Minimal and near-minimal critical sets in back-circulant latin squares

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Abstract

A *critical set* in a latin square is a subset of its elements with the following properties:

1) No other latin square exists which also contains that subset.

2) No element may be deleted without destroying property 1.

Let $scs(n)$ denote the smallest possible cardinality of a critical set in an $n \times n$ latin square. It is conjectured that $scs(n) = \lfloor n^2/4 \rfloor$, and that only the back-circulant latin square contains a critical set of this size. These conjectures have been proven for $n \leq 7$. In this paper, we further conjecture that in a back-circulant latin square of size $n > 4$, the critical set of size $\lfloor n^2/4 \rfloor$ is unique up to isomorphism, and that no critical set of size $\lfloor n^2/4 \rfloor + 1$ exists if n is even. These conjectures are proven for all $n \leq 12$ except $n = 11$.

1 Definitions

A *latin square* of order n is an $n \times n$ array $A = (a_{rc})$ containing elements from a set $V = \{v_i : 1 \leq i \leq n\}$ such that each element of V occurs exactly once in each row and once in each column. The triple $(r, c; e)$ indicates that the entry $a_{rc} = v_e$ in the latin square. For convenience, we will adopt the convention in the remainder of the paper that the set of elements is $V = \{1, 2, \ldots, n\}$ and that the rows and columns of A are also indexed by $\{1, 2, \ldots, n\}$.

A *partial latin square* is a set of k entries $P = \{(r_i, c_i; e_i) : 1 \leq i \leq k\}$ such that each element occurs at most once in each row and column. (Note that P is not necessarily a subset of the set of entries S of a latin square.) A *uniquely completable*

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set U is a partial latin square which is a subset of the entries of exactly one $n \times n$ latin square A. If no proper subset of U is a uniquely completable set, then U is also a *critical set* of A. For example, $U = \{(1,2;2),(1,4;4),(2,3;1),(3,4;2)\}$ is a uniquely completable set and also a critical set of the 4×4 latin square A shown in Figure 1.

$$\begin{array}{cccc} . & 2 & . & 4 \\ . & . & 1 & . \\ . & . & . & 2 \\ . & . & . & . \end{array} \quad \longrightarrow \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{array}$$
$$U \qquad\qquad\qquad A$$

Figure 1: A critical set and its unique completion

If the set of entries in a latin square A is $S = \{(r,c;e) \mid a_{rc} = v_e\}$, and $\pi(a,b,c)$, where $\{a,b,c\} = \{1,2,3\}$, is the permutation $\pi(a,b,c) : (x_1\ x_2\ x_3) \rightarrow (x_a\ x_b\ x_c)$, then the set of entries $S_{\pi(a,b,c)} = \{\pi(a,b,c) : (r,c;e) \mid (r,c;e) \in S\}$ forms a latin square $A_{\pi(a,b,c)}$ known as a *conjugate* of A or more specifically the *(a,b,c)-conjugate* of A. Figure 2 shows $U_{\pi(2,3,1)} = \{(2,2;1),(4,4;1),(3,1;2),(4,2;3)\}$, a conjugate of the critical set U from Figure 1, and its unique completion $A_{\pi(2,3,1)}$.

$$\begin{array}{cccc} . & . & . & . \\ . & 1 & . & . \\ 2 & . & . & . \\ . & 3 & . & 1 \end{array} \quad \longrightarrow \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \\ 2 & 4 & 1 & 3 \\ 4 & 3 & 2 & 1 \end{array}$$
$$U_{\pi(2,3,1)} \qquad\qquad A_{\pi(2,3,1)}$$

Figure 2: Conjugates of the critical set and latin square from Figure 1

It follows immediately from the definitions that if U is a uniquely completable (critical) set of a latin square A then $U_{\pi(a,b,c)}$ is a uniquely completable (critical) set of the conjugate latin square $A_{\pi(a,b,c)}$. Since the operations of row rearrangement, column rearrangement, element renumbering, and conjugation preserve uniquely completable sets and critical sets, we will call two uniquely completable sets, or critical sets *equivalent* if they can be transformed into one another by this set of operations. The corresponding latin squares are *main class isotopic* [4]. In Figure 2, U is equivalent to $U_{\pi(2,3,1)}$ and A is main class isotopic to $A_{\pi(2,3,1)}$.

