On even $[2, b]$-factors in graphs

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Abstract

For each even integer $b \geq 2$ we prove that a graph G with n vertices has an even $[2, b]$-factor if G is 2-edge connected and each vertex of G has degree at least $\max\{3, \frac{2n}{b+2}\}$.

1 Introduction

Tutte's f-factor theorem [16, 4] has evolved in many directions. Surveys are given in [1].

Lovász derived an extensive $[g, f]$-factor theory [11, 12, 13] which has been continued by other authors [5, 10].

Connected factors are treated in [6, 8, 9]. Odd factors have been treated by Amahashi, Yuting, Kano, Topp and Vestergaard.

Amahashi [2] extended Tutte's 1-factor theorem to $\{1, 3, 5, ..., 2t-1\}$ factors, and Yuting, Kano [17] generalized this further: for an integer valued function f given on $V(G)$ they define H to be a $[1, f]$-odd factor of G if for every vertex x in G, $d_H(x)$ is odd and satisfies $1 \leq d_H(x) \leq f(x)$. They then prove that G has a $[1, f]$-odd factor if and only if deletion of any set S of vertices leaves a graph whose number $o(G-S)$ of odd components is not larger than $\sum_{x \in S} f(x)$, i.e.

$$G \text{ has a } [1,f]\text{-odd factor } \Leftrightarrow o(G-S) \leq \sum_{x \in S} f(x). \quad (*)$$

Using Sumner's theory [14] on minimal barriers, Topp and Vestergaard [15] proved that it is not necessary to test (*) for all subsets S of $V(G)$, but only for some of them. As one consequence they show that if G is of even order n and if no vertex v in G is the center of an induced $K_{1,nf(v)+1}$-star, then G has a $[1,f]$-odd factor.

In this paper we shall consider even factors. In general, existence of even factors is not deducible from the existence of odd factors.

2 Notation

We consider graphs without loops or multiple edges. A graph G has vertex set $V(G)$ and edge set $E(G)$. The *order* of G is $|G| = |V(G)| = n$. For subsets X, Y of $V(G)$ we denote by $e_G(X,Y)$ the number of edges in G having one end-vertex in X and the other in Y. Thus $e_G(v, V(G)-v) = d_G(v)$ is the *degree* of v and $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ is the smallest degree in G.

A subgraph of G containing all of $V(G)$ but possibly not all of $E(G)$ is called a *spanning subgraph* of G or a *factor* in G.

Let g, f be mappings from $V(G)$ into the nonnegative integers $\mathbb{Z}_0^+$ and let $g(v) \leq f(v), \forall\, v \in V(G)$. Then F is called a $[g,f]$-*factor* of G if F is a factor of G with $g(v) \leq d_F(v) \leq f(v), \forall v \in V(G)$. A factor F satisfying $d_F(v) \equiv 0 \pmod 2, \forall v \in V(G)$, is called even.

An edge $e \in E(G)$ is a *bridge* if $G-e$ has more components than G and $v \in V(G)$ is a cut-vertex if $G-v$ has more components than G. A graph with at least 3 vertices is 2-*edge connected* if it is connected and has no bridge; G is 2-*vertex connected* if G is connected and has no cut-vertex.

A *block* in a graph with no isolated vertex is either a bridge together with its two end-vertices, or it is a maximal 2-vertex connected subgraph of G. The latter is called a *proper block* of G.

Consider functions g, f on $V(G)$ with $g(v) \leq f(v)$ for each $v \in V(G)$, and an ordered pair X, Y of disjoint subsets of $V(G)$. A component C of $G-(X \cup Y)$ is called *odd* if $\sum_{v \in V(C)} f(v) + e_G(V(C), Y)$ is an odd number. The number of odd components in $G-(X \cup Y)$ is denoted by $h_G(X,Y)$. When clear from the context we may omit reference to G.

3 Complete bipartite graphs

Let us observe that $K_{1,q}$ has no $[2,b]$-factor; and $K_{2,q}$ has no proper $[2,b]$-factor, and so, it has an even one if and only if q is even and $q \leq b$.

Existence of an even factor with degrees bounded by the constant b is characterized in Theorem 1 below.

