Resolving $P(v,3,\lambda)$ designs into regular P_3-configurations*

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Abstract

There is one nontrivial regular configuration on two paths of three vertices, and one on three paths. Path designs which are resolvable into copies of these configurations are shown to exist whenever basic numerical conditions are met, with a few possible exceptions.

1 Introduction

Let $P(v,3,\lambda)$ be a P_3-design of λK_v, where P_3 is the path with three vertices and two edges. A *regular P_3-configuration* on p vertices with regularity ρ on ℓ blocks is a pair $(P,\mathcal{L})$ where $\mathcal{L}$ is a collection of P_3 with vertices of P so that every $p \in P$ is in exactly ρ elements of $\mathcal{L}$, and $|\mathcal{L}| = \ell$.

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Each regular P_3-configuration $(P, \mathcal{L})$ is a multigraph G of λK_v, where we put $V(G) = P$ and e is an edge μ-times repeated of G if and only if there are μ paths of $\mathcal{L}$ having e as an edge. We say that G is the *underlying graph* of the regular P_3-configuration $(P, \mathcal{L})$.

Example 1. The regular P_3-configuration $P = \{1, \dots, 6\}$, $\mathcal{L} = \{312, 514, 624, 536\}$, has the underlying graph G with $V(G) = P$ and $E(G) = \{12, 13, 14, 15, 24, 26, 35, 36\}$.

This graph G is also the underlying graph of the following regular P_3-configuration $\mathcal{L}' = \{314, 215, 426, 536\}$. $\mathcal{L}$ and $\mathcal{L}'$ are not isomorphic.

Example 2. The *smallest* nontrivial regular P_3-configuration is the following one: $P = \{1, 2, 3\}$ and $\mathcal{L} = \{123, 213\}$. We denote by Γ its underlying graph. Since Γ is a triangle with one of its edges two times repeated, we denote the graph Γ by $\{1 = 2, 3\}$, where $1 = 2$ means that the edge $\{1, 2\}$ is two times repeated. Moreover, when there is no confusion, we write $1 = 2\ 3$ instead of $\{1 = 2, 3\}$.

In [3], the following problem is posed: Let G be the underlying graph of a regular P_3-configuration $(P, \mathcal{L})$. Find a resolvable G-decomposition of λK_v, for each admissible v and λ.

Example 3. The following is a resolvable Γ-decomposition of $2K_9$.
$\mathcal{L} = \{1 = 2\ 6, 4 = 5\ 7, 8 = 9\ 3\} \cup \{1 = 3\ 5, 4 = 6\ 9, 7 = 8\ 2\} \cup \{2 = 3\ 4, 5 = 6\ 8, 7 = 9\ 1\} \cup \{1 = 4\ 7, 2 = 5\ 8, 3 = 6\ 9\} \cup \{4 = 8\ 3, 5 = 9\ 1, 6 = 7\ 2\} \cup \{1 = 8\ 6, 2 = 9\ 4, 3 = 7\ 5\}$.

Here is another example with the same parameters:
$\mathcal{L}' = \{1 = 2\ 3, 4 = 5\ 6, 7 = 8\ 9\} \cup \{3 = 6\ 1, 4 = 7\ 2, 5 = 8\ 9\} \cup \{1 = 7\ 5, 2 = 8\ 3, 6 = 9\ 4\} \cup \{1 = 4\ 8, 2 = 5\ 6, 3 = 7\ 9\} \cup \{1 = 9\ 5, 6 = 7\ 2, 3 = 4\ 8\} \cup \{6 = 8\ 1, 2 = 9\ 4, 3 = 5\ 7\}$.
$\mathcal{L}$ has an automorphism group of order 12 generated by (164853)(297) and (15)(68)(79), which is intransitive with two element orbits, of sizes 3 and 6. $\mathcal{L}'$ has an automorphism group of order 18 generated by (125367)(498) and (129)(345)(678) which is transitive on the elements. The two decompositions are also easily distinguished by the graph of doubled edges. These are, in fact, the only two nonisomorphic decompositions with these parameters.