A *back-circulant* latin square (see Figure 3) contains the set of entries $S = \{(r,c;e) \mid e \equiv r + c - 1 \bmod n\}$.

2 Background

Among the most interesting problems concerning critical sets in latin squares are the determination of $scs(n)$ and $lcs(n)$, the sizes of the smallest and largest critical sets in $n \times n$ latin squares. In this paper we will be concerned only with $scs(n)$. For convenience, we make the following definition.

Definition 1 The set of elements forming a $\lceil (n-1)/2 \rceil \times \lceil (n-1)/2 \rceil$ triangle in the upper left corner and a $\lfloor (n-1)/2 \rfloor \times \lfloor (n-1)/2 \rfloor$ triangle in the lower right corner of a back-circulant $n \times n$ latin square, as shown in Figure 3, will be called the *standard* critical set.

The first results were obtained by Nelder [11], Curran and van Rees [6] and Smetaniuk [12] in 1977-79. It was shown that $scs(n) \leq \lfloor n^2/4 \rfloor$ by proving that the set of elements forming the standard critical set is a uniquely completable set, and also a critical set when n is even. It was later shown to be a critical set for odd values of n by Cooper, Donovan, and Seberry [5]. Critical sets in other main classes of small latin squares were investigated by Howse [7]. Exhaustive backtrack methods showed that for the small cases where $n \leq 5$ the standard critical set was in fact the smallest possible critical set. It has recently been shown that this is also true for $n = 6$ (Bate and van Rees [2]), and $n = 7$ (Adams and Khodkar [1]). We summarize these results in Lemma 1. It has been conjectured [10] that equality holds for all values of n.

Lemma 1 *$scs(n) = \lfloor n^2/4 \rfloor$ for $n \leq 7$*

$$\begin{array}{ccccc} \underline{\mathbf{1}} & \underline{\mathbf{2}} & 3 & 4 & 5 \\ \underline{\mathbf{2}} & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & \underline{\mathbf{3}} \\ 5 & 1 & 2 & \underline{\mathbf{3}} & \underline{\mathbf{4}} \end{array} \qquad \begin{array}{cccccc} \underline{\mathbf{1}} & \underline{\mathbf{2}} & \underline{\mathbf{3}} & 4 & 5 & 6 \\ \underline{\mathbf{2}} & \underline{\mathbf{3}} & 4 & 5 & 6 & 1 \\ \underline{\mathbf{3}} & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & \underline{\mathbf{4}} \\ 6 & 1 & 2 & 3 & \underline{\mathbf{4}} & \underline{\mathbf{5}} \end{array}$$

Figure 3: The standard critical sets (underlined) in 5×5 and 6×6 back-circulant latin squares

3 Small values of n

For latin squares of order $n \leq 6$, an exhaustive backtrack search is sufficient to enumerate the critical sets of a particular size. In Figure 4, the number of minimal critical sets of size $\lfloor n^2/4 \rfloor$ and the number of near-minimal critical sets of size $\lfloor n^2/4 \rfloor + 1$ are listed for back-circulant latin squares of order $3 \leq n \leq 6$. These form groups of equivalent critical sets, and the number of inequivalent critical sets of each size is also given.

In this paper, we will investigate two of the most interesting features of this table. First, note that for all $3 \leq n \leq 6$, there is only one inequivalent critical set of minimal size. By enumerating all inequivalent critical sets of size $\lfloor n^2/4 \rfloor$ in the back-circulant latin squares of order $7 \leq n \leq 12$ (except $n = 11$), we will show that this pattern continues, giving strong evidence for a conjecture that it is true for all n.

Second, there are no critical sets at all of size 10 (one more than the minimum) for $n = 6$. We will also enumerate all inequivalent critical sets of size $\lfloor n^2/4 \rfloor + 1$ in

n	size	number	inequivalent
3	2	9	1
	3	18	1
4	4	32	1
	5	576	4
5	6	50	1
	7	1000	4
6	9	72	1
	10	0	0

Figure 4: The number of minimal or near-minimal critical sets, and the number of inequivalent families, in $n \times n$ back-circulant latin squares

the back-circulant latin squares of even orders $n = 8$, $n = 10$, and $n = 12$, giving strong evidence for a conjecture that this pattern also continues for even values of n ($n > 4$). (In [3], Cavenagh, Donovan, and Khodkar have recently shown that critical sets of these sizes do exist for all odd values of n.)

