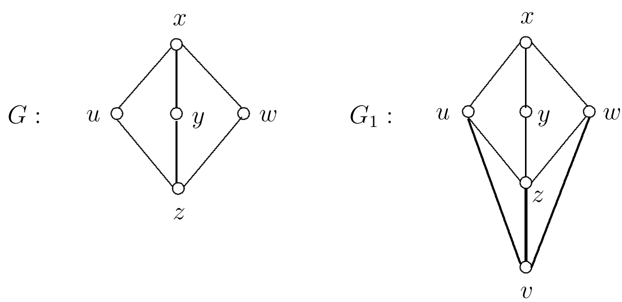

Figure 4: Graphs G and G_1

However, if a vertex v is added to a connected graph G such that more than one edge is incident with v, then the resolving connected number of the resulting graph can actually decrease significantly, as we show next.

Proposition 3.2 *For each positive integer N, there exist connected graphs G and G_1 such that G_1 is obtained from G by adding a vertex with more than one edge incident with v and*

$$cr(G_1) \leq cr(G) - N.$$

Proof. Let G be the graph obtained from the path $P_{2n} : v_1, v_2, \cdots, v_{2n}$, where $n \geq 3$, by adding the four new vertices x_i, y_i for $i = 1, 2$ and the new edges x_iv_1, y_iv_{2n} for $i = 1, 2$. The graph G is shown in Figure 5. By Observation 1.1, every cr-set of G contains at least one vertex from each of $\{x_1, x_2\}$ and $\{y_1, y_2\}$. This implies that the set $V(P_{2n})$ belongs to every cr-set of G and so $cr(G) \geq 2n + 2$. Since $S = V(P_{2n}) \cup \{x_1, y_1\}$ is a connected resolving set, $cr(G) \leq |S| = 2n + 2$. Hence $cr(G) = 2n + 2$. Now let G_1 be the graph obtained from G by adding a new vertex u and the four new edges uv_i for $i \in \{1, 2, 2n - 1, 2n\}$. The graph G_1 is also shown in Figure 5.

By Observation 1.1, every cr-set of G_1 contains at least one vertex from each of $\{x_1, x_2\}$ and $\{y_1, y_2\}$. Thus, any connected subgraph of G_1 containing a cr-set of G_1 must have order at least 5. This implies that $cr(G_1) \geq 5$. On the other hand, $\{u, v_1, v_{2n}, x_1, y_1\}$ is a connected resolving set of G_1 and so $cr(G_1) = 5$. Thus

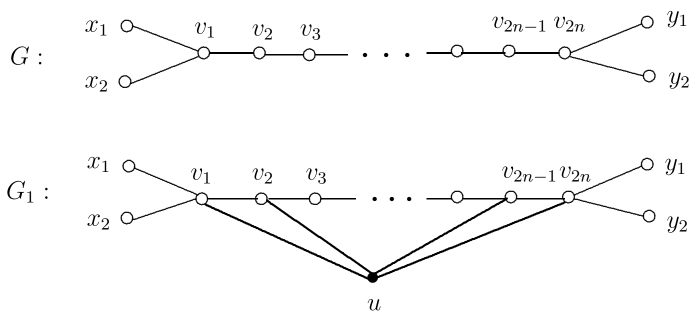

Figure 5: Graphs G and G_1

$cr(G) - cr(G_1) = 2n - 3$. If we choose n such that $n \geq \lceil \frac{N+3}{2} \rceil$, then the result follows. ■

Finally, we show that if a vertex v is added to a connected graph G such that more than one edge is incident with v, then the connected resolving number of the resulting graph can increase significantly. For example, let H be the graph obtained from the path $P_n : v_1, v_2, \cdots, v_n$, where $n \geq 3$, by adding the two new vertices x, y and let H_1 is the graph obtained from H by adding a new vertex v such that v is adjacent to v_{n-1} and v_n. The graphs H and H_1 are shown in Figure 6. Since $\{x, v_1, v_2\}$ is a cr-set of H and $\{x, v_1, v_2, \cdots, v_n\}$ is a cr-set of H_1, we have $cr(H) = 3$ and $cr(H_1) = n + 1$.