Example 4. The following is a resolvable Γ-decomposition of $3K_9$.
$\mathcal{L} = \{1 = 2\ 6, 4 = 5\ 7, 8 = 9\ 3\} \cup \{1 = 3\ 5, 4 = 6\ 9, 7 = 8\ 2\} \cup \{2 = 3\ 4, 5 = 6\ 8, 7 = 9\ 1\} \cup \{1 = 4\ 2, 5 = 9\ 6, 7 = 3\ 8\} \cup \{4 = 8\ 5, 2 = 9\ 7, 6 = 3\ 1\} \cup \{1 = 8\ 9, 2 = 5\ 3, 6 = 7\ 4\} \cup \{1 = 7\ 4, 2 = 8\ 5, 3 = 9\ \} \cup \{1 = 5\ 9, 2 = 6\ 7, 4 = 3\ 8\} \cup \{5 = 7\ 3, 6 = 8\ 1, 4 = 9\ 2\}$.

The major goal is to find a resolvable G-decomposition of λK_v, where G is the underlying graph of each regular P_3-configuration having at most three lines (or paths). We settle this problem with few exceptions here.

2 Two Paths

There is only one nontrivial regular P_3-configuration with $\ell \leq 2$. This is the configuration whose underlying graph is Γ, given in Example 2.

Theorem 1 *(Necessary conditions). If there is a resolvable* Γ*-decomposition of* λK_v*, then* v *and* λ *satisfy one of the following conditions:*

I. $v \equiv 9 \pmod{12}$, $\lambda \equiv 2 \pmod 4$, $\lambda \geq 2$;

II. $v \equiv 3 \pmod 6$, $\lambda \equiv 4 \pmod 8$, $\lambda \geq 4$;

III. $v \equiv 0 \pmod 3$, $\lambda \equiv 0 \pmod 8$, $\lambda \geq 8$;

IV. $v \equiv 9 \pmod{24}$, $\lambda \equiv 1 \pmod 2$, $\lambda \geq 3$.

Lemma 1 *The necessary conditions are sufficient when* $\lambda \equiv 0 \pmod 4$.

Proof. When $v \equiv 3 \pmod 6$, $\lambda \equiv 4 \pmod 8$, and $\lambda \geq 4$, replace every triple of a Kirkman triple system of order v by $\lambda/4$ triples; then replace every triple by a Γ-decomposition of $4K_3$. When $v \equiv 0 \pmod 3$, $v \neq 6$, $\lambda \equiv 0 \pmod 8$, and $\lambda \geq 8$, perform the same operation on a resolvable triple system of order v and index 2 first replicating each triple $\lambda/8$ times. For $v = 6$, take the following base blocks $\pmod 5$: $0 = 1\infty$, $0 = \infty 2$, $0 = \infty 1$, $2 = 4\ 3$, $1 = 3\ 4$, $2 = 4\ 3$. $\square$

A Γ-*frame* $(V, \mathcal{G}, \mathcal{A})$ is defined as follows. Let V, $|V| = n$, and let $\mathcal{G}$ be a partition of V into $G_1, G_2, \ldots, G_q$ such that $|G_i| = t_i$ for $i = 1, 2, \ldots, q$. Let $\mathcal{A}$ be a Γ-decomposition of $\lambda K_{t_1,t_2,\ldots,t_q}$ such that it is possible to partition $\mathcal{A}$ into *holey parallel classes*, where the hole is G_i, $i = 1, 2, \ldots, q$. We say that $(V, \mathcal{G}, \mathcal{A})$ is a Γ-frame of type $t_1 t_2 \ldots t_q$.

Example 5. The following is a Γ-frame of type 2^4, $\lambda = 2$.
$V = \{a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1\}$, $G_1 = \{a_0, a_1\}$, $G_2 = \{b_0, b_1\}$, $G_3 = \{c_0, c_1\}$, $G_4 = \{d_0, d_1\}$. The holey parallel classes, $\pmod{(-, 2)}$, are:
$A_1 = \{c_0 = d_1\ a_0\} \cup \{a_1 = c_0\ d_0\}$;
$A_2 = \{d_0 = b_1\ a_0\} \cup \{b_0 = a_0\ d_0\}$;
$A_3 = \{b_1 = c_0\ a_0\}$;
$A_4 = \{c_0 = b_0\ d_0\}$.

The number of holey parallel classes associated to each hole G_i is not constant. When this number is a constant, the frame is *uniform*, and when not a constant (as here) it is *nonuniform*.