We will make use of an exhaustive backtrack search algorithm, more fully described in [2], which will find all critical sets of a given size in a particular latin square. Since the search space is prohibitively large for $n > 6$, our program allows two arbitrary sets of entries U and R to be specified, where all of the elements of U will automatically be used in the critical sets, and all of the elements of R will be rejected from consideration and will not appear in any critical set. By making extensive use of the automorphism groups of the latin squares, we are able to produce suitable sets of this kind and thus reduce the level of computation required to acceptable levels.

4 Preliminary results

In [1], Adams and Khodkar define and prove the following. Let the *shape* of a partial latin square P be the set of cells $\{(r, c) \mid (r, c; e) \in P\}$. Two partial latin squares P and P' with the same shape are *mutually balanced* if the set of elements in row r_i (and column c_i) of P is equal to the set of elements in row r_i (and column c_i) of P', and are *disjoint* if no entry in P contains the same element as corresponding entry in P'. If P and P' are disjoint and mutually balanced partial latin squares of the same shape, then P is a *latin interchange* and P' is its *disjoint mate* (and vice-versa). The Lemmas below follow immediately.

Lemma 2 *A partial latin square U contained in a latin square L is a uniquely completable set if and only if for any latin interchange $P \in L$, $|U \cap P| \geq 1$.*

Lemma 3 *A uniquely completable set C contained in a latin square L is a critical set if and only if for any element $X = (r, c; e) \in C$, there exists a latin interchange $P \in L$ such that $P \cap C = \{X\}$.*

The following lemma appears in [2]:

Lemma 4 *If S is a latin subsquare of a latin square L, and U is a uniquely completable set of L, then $U \cap S$ is a uniquely completable set of S.*

For latin squares of order $n \geq 7$, simple exhaustive backtrack searches become infeasible. We will make extensive use of the automorphism groups of these squares in order to greatly reduce the search time required. We first create a graph corresponding to the latin square in the following standard way: Create $3n$ vertices r_i, c_i, and e_i $(1 \leq i \leq n)$ corresponding to the rows, columns, and elements of the latin square, respectively. Create n^2 vertices v_{ij} corresponding to the entries a_{ij} in the square. Create $3n^2$ edges by connecting v_{ij} to r_i, c_j, and $e_{a_{ij}}$.

The program *Groups and Graphs* [8, 9], given such a graph, will very quickly produce its automorphism group and the orbits of the vertices. This will allow us to quickly identify equivalent rows, columns, elements, and entries in the latin square. Note that the vertices corresponding to the entries have degree 3 and the vertices corresponding to the rows, columns, and elements have degree n, preventing automorphisms which map entries onto rows, columns, or elements (for $n > 3$), but allowing automorphisms to map rows, columns, and elements onto each other which provides the desired conjugation operations.

When a set of entries U has been chosen to be included in a potential critical set, or to be singled out for any other reason, an additional vertex x may be added with edges joining x to each vertex in U. The "special" entries will be forced into separate orbits since they will now have degree 4 rather than degree 3, and no new automorphisms can be created as long as $n > 4$. (It is possible but unlikely that x itself will become equivalent to an existing vertex. For example in the case that U consists of all of the entries in row 1, x will be equivalent to r_1. Connecting x to a vertex y of degree 1 will solve this problem.)

Groups and Graphs is also capable of accepting a text file describing a large number of graphs, and, by producing a certificate for each, reducing them to a smaller number of inequivalent graphs in a relatively short period of time. We will also make frequent use of this operation.

5 n = 7

We are looking for a critical set of size $\lfloor 7^2/4 \rfloor = 12$ in a latin square of order 7, and so it must contain at least 2 entries from some row r. All entries are initially in the same orbit of the automorphism group, and so we may choose (1,1;1) as the first element to be used without loss of generality. Once it has been chosen, all other elements in row 1 are still in the same orbit of the automorphism group, and so we may choose (1,2;2) as the second element again without loss of generality.