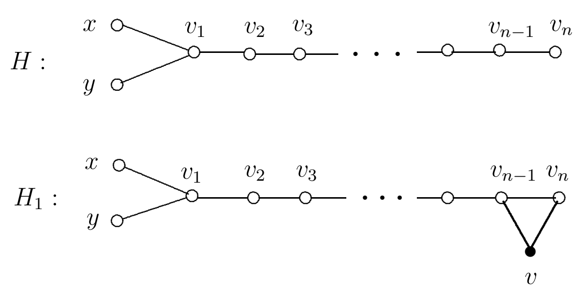

Figure 6: Graphs H and H_1

As another example, we consider the connected resolving number of the wheel $W_n = C_n + K_1$ for $n \geq 3$, that is, W_n is obtained from the cycle C_n of order n by adding a new vertex and joining this new vertex to every vertex of C_n. In [1], it was shown that $\dim(C_n) = 2$ and $\dim((W_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$ if $n \geq 7$, implying that the dimension of of the wheel W_n increases with n for $n \geq 7$. This is also true for the connected resolving numbers of C_n and W_n for $n \geq 7$. For $n \geq 3$, $cr(C_n) = 2$. Clearly, $cr(W_3) = 3$, $cr(W_4) = cr(W_5) = 2$, and $cr(W_6) = 3$. However, for $n \geq 7$, the connected resolving number of W_n increases with n, as we now show.

Theorem 3.3 *For $n \geq 7$,*

$$cr(W_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor + 1.$$

Proof. In $W_n = C_n + K_1$, let $C_n : v_1, v_2, \cdots, v_n, v_1$, where $n \geq 7$, and let v be the central vertex. Since $d(v, v_i) = 1$ for all i with $1 \leq i \leq n$, it follows that v does not belong to any basis of W_n. Let W be a basis of W_n. Then $W \cup \{v\}$ is a connected resolving set and so $cr(W_n) \leq |W| + 1 = \left\lfloor \frac{2n+2}{5} \right\rfloor + 1$.

Next we show that $cr(W_n) \geq \left\lfloor \frac{2n+2}{5} \right\rfloor + 1$. Assume, to the contrary, that $cr(W_n) \leq \left\lfloor \frac{2n+2}{5} \right\rfloor$. Let W be a connected resolving set of W_n with $|W| = \left\lfloor \frac{2n+2}{5} \right\rfloor$. Since $|W| = \dim(W_n)$, it follows that W is a basis of W_n and so $v \notin W$. On the other hand, $\langle W \rangle$ is connected in W_n and so $\langle W \rangle$ is a path of order $\left\lfloor \frac{2n+2}{5} \right\rfloor$ in C_n. Without loss of generality, assume that $\langle W \rangle : v_1, v_2, \cdots, v_{\lfloor \frac{2n+2}{5} \rfloor}$. Since $n - \left\lfloor \frac{2n+2}{5} \right\rfloor \geq 4$ for $n \geq 7$, there exist four consecutive vertices v_j, v_{j+1}, v_{j+2}, v_{j+3} of C_n, where $1 \leq j \leq n$ and addition is performed modulo n, that are not in W. Then $r(v_{j+1}|W) = (2, 2, \cdots, 2)$ $= r(v_{j+2}|W)$, which is a contradiction. Therefore, $cr(W_n) \geq \left\lfloor \frac{2n+2}{5} \right\rfloor + 1$. ∎

4 Graphs with a unique cr-set or various cr-sets

In this section we show that for every integer $k \geq 2$, there exists a graph with a unique *cr*-set of cardinality k. For positive integers d and n with $d < n$, define $f(n, d)$ as the least positive integer k such that $k + d^k \geq n$. It was shown [2] that if G is a connected graph of order $n \geq 2$ and diameter d, then $\dim(G) \geq f(n, d)$. Since $cr(G) \geq \dim(G)$ for every graph G, we have the following.