Example 6. The following is a uniform Γ-frame of type 4^4, $\lambda = 2$.
$V = \{a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, c_0, c_1, c_2, c_3, d_0, d_1, d_2, d_3\}$,
$G_1 = \{a_0, a_1, a_2, a_3\}$, $G_2 = \{b_0, b_1, b_2, b_3\}$, $G_3 = \{c_0, c_1, c_2, c_3\}$, $G_4 = \{d_0, d_1, d_2, d_3\}$.
The holey parallel classes, $\pmod{(-, 4)}$, are:
$A_1 = \{a_0 = b_0\ c_1\} \cup \{a_0 = b_2\ c_1\} \cup \{c_0 = b_0\ a_1\}$;

$A_2 = \{d_2 = b_0\ a_3\} \cup \{d_0 = b_0\ a_3\} \cup \{a_0 = d_0\ b_3\}$;
$A_3 = \{a_0 = c_0\ d_1\} \cup \{a_0 = c_2\ d_3\} \cup \{a_0 = d_2\ c_3\}$;
$A_4 = \{d_2 = c_0\ b_3\} \cup \{d_0 = c_0\ b_1\} \cup \{b_0 = c_2\ d_1\}$.

Example 7. The following is a nonuniform Γ-frame of type 2^4, $\lambda = 3$.
$V = \{a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1\}$, $G_1 = \{a_0, a_1\}$, $G_2 = \{b_0, b_1\}$, $G_3 = \{c_0, c_1\}$, $G_4 = \{d_0, d_1\}$. The holey parallel classes are, $\pmod{(-, 2)}$, are:
$A_1 = \{a_0 = d_0\ c_0\} \cup \{a_1 = d_0\ c_0\}$;
$A_2 = \{a_0 = b_0\ d_0\} \cup \{a_0 = b_1\ d_1\}$;
$A_3 = \{a_0 = c_0\ b_0\} \cup \{a_0 = c_1\ b_1\}$;
$A_4 = \{b_0 = d_1\ c_0\} \cup \{d_0 = c_1\ b_0\} \cup \{b_0 = c_1\ d_1\}$.

Lemma 2 *There is a resolvable Γ-decomposition of $2K_v$ when $v \equiv 9 \pmod{12}$ except possibly for $v = 69$.*

Proof. First we treat some small cases.

$v = 9$: See Example 3.

For the next three cases, base parallel classes are given. These are to be developed modulo $(v-1)/2$, with the first yielding $(v-1)/4$ parallel classes and the second yielding $(v-1)/2$.

$v = 21$: $0_0 = 5_0\ \infty$, $0_1 = 1_1\ 1_0$, $5_1 = 6_1\ 6_0$, $2_1 = 4_1\ 3_0$, $7_1 = 9_1\ 8_0$, $3_1 = 7_0\ 9_0$, $8_1 = 2_0\ 4_0$ and $\infty = 0_1\ 0_0$, $2_1 = 8_1\ 5_1$, $7_0 = 4_1\ 9_1$, $1_0 = 2_0\ 6_1$, $3_0 = 9_0\ 1_1$, $5_0 = 8_0\ 3_1$, $7_1 = 4_0\ 6_0$.

$v = 33$: $0_0 = 8_0\ \infty$, $1_0 = 0_1\ 1_1$, $9_0 = 8_1\ 9_1$, $2_1 = 4_1\ 6_0$, $10_1 = 12_1\ 14_0$, $3_1 = 6_1\ 2_0$, $11_1 = 14_1\ 10_0$, $3_0 = 4_0\ 5_1$, $11_0 = 12_0\ 13_1$, $7_1 = 13_0\ 15_0$, $15_1 = 5_0\ 7_0$ and $\infty = 0_1\ 0_0$, $1_1 = 5_1\ 10_1$, $2_1 = 8_1\ 3_1$, $1_0 = 6_1\ 15_1$, $9_0 = 4_1\ 12_1$, $3_0 = 15_0\ 7_1$, $6_0 = 12_0\ 9_1$, $11_1 = 5_0\ 14_0$, $13_1 = 4_0\ 11_0$, $14_1 = 7_0\ 2_0$, $10_0 = 13_0\ 8_0$.