Theorem 1 *For $3 \leq p \leq q$ let $K_{p,q}$ be a complete, bipartite graph and let $b \geq 2$ be an even integer. Then the graph $K_{p,q}$ has an even $[2,b]$-factor if and only if $q \leq \frac{b}{2}p$.*

Remark: As b is even, the inequality $q \leq \frac{b}{2}p$ is equivalent to $b \geq 2\lceil \frac{q}{p} \rceil$.

Proof: $\Downarrow$: Let F be an even $[2,b]$-factor of $K_{p,q}$. Then $2q \leq |E(F)| \leq bp$ and $q \leq \frac{b}{2}p$ follows.

$\Uparrow$: Assume $q \leq \frac{b}{2}p$. Let $q = rp + s, 0 \leq s < p$. Necessarily $1 \leq r \leq \frac{b}{2}$, and if $r = \frac{b}{2}$ then $s = 0$.

Let $x_1, x_2, \ldots, x_p$ be the vertices of one colour class, and $y_1, \ldots, y_p; y_{p+1}, y_{p+2}, \ldots, y_{2p}; y_{2p+1}, \ldots, y_{3p}; \ldots; y_{(r-1)p+1}, \ldots, y_{rp}; y_{rp+1}, y_{rp+2}, \ldots, y_{rp+s}$ the vertices of the other colour class of $K_{p,q}$.

For $s \geq 2$, form the $r+1$ cycles:

$$\begin{aligned}
C_1 &= x_1 y_1 x_2 y_2 \ldots x_p y_p \\
C_2 &= x_1 y_{p+1} x_2 y_{p+2} \ldots x_p y_{2p} \\
&\vdots \\
C_i &= x_1 y_{(i-1)p+1} x_2 y_{(i-1)p+2} \ldots x_p y_{ip} \\
&\vdots \\
C_r &= x_1 y_{(r-1)p+1} x_2 y_{(r-1)p+2} \ldots x_p y_{rp} \\
C_{r+1} &= x_1 y_{rp+1} x_2 y_{rp+2} \ldots x_s y_{rp+s}.
\end{aligned}$$

The union $F = \bigcup_{i=1}^{r+1} C_i$ is an even $[2,b]$-factor of $K_{p,q}$ because $V(F) = V(G)$, and all vertices have in F even degree at least two and at most b: Certainly $d_F(y_i) = 2, 1 \leq i \leq q$, and for $x_j, 1 \leq j \leq p$, we have, since $s > 0$ implies $r < \frac{b}{2}$, that $d_F(x_j) \leq 2r + 2 = 2\lceil \frac{q}{p} \rceil = b$.

For $s = 1$, we have $b \geq 2(r+1)$,

1. *if $p \geq 4$,* in the preceding definition we replace the cycle C_{r+1} by the cycle $C'_{r+1} = x_2 y_{q-1} x_3 y_q$, and $F = (\bigcup_{i=1}^{r} C_i) \cup C'_{r+1}$; we have $d_F(y_i)$ is 2 or 4, for each i; and for each j, we have $d_F(x_j) \leq 2(r-1) + 4 = 2r + 2 \leq b$;

2. *if $p = 3$,* then let $C''_r = x_1 y_{q-2} x_2 y_{q-3}$ and $C''_{r+1} = x_1 y_q x_3 y_{q-1}$.

So, for $s = 1$ with $F = (\bigcup_{i=1}^{r-1} C_i) \cup C''_r \cup C''_{r+1}$ we have $d_F(y_i) = 2$ for each i; and for each j, we have $d_F(x_j) \leq 2(r-1) + 4 = 2r + 2 \leq b$.

For $s = 0$ we have $q = rp, r \leq \frac{b}{2}$, and $F = \bigcup_{i=1}^{r} C_i$ is an even $[2,b]$-factor of $K_{p,q}$ since $d_F(x_j) = 2r \leq b, 1 \leq j \leq p$.

This proves Theorem 1. □

So we conclude as follows.

Corollary *For $3 \leq p \leq q$, the least even integer b such that the bipartite graph $K_{p,q}$ has an even $[2,b]$ factor is $b = 2\lceil \frac{q}{p} \rceil$.*

Generalization Above, with $G = K_{p,q}, 3 \leq p \leq q$, we have $p+q=n$ and $\delta(G) = p$. The conditions $p \geq \frac{2q}{b}, p \geq 3$ translate into $\delta \geq \max\{3, \frac{2n}{b+2}\}$ which in the following section as a generalization is proven to be a sufficient condition for any 2-edge connected graph to contain an even $[2,b]$-factor. Furthermore, for $q = \frac{b}{2}p$ the graphs $K_{p,q}$ demonstrate that the condition $\delta \geq \max\{3, \frac{2n}{b+2}\}$ is strict.