Lemma 4.1 *For a connected graph G of order $n \geq 2$ and diameter d,*

$$cr(G) \geq \dim(G) \geq f(n, d).$$

We show now that for each integer $k \geq 2$, there exists a graph G containing a unique *cr*-set of cardinality k. The graph in the following proof is a modification of the one constructed in [4].

Theorem 4.2 *For $k \geq 2$, there exists a graph with a unique cr-set of cardinality k.*

Proof. Let $G_1 = K_{2^k}$ with vertex set $U = \{u_0, u_1, \cdots, u_{2^k-1}\}$ and let $G_2 = K_k$ with vertex set $W = \{w_{k-1}, w_{k-2}, \cdots, w_0\}$. Then the graph G is obtained from G_1 and G_2 by adding edges between U and W as follows. Let each integer j $(0 \leq j \leq 2^k - 1)$ be expressed in its base 2 (binary) representation. Thus, each such j can be expressed as a sequence of k coordinates, that is, a k-vector, where the rightmost coordinate represents the value (either 0 or 1) in the 2^0 position, the coordinate to its immediate left is the value in the 2^1 position, etc. For integers i and j, with $0 \leq i \leq k-1$ and $0 \leq j \leq 2^k - 1$, we join w_i and u_j if and only if the value in the 2^i position in the binary representation of j is 1. For $k = 3$, the edges joining W and U in the graph

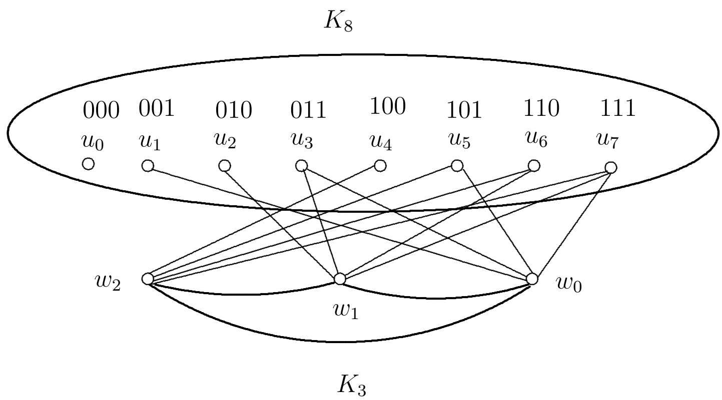

Figure 7: The graph G for $k = 3$

G just constructed are shown in Figure 7. By an argument similar to the one used in [4], we show next that W is the unique cr-set of G.

We first show that W is a cr-set of G. Since G has diameter 2 and order $k + 2^k$, it follows by Lemma 4.1 that $cr(G) \geq k$. Also, since G has diameter 2, the distance between every two distinct vertices of G is 1 or 2. We claim that W is a connected resolving set for G. Since $\langle W \rangle$ is complete, it suffices to show that W is a resolving set. To do this we need only show that the vertices of U have distinct representations with respect to W. The representation for each u_j $(0 \leq j \leq 2^k - 1)$ can be expressed as

$$r(u_j|W) = (2 - a_{k-1}, 2 - a_{k-2}, \cdots, 2 - a_0)$$

where a_m $(0 \leq m \leq k-1)$ is the value in the 2^m position of the binary representation of j. Since the binary representations $a_{k-1}a_{k-2}\cdots a_1a_0$ are distinct for the vertices of U, their representations $(2 - a_{k-1}, 2 - a_{k-2}, \cdots, 2 - a_0)$ are distinct as well. Hence W is a resolving set of G and $cr(G) \leq k$. Thus $cr(G) = k$. Since W is a connected resolving set and $|W| = k$, we conclude that W is a cr-set for G, as claimed.