$v = 45$: $0_0 = 11_0\ \infty$, $0_1 = 1_1\ 1_0$, $11_1 = 12_1\ 12_0$, $2_1 = 4_1\ 3_0$, $13_1 = 15_1\ 14_0$, $3_1 = 6_1\ 2_0$, $14_1 = 17_1\ 13_0$, $5_1 = 9_1\ 7_0$, $16_1 = 20_1\ 18_0$, $4_0 = 5_0\ 7_1$, $15_0 = 16_0\ 18_1$, $8_1 = 19_0\ 10_0$, $19_1 = 8_0\ 21_0$, $10_1 = 17_0\ 20_0$, $21_1 = 6_0\ 9_0$ and $\infty = 0_1\ 0_0$, $1_1 = 7_1\ 14_1$, $2_1 = 10_1\ 15_1$, $3_1 = 13_1\ 18_1$, $1_0 = 6_1\ 17_1$, $2_0 = 4_0\ 20_1$, $3_0 = 13_0\ 9_1$, $5_0 = 9_0\ 19_1$, $6_0 = 20_0\ 12_1$, $7_0 = 14_0\ 11_1$, $10_0 = 15_0\ 5_1$, $4_1 = 19_0\ 16_0$, $8_1 = 17_0\ 11_0$, $16_1 = 8_0\ 21_0$, $21_1 = 12_0\ 18_0$.

Now use 4-GDDs (see [1]) of types 2^n (for $n \geq 7$, $n \equiv 1 \pmod 3$), $2^n 5$ (for $n \geq 9$, $n \equiv 0 \pmod 3$) and the uniform Γ-frame of type 4^4 given in Example 6, to construct uniform Γ-frames of types 8^n (for $n \geq 7$, $n \equiv 1 \pmod 3$), $8^n 20$ (for $n \geq 9$, $n \equiv 0 \pmod 3$). The existence of a resolvable Γ-decomposition of $2K_9$ and of a resolvable Γ-decomposition of $2K_{21}$ complete the proof. $\square$

It remains to treat cases with odd index λ, and for this it suffices to settle existence when $\lambda = 3$. The main ingredient is a somewhat large frame:

Example 8. A uniform Γ-frame of type 8^7 and index 3. The point set is the integers modulo 56, with groups determined by the congruence classes modulo 7. Consider the nine pairs of copies of Γ given by: $5{=}8\ \ 13$, $2{=}3\ \ 4$; $4{=}31\ \ 20$, $1{=}26\ \ 51$; $5{=}24\ \ 37$, $4{=}13\ \ 22$; $2{=}52\ \ 15$, $4{=}40\ \ 27$; $1{=}9\ \ 24$, $6{=}40\ \ 18$; $1{=}19\ \ 46$, $2{=}6\ \ 17$; $2{=}34\ \ 17$, $5{=}15\ \ 25$; $3{=}5\ \ 8$, $2{=}46\ 41$; $1{=}17\ \ 40$, $6{=}32\ \ 2$. Adding each multiple of 7 modulo 56 to the pair of graphs given results, in each case, in a holey parallel class for the group consisting of the multiples of 7. Hence this group has nine holey parallel classes. Then adding i modulo 7 to each element in each of these holey parallel classes produces the nine holey parallel classes for the group of elements which is i modulo 7.

This frame leads to the final result on sufficiency:

Lemma 3 *A resolvable Γ-decomposition of $3K_v$ exists whenever $v \equiv 9 \pmod{24}$ except possibly when $v = 153$.*

Proof. A uniform Γ-frame of type 8^4 and index 3 can be obtained using a resolvable transversal design RTD(4,4) (see [1]) and the nonuniform Γ-frame of type 2^4 in Example 5. Now give weight 8 to the elements of a 4-GDD on $x = (v-1)/8$ elements; its type is 1^x when $x \equiv 1, 4 \pmod{12}$, and $1^{x-7}7^1$ when $x \equiv 7, 10 \pmod{12}$ and $x \geq 22$. Fill in the holes using resolvable Γ-decompositions of orders 9, 33, and 57. □

Summarizing, we have the following:

Theorem 2 *The necessary conditions of Theorem 1 are sufficient except possibly when $v = 153$ and λ is odd, or $v = 69$ and $\lambda \equiv 2 \pmod{4}$.*

3 Three Paths

There is only one nontrivial regular P_3-configuration with $\ell = 3$. This configuration is given by $P = \{1, 2, 3\}$, $\mathcal{L} = \{123, 123, 213\}$. We denote by $\Lambda = \{1 \equiv 2 = 3\}$ its underlying graph, where $1 \equiv 2$ means that the edge $\{1, 2\}$ is three times repeated and $2 = 3$ means that the edge $\{2, 3\}$ is two times repeated. Moreover, when there is no confusion, we write $1 \equiv 2 = 3$ instead of $\{1 \equiv 2 = 3\}$.