4 General graphs

Below, we cite a theorem by Lovász characterizing graphs having an even $[g,f]$-factor and a fortiori an even $[2,b]$-factor. We use Lovász's theorem to derive Theorem 2, which only gives a sufficient condition for G to contain an even $[2,b]$-factor. However, Theorem 2 has the advantage of being easy to apply.

Lovász' parity $[g,f]$-factor Theorem [11, 3]. *Let G be a graph, let g and f map $V(G)$ into the nonnegative integers such that $g(v) \leq f(v), \forall\, v \in V(G)$, and $g(v) \equiv f(v) (\text{mod } 2), \forall\, v \in V(G)$. Then G contains a $[g,f]$-factor F such that $d_F(v) \equiv f(v) (\text{mod } 2), \forall\, v \in V(G)$, if and only if, for every ordered pair X, Y of disjoint subsets of $V(G)$*

$$\sum_{y \in Y} d_G(y) - \sum_{y \in Y} g(y) + \sum_{x \in X} f(x) - h(X,Y) - e(X,Y) \geq 0.$$

Let $b \geq 2$ be an even integer and in the theorem above, let $g(v) = 2, f(v) = b, \forall\, v \in V(G)$. Then we immediately obtain

Corollary *G contains an even $[2,b]$-factor if*

$$\sum_{y \in Y} d_G(y) - 2|Y| + b|X| - h(X,Y) - e(X,Y) \geq 0 \qquad (**)$$

for all ordered pairs X, Y of disjoint subsets of $V(G)$.

In Theorem 2 below we describe an important class of graphs which satisfy (**).

Theorem 2 *Let $b \geq 2$ be an even integer and let G be a 2-edge connected graph with n vertices and with minimum degree $\delta(G) \geq \max\{3, \frac{2n}{b+2}\}$. Then G contains an even $[2,b]$-factor.*

We generalize this result in the following form.

Corollary *Let $b \geq 2$ be an even integer and let G be a graph such that*

(i) each vertex of G belongs to a proper block of G, and

(ii) each block B in G satisfies $\delta(B) \geq \max\{3, \frac{2|B|}{b+2}\}$, and

(iii) each cut vertex in G has degree at most b.

Then G has an even $[2, b]$-factor.

The corollary follows immediately by applying Theorem 2 to each block of G. We shall prove Theorem 2 by demonstrating that (**) holds.

Proof: Let X, Y be any pair ($X = \emptyset$ or $Y = \emptyset$ may occur) of disjoint subsets of $V(G)$. Certainly

$$\sum_{y \in Y} d_G(y) \geq e_G(Y, V(G) - Y) \geq e_G(X, Y) + h(X, Y) \tag{1}$$

and we can find the following inequality.

$$\sum_{y \in Y} d_G(y) - 2|Y| + b|X| - h(X, Y) - e_G(X, Y) \geq -2|Y| + b|X|. \tag{2}$$

Thus, if $-2|Y| + b|X| \geq 0$, inequality (**) and hence Theorem 2 holds. We may therefore assume that for some pairs X, Y we have

$$-2|Y| + b|X| < 0. \tag{3}$$

For pairs X, Y with $|X| \geq \delta(G) = \delta$ we can use (3) together with $|X| + |Y| \leq n$ (as $X \cap Y = \emptyset$) to obtain

$$\delta \leq |X| < \frac{2}{b}|Y| \leq \frac{2}{b}(n - |X|) \leq \frac{2}{b}(n - \delta) \tag{4}$$

giving

$$\delta < \frac{2n}{b+2}, \tag{5}$$

but that contradicts the hypothesis $\delta \geq \frac{2n}{b+2}$, so no pair X, Y satisfying (3) can have $|X| \geq \delta(G)$. We thus henceforth have

$$-2|Y| + b|X| < 0 \text{ and } |X| \leq \delta - 1. \tag{6}$$

Case 1 $|Y| \geq b + 1$:

There are at most $|X||Y|$ edges between X and Y, so

$$e(X, Y) \leq |X||Y|. \tag{7}$$

Each odd component of $G - (X \cup Y)$ contains at least one vertex, so

$$h(X, Y) \leq n - |X| - |Y|. \tag{8}$$

Define

$$\tau = \sum_{y \in Y} d(y) - 2|Y| + b|X| - h(X, Y) - e(X, Y). \tag{9}$$

Using (7), (8), and $d_G(y) \geq \delta$, we obtain

$$\tau \geq \delta|Y| - 2|Y| + b|X| - n + |X| + |Y| - |X||Y|, \tag{10}$$

$$\tau \geq (\delta - 1)|Y| + ((b+1) - |Y|)\,|X| - n. \tag{11}$$

Since $b + 1 - |Y| \leq 0$ and $|X| \leq \delta - 1$, we obtain

$$\tau \geq (\delta - 1)|Y| + (b + 1 - |Y|)(\delta - 1) - n, \tag{12}$$

$$\tau \geq (b+1)(\delta - 1) - n. \tag{13}$$

By hypothesis $\delta \geq \frac{2n}{b+2}$, so

$$\tau \geq (b+1)\left(\frac{2n}{b+2} - 1\right) - n \tag{14}$$

and

$$\tau \geq \frac{b}{b+2}n - b - 1. \tag{15}$$

For $n \geq b + 4$ we obtain

$$\tau \geq \frac{b-2}{b+2}, \tag{16}$$

and as $b \geq 2$ we have that $\tau \geq 0$.

That is, (**) holds for $n \geq b + 4$.

If $n \leq b + 3$ we use $\delta \geq 3$ in (13) to obtain the continuation

$$\tau \geq (b+1)2 - (b+3) = b - 1 \geq 1 \geq 0. \tag{17}$$

Thus (**), and hence Theorem 1, is proven in Case 1.

Case 2 $|Y| \leq b$ (and still $-2|Y| + b|X| \leq 0, |X| \leq \delta - 1$):

From $|X| < \frac{2}{b}|Y| \leq 2$ we get that $|X|$ equals 0 or 1.

Let $h_1 = h_1(X, Y)$ be the number of odd components C of $G - (X \cup Y)$ with $e(C, Y) = 1$, and let $h_2 = h_2(X, Y)$ be the number of odd components C of $G - (X \cup Y)$ having $e(C, Y) > 1$, i.e. $e(C, Y) \geq 3$. Then $h(X, Y) = h_1 + h_2$.

Case 2.1 $|Y| \leq b$ and $|X| = 0$:

From $X = \emptyset$ we infer $h_1 = 0$, since a single edge between Y and an h_1-component C of $G - Y$ would be a bridge of G; but that contradicts the hypothesis that G is 2-edge connected. Thus, $h(X, Y) = h_2$. Furthermore $X = \emptyset$ implies by definition that $e(X, Y) = \emptyset$. We use this and $\sum_{y \in Y} d_G(y) \geq 3h_2$ to obtain

$$\sum_{y \in Y} d(y) - 2|Y| + b|X| - h(X, Y) - e(X, Y) \geq 3h_2 - 2|Y| - h_2. \tag{18}$$

If $|Y| \leq h_2$, we see immediately that (**) holds. Otherwise, $|Y| > h_2$, and together with $\delta(G) \geq 3$ we obtain

$$\sum_{y \in Y} d_G(y) - 2|Y| - h_2 \geq |Y| - h_2 > 0. \tag{19}$$

Thus (**) holds in Case 2.1.

Case 2.2 $|Y| \leq b$ and $|X| = 1$:

As $\sum_{y\in Y} d_G(y) \geq h_1 + 3h_2 + e(X,Y), \quad h(X,Y) = h_1 + h_2$ we have

$$\begin{aligned} \sum_{y \in Y} d_G(y) &- 2|Y| + b - h(X,Y) - e(X,Y) && (20) \\ &\geq h_1 + 3h_2 + e(X,Y) - 2|Y| + b - h_1 - h_2 - e(X,Y) && (21) \\ &= 2h_2 - 2|Y| + b. && (22) \end{aligned}$$

For $|Y| \leq h_2 + b/2$ we see that (**) holds.