Once (1,1;1) and (1,2;2) have been chosen, an automorphism group of order 4 remains in which (2,2;3), (2,7;1), (7,1;7) and (7,3;2) are in the same orbit. The critical set may contain one or more of these elements, in which case (2,2;3) may be chosen without loss of generality (case 1), or it may contain none of them (case 2).

In case 1, an automorphism group of order 2 remains in which (1,3;3) and (7 2;1) are in the same orbit. The critical set may contain one or both of these elements, in which case (1,3;3) may be chosen without loss of generality (case 1.1), or it may contain neither of them (case 1.2). In case 2, an automorphism group of order 4 remains in which (2,4;5), (2,5;6), (7,5;4) and (7,6;5) are in the same orbit, yielding case 2.1 in which (2,4;5) is chosen, and case 2.2 in which all four of these elements are rejected.

In cases 1.1 and 2.1 the automorphism group is now of order 1, and so this technique cannot be applied again. In case 1.2, (1,4;4) and (6,2;7) are equivalent, and the same method gives case 1.2.1 (which cannot be further refined) and case 1.2.2, which can be refined once more using the equivalent elements (1,5;5) and (5,2;6). In case 2.2, an automorphism group of order 4 remain in which (2,3;4), (2,6;7), (7,4;3), and (7,7;6) are in the same orbit. Case 2.2.1, in which (2,3;4) is used, cannot be refined further. In Case 2.2.2, in which all four of these elements are eliminated, we find that all elements in rows 2 and 7 have now been rejected with the exception of (2,1;2) and (7,2;1), and that these two are equivalent. At least one of them must be included in any critical set, since otherwise we would have no element from either of those two rows, and that is impossible by Lemma 2 since two complete rows of a latin square always form a latin interchange (simply interchange the rows to obtain a disjoint mate). So case 2.2.2 may be finished off by including (2,1;2) without loss of generality.

A total of 7 cases (each consisting of the pair of sets U and R described earlier) were generated. A backtrack search was used to find all possible critical sets with these 7 cases as starting points. A total of 763,565,303 potential critical sets were generated and checked in approximately 16 hours of CPU time on a PowerPC 750 (G3). Only 2 critical sets were found, and both were shown to be equivalent to the standard critical set by using *Groups and Graphs*. Since it is shown in [1] that no latin square of order 7, other than the back-circulant square, contains a critical set of size 12, Lemma 5 has been proved.

Lemma 5 *Any critical set of size* $scs(7) = 12$ *in a latin square of order* 7 *is equivalent to the standard critical set.*

6 n = 8

The back-circulant latin square of order 8 contains four disjoint back-circulant latin subsquares of order 4. By row and column rearrangement and renumbering, they can be placed in four separate quadrants as shown in Figure 5.

We are looking for a critical set of size $\lfloor 8^2/4 \rfloor = 16$. By Lemma 4, each quadrant must contain a uniquely completable set. By Lemma 1, the minimum size of such a set is 4. Therefore each quadrant must contain exactly 4 elements of the critical set. Since the critical set of size 4 in a back-circulant latin square of order 4 is unique up to equivalence, the elements (1 2;2), (2,1;2), (2,2;3), and (3,3;1) may be chosen in the upper left quadrant without loss of generality. There are 32 critical sets of size

$$
\begin{array}{cccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \\
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\
4 & 1 & 2 & 3 & 8 & 5 & 6 & 7 \\
\hline
5 & 6 & 7 & 8 & 2 & 3 & 4 & 1 \\
6 & 7 & 8 & 5 & 3 & 4 & 1 & 2 \\
7 & 8 & 5 & 6 & 4 & 1 & 2 & 3 \\
8 & 5 & 6 & 7 & 1 & 2 & 3 & 4
\end{array}
$$

Figure 5: The 8×8 back-circulant latin square written as four quadrants

4 in a back-circulant latin square of order 4 (all of which are equivalent). There are therefore $32 \times 32 = 1024$ ways to choose 4 elements from the top right quadrant and 4 elements from the lower left quadrant, giving the first 12 elements of 1024 potential critical sets. The program *Groups and Graphs* was used to determine that 104 of these are inequivalent. A simple backtrack search was performed on each of these to determine whether or not an additional 4 elements could be added in the lower right quadrant to produce a critical set of size 16. Only two solutions were found, both equivalent to the standard critical set, proving Lemma 6.