Theorem 3 *(Necessary conditions). If there is a resolvable Λ-decomposition of λK_v, then v and λ satisfy one of the following conditions:*

I. $v \equiv 0 \pmod{3}$, $\lambda \equiv 0 \pmod{4}$, $\lambda \geq 4$;

II. $v \equiv 3 \pmod{6}$, $\lambda \equiv 2 \pmod{4}$, $\lambda \geq 6$;

III. $v \equiv 9 \pmod{12}$, $\lambda \equiv 1 \pmod{2}$, $\lambda \geq 3$.

Now we treat sufficiency.

Lemma 4 *When $v \equiv 0 \pmod{6}$ and $v \geq 12$, a resolvable Λ-decomposition of λK_v exists provided that $\lambda \equiv 0 \pmod{4}$.*

Proof. According to [2], given a $S_2(2,3,v)$, $(V,\mathcal{B})$, it is always possible to partition $\mathcal{B}$ into $\mathcal{B}_1$ and $\mathcal{B}_2$ so that $|\mathcal{B}_1| = |\mathcal{B}_2|$, $|\mathcal{B}_1 \cap \mathcal{B}_2| = 0$, $\mathcal{B}_1 \cup \mathcal{B}_2 = \mathcal{B}$ and there is a $1-1$ mapping

$$f : \mathcal{B}_1 \to \mathcal{B}_2$$

such that B and $f(B)$ share at least one edge for each block $B \in \mathcal{B}_1$. $\square$

There is no resolvable Λ-decomposition of $4K_6$.

Lemma 5 *When* $v \equiv 3 \pmod 6$, $\lambda \equiv 0 \pmod 2$, *and* $\lambda \geq 4$, *there is a resolvable* Λ*-decomposition of* λK_v.

Proof. There is a resolvable Λ-decomposition of $4K_3$ and a resolvable Λ-decomposition of $6K_3$. Form a Kirkman triple system on v elements, and replace each triple by a copies of the first decomposition and b copies of the second, with $4a + 6b = \lambda$. $\square$

Lemma 6 *If* $v \equiv 9 \pmod{12}$ *and* $\lambda \geq 3$, *there is a resolvable* Λ*-decomposition of* λK_v *except possibly when* $\lambda = 5$.

Proof. Whenever a resolvable Γ-decomposition of $2K_v$ exists, a resolvable Λ-decomposition of $3K_v$ is obtained as follows. Partition the single edges appearing in copies of Γ into two classes, black and white, so that every Γ has one black and one white singleton edge. Then replace the doubly repeated edges in copies of Γ by triply repeated edges, replace black edges by doubly repeated edges, and leave white edges as singletons.

This handles all cases with $\lambda = 3$; using the preceding lemmas, union with solutions with $\lambda \in \{4, 6\}$ settles all cases with $\lambda \geq 7$. $\square$

To complete the cases with $\lambda = 5$, we require a further frame. This somewhat large frame was found by first selecting five starter blocks of size three on the integers modulo 16, which generate eighty triples in total. Triples were chosen so that each pair occurred in either two or three triples. The resulting collection of triples is frame resolvable, by choosing every fourth triple under the action of the cyclic group to form a frame parallel class. Hence the task is to replace triples by copies of Λ so that the resulting design has index $\lambda = 5$. All pairs occurring in two of the eighty triples are forced to appear as a tripled edge in one copy of Γ and doubled in another.

We developed a simple hillclimbing method, which starts with a "random" assignment of copies of Γ to the triples and then repeatedly modifies the selection of triple, double, and single edges in one Γ to reduce the discrepancy between the current assignment and a design with index five. This simple minded method succeeded in producing a solution.

Example 9. The following is a uniform Λ-frame of type 4^4 with $\lambda = 5$.