For $|Y| > h_2 + b/2$ we use $b - e(X,Y) \geq b - |Y| \geq 0$ to obtain that

$$\sum_{y \in Y} d(y) - 2|Y| + b - h_1 - h_2 - e(X,Y) \geq (\delta - 2)|Y| - h_1 - h_2. \tag{23}$$

As $|X| = 1$ and $\delta \geq 3$ we observe that each h_1-component C of $G - (X \cup Y)$ contains at least two vertices. Let c' be the unique vertex in C which has a neighbour in Y and let $c \in C \setminus c'$. Then $e(c, X \cup Y) \leq 1$ and c has at least $\delta - 1$ neighbours in C. So C contains at least δ vertices. Therefore $h_1 \leq \frac{n-|Y|-h_2-1}{\delta}$. Using this and $-\frac{n}{\delta} \geq -\frac{b+2}{2}, |Y| \geq h_2 + \frac{b+1}{2}$ in (22) we obtain

$$(\delta - 2)|Y| - h_1 - h_2 \geq (\delta - 2)\left(h_2 + \frac{b+1}{2}\right) - \frac{n - |Y| - h_2 - 1}{\delta} - h_2 \tag{24}$$

$$\geq (\delta - 3)\left(h_2 + \frac{b}{2} + \frac{1}{2}\right) - \frac{1}{2} + \frac{|Y| + h_2 + 1}{\delta}. \tag{25}$$

This expression is nonnegative if $\delta \geq 4$, and if $\delta = 3$ we use $|Y| \geq \frac{b+1}{2} \geq \frac{3}{2}$ to obtain $\frac{|Y|}{3} \geq \frac{1}{2}$ and we get the same conclusion.

Thus Case 2.2, and with that Theorem 2, is proven. □

In Theorem 2 it is necessary to demand $\delta(G) \geq 3$ as shown by the following example.

Example 1 G has $n = 14$ vertices such that one vertex v has 11 neighbours, all of degree 2. Three of them, x, y, z, also have another common neighbour $w, d_G(w) = 3$, and four of them share a common neighbour $u, d_G(u) = 4$.

This graph G has $n = 14, \delta(G) = 2$, is 2-edge connected, and with $b = 12$ it satisfies $\delta(G) \geq \frac{2n}{b+2}$ as $2 \geq \frac{2 \cdot 14}{12+2}$; but G has no even $[2,12]$-factor F since each of

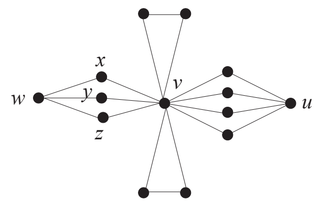

Figure 1: Example 1.

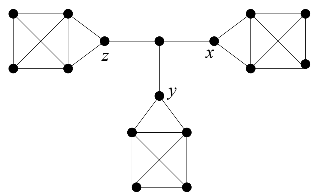

Figure 2: Example 2.

x, y, z must be in F with degree 2, but then w will be in F with degree 3, which is not an even number.

A graph with bridges may have an even factor; this is the case for two circuits joined by an edge, but in Theorem 2 the condition that G is 2-edge connected cannot be omitted.

Example 2 Let G be the graph on 16 vertices consisting of one vertex with exactly 3 neighbours x, y, z such that the remaining 12 vertices form 3 disjoint K_4's, and x is joined by two edges to one K_4, y by two edges to the second K_4 and z by two edges to the third K_4. Let $b = 4$; we have $n = 16, \delta = 3$ and $3 = \delta \geq \frac{16}{4+2}$, but G has no even factor.

Other conditions: Considering degree sums $\sigma_k(G) = \min\{d_G(v_1) + d_G(v_2) + \ldots + d_G(v_k) \mid v_1, \ldots, v_k$ is a set of independent vertices$\}$, it might for $k = 2$ be conjectured that $\sigma_2(G) \geq \max\{6, \frac{4n}{b+2}\}$ implies existence of an even $[2, b]$-factor.

Another condition, suggested by an anonymous referee, is that $\delta(G) \geq 3$ and $\sigma_{k+1}(G) \geq n$ implies that G has an even $[2, 2k]$-factor. This is a generalization of Theorem 2 since certainly $\sigma_{k+1}(G) \geq n$ is satisfied if $\delta \geq \max\{3, \frac{n}{k+1}\}$ and by Theorem 2 that gives an even $[2, 2k]$-factor of G.

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