It remains only to show that G has no other cr-set. First, we make an observation. If U' a subset of U, then $|U'| = k$ and so $|U - U'| = 2^k - k \geq 2$. Since the distance between every two distinct vertices of U is 1, there are at least two vertices having the same representations with respect to U' and so U' is not a resolving set. Hence every cr-set of G contains at least one vertex of W. In what follows, it is now useful to reorder the set W as $W = \{w_0, w_1, \cdots, w_{k-1}\}$, namely w_0, being in position 0, w_1 in position 1, etc. However, in any representation, we continue to refer to 2^0 positions, 2^1 positions, etc., listed from the right. Suppose that $S = W' \cup U'$, where $W' \subseteq W$, $U' \subseteq U$, $|W'| = k - j$, and $|U'| = j$, where $1 \leq j \leq k - 1$. We now order the set S by placing each vertex of W' in the same position it occupied in W and where the elements of U' are ordered arbitrarily to occupy the vacant positions of S. Let $w \in W - W'$, where w occupied position i in W. If $u \in U - U'$, then the representation of u with respect to S has 1 in the 2^i position. In fact, since

$|W - W'| = |U'| = j$, every vertex $u \in U - U'$ has 1 in each of j specific coordinates in its representation with respect to S. So there are 2^{k-j} distinct representations of the vertices of $U - U'$, and there exactly 2^j vertices of each representation. If $j = 1$, then there are two vertices of $U - U'$ with the same representation with respect to S. If $j \geq 2$, then $2^j - j \geq 2$ and for each of 2^{k-j} distinct representations of the vertices of U with respect to W', there are at least two vertices of $U - U'$ with the same representation with respect to S. This is a contradiction. ■

The result can now be extended to the following.

Theorem 4.3 *For every pair r, k of integers with $k \geq 2$ and $0 \leq r \leq k$, there exists a connected graph G such that $cr(G) = k$ and exactly r vertices of G belong to every cr-set of G.*

Proof. For $r = 0$, let $G = K_{k+1}$. Since every k vertices of G form a *cr*-set, no vertex of G belong to every *cr*-set. Thus $cr(G) = k$ and $r = 0$. For $r = 1$, let $G = K_{1,k}$. Since every *cr*-set consists of the central vertex v of G and any $k - 1$ end-vertices of G, it follows that v belongs to every *cr*-set of G and no other vertex belongs to every *cr*-set of G. Hence $cr(G) = k$ and $r = 1$.

For $r = k$, the graph G constructed in the proof of Theorem 4.2 has a unique *cr*-set W containing k vertices. Thus the k-vertices in W are the only vertices of G belonging to every *cr*-set of G. Thus G has the desired properties. For $r = k-1 \geq 2$, take the construction of the graph in the proof of Theorem 4.2 for $|W| = k - 1$ and take two copies of u_{2^k-1}, say x and y, each of which has the same neighborhood as u_{2^k-1}. Then the resulting graph G has connected resolving number k. Moreover, G has exactly two *cr*-sets $W_1 = W \cup \{x\}$ and $W_2 = W \cup \{y\}$. Thus the $k-1$ vertices in W are the only vertices of G belonging to every *cr*-set of G. Thus $r = k-1 = cr(G)-1$.

For $2 \leq r \leq k - 2$, let G be the graph obtained from the path $P_r : u_1, u_2, \cdots, u_r$ by adding the $k - r + 2$ new vertices $v_1, v_2, \cdots, v_{k-r+2}$ and joining (1) each of v_1 and v_2 to u_1 and (2) each of v_i, where $3 \leq i \leq k - r + 2$, to u_r. Then $cr(G) = k$. By observation 1.1, every *cr*-set of G contains at least one vertex from $\{v_1, v_2\}$ and at least $k - r - 1 \geq 1$ vertices from $\{v_3, v_4, \cdots, v_{k-r+2}\}$. In order to form a connected resolving set, the r vertices of P_r must belong to every *cr*-set of G. Moreover, if $v \in V(G) - V(P)$, then v is an end-vertex of G and there is a *cr*-set of G does not contain v and so v does not belong to every *cr*-set of G. Therefore, exactly r vertices of G belong to every *cr*-set of G. ■

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