$6\equiv3=1$	$10\equiv5=7$	$14\equiv11=9$	$2\equiv15=13$
$9\equiv2=3$	$13\equiv6=7$	$1\equiv10=11$	$5\equiv14=15$
$7\equiv5=2$	$11\equiv6=9$	$15\equiv13=10$	$3\equiv1=14$
$3\equiv9=10$	$11\equiv1=2$	$7\equiv13=14$	$15\equiv5=6$
$2\equiv3=5$	$7\equiv6=9$	$10\equiv11=13$	$15\equiv14=1$
$0\equiv6=7$	$4\equiv10=11$	$8\equiv14=15$	$12\equiv2=3$
$7\equiv2=4$	$11\equiv8=6$	$15\equiv10=12$	$3\equiv14=0$
$14\equiv7=8$	$2\equiv11=12$	$6\equiv15=0$	$10\equiv3=4$
$12\equiv10=7$	$0\equiv11=14$	$4\equiv2=15$	$8\equiv6=3$
$7\equiv8=10$	$12\equiv11=14$	$0\equiv15=2$	$4\equiv3=6$
$7\equiv0=1$	$3\equiv12=13$	$15\equiv8=9$	$11\equiv4=5$
$12\equiv13=15$	$0\equiv1=3$	$4\equiv5=7$	$9\equiv8=11$
$5\equiv3=0$	$9\equiv4=7$	$13\equiv11=8$	$1\equiv12=15$
$8\equiv3=5$	$12\equiv7=9$	$0\equiv13=11$	$4\equiv15=1$
$1\equiv7=8$	$5\equiv11=12$	$9\equiv15=0$	$13\equiv3=4$
$5\equiv0=2$	$9\equiv6=4$	$13\equiv8=10$	$1\equiv14=12$
$6\equiv1=4$	$10\equiv8=5$	$14\equiv9=12$	$2\equiv13=0$
$8\equiv1=2$	$12\equiv5=6$	$0\equiv9=10$	$4\equiv13=14$
$1\equiv2=4$	$5\equiv6=8$	$9\equiv10=12$	$14\equiv13=0$
$2\equiv8=9$	$6\equiv12=13$	$10\equiv0=1$	$14\equiv4=5$

Lemma 7 *There is a resolvable* Λ*-decomposition of* $5K_v$ *when* $v \equiv 9 \pmod{12}$ *except possibly for* $v \in \{33, 45, 69\}$*.*

Proof. First we treat some small cases.

$v = 9$:

We form ten parallel classes on the elements $Z_3 \times Z_3$. The first three are formed by developing the block $(0,1) \equiv (1,2) = (0,0)$ modulo $(3,-)$ to form a parallel class. Then develop modulo $(-,3)$ to form three parallel classes. The next three are formed by developing the block $(2,0) \equiv (1,1) = (0,0)$ modulo $(-,3)$; then develop modulo $(3,-)$ to form three parallel classes. The final four are formed by placing a resolvable Λ-decomposition of $4K_3$ on each of the triples with constant first or second coordinate.

$v = 21$:

Form a base parallel class on $Z_7 \times \{0,1,2\}$: $1_0 \equiv 0_0 = 2_1$, $1_1 \equiv 2_0 = 4_0$, $6_1 \equiv 3_0 = 6_0$, $6_2 \equiv 5_0 = 0_2$, $4_2 \equiv 0_1 = 3_1$, $4_1 \equiv 5_2 = 2_2$, $3_2 \equiv 1_2 = 5_1$. Develop this under the mapping $i_j \mapsto (2i \bmod 7)_{(j+1) \bmod 3}$ to form three base parallel classes in total, and develop each modulo $(7,-)$ to obtain 21 parallel classes. Place two parallel classes on the triples with constant first coordinate using a resolvable Λ-decomposition of $4K_3$. The final two parallel classes are handled in a similar way, placed on triples of the form $(i_0, (2i)_1, (4i)_2)$ for $i \in Z_7$.

Now use 4-GDDs (see [1]) of types 2^n (for $n \geq 7$, $n \equiv 1 \pmod 3$), $2^n 5$ (for $n \geq 9$, $n \equiv 0 \pmod 3$) and the uniform Γ-frame of type 4^4 given in Example 9, to construct uniform Γ-frames of types 8^n (for $n \geq 7$, $n \equiv 1 \pmod 3$), $8^n 20$ (for $n \geq 9$, $n \equiv 0 \pmod 3$). The existence of a resolvable Γ-decomposition of $5K_9$ and of a resolvable Γ-decomposition of $5K_{21}$ complete the proof. □

This leaves only a few small exceptions for the existence of resolvable Λ-decompositions. Summarizing, we have the following:

Theorem 4 *The necessary conditions of Theorem 2 are sufficient except possibly when $v = 6$ or $\lambda = 5$ and $v \in \{33, 45, 69\}$.*

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