Lemma 6 *Any critical set of size* 16 *in a back-circulant latin square of order* 8 *is equivalent to the standard critical set.*

Consider a critical set of size 17 in a back-circulant latin square of order 8. Since there must be at least 4 elements in each quadrant, there must be exactly 4 elements in three quadrants, and exactly 5 elements in the fourth. Without loss of generality, we may choose to place 5 elements in the lower right quadrant. Even with the lower right quadrant differentiated from the other three, we may still choose the elements (1 2;2), (2,1;2), (2,2;3), and (3,3;1) without loss of generality, and there are still 104 inequivalent ways to place two critical sets of size four in the upper right and lower left. The backtrack search was extended to look for an additional 5 elements that could be added in the lower right quadrant to produce a critical set of size 17. The expected 24 solutions were generated, consisting of the two solutions of size 16 with each of the other 12 elements from the lower right quadrant added to them. Since they all contain a critical set as a proper subset, they are not critical sets themselves, proving Lemma 7.

Lemma 7 *There are no critical sets of size* 17 *in a back-circulant latin square of order* 8.

7 n = 9

The back-circulant latin square of order 9 contains nine disjoint back-circulant latin subsquares of order 3. By row and column rearrangement and renumbering, they can be placed as shown in Figure 6.

1	2	3	4	5	6	7	8	9
2	3	1	5	6	4	8	9	7
3	1	2	6	4	5	9	7	8
4	5	6	7	8	9	2	3	1
5	6	4	8	9	7	3	1	2
6	4	5	9	7	8	1	2	3
7	8	9	2	3	1	5	6	4
8	9	7	3	1	2	6	4	5
9	7	8	1	2	3	4	5	6

Figure 6: The 9×9 back-ciculant latin square written as disjoint subsquares

We are looking for a critical set of size $\lfloor 9^2/4 \rfloor = 20$. By Lemma 4, the elements in each subsquare must form a uniquely completable set. By Lemma 1, the minimum size of such a set is 2. Therefore at least seven subsquares must contain exactly 2 elements of the critical set. Without loss of generality, we may assume than the one or two subsquares which contain more than 2 elements of the critical set are in the 4 subsquares in the bottom right corner, and that the five subsquares in the first three rows and/or columns each contain exactly 2 elements, which much form critical sets of those subsquares.

Even under this assumption, all 9 critical sets of size 2 in the upper-left subsquare are equivalent, and so we may choose (1,1;1) and (2 2;3) without loss of generality. There are $9 \times 9 = 81$ ways to choose critical sets of size 2 to be placed in the top-centre and centre-left subsquares. *Groups and Graphs* was used to show that of these, 15 are inequivalent. Critical sets of size 2 in the top-right and bottom-left subsquares can be added to each of these in $9 \times 9 = 81$ ways to give a total of $15 \times 81 = 1215$ ways to place 10 elements in the five subsquares in the first three rows and/or columns. *Groups and Graphs* was used to show that of these, 567 are inequivalent.

To complete the desired critical set, 10 additional elements must be added in the lower right, comprising a uniquely completable set in each of the four 3×3 subsquares. Unfortunately, this can be done in 2,774,574 ways, yielding a total of 1,573,183,458 potential critical sets, which is too large to be handled easily. The following technique can be used to reduce the number of potential critical sets to 14.

There is a convenient set of 1458 latin interchanges of size 12 in the back-circulant latin square of order 9, which can be generated as follows. Choose two of the appearances of some element e which are not in the same 3×3 subsquare. Let these be $(r_1, c_1; e)$ and $(r_2, c_2; e)$. This can be done in $9(\binom{9}{2} - 3\binom{3}{2}) = 243$ ways. Add the elements from the other two intersections of r_1, r_2 and c_1, c_2 and let them be $(r_1, c_2; e_1)$ and $(r_2, c_1; e_2)$. Now form a "chain" joining these two new elements by alternately moving row-wise and column-wise between e_1 and e_2, which in a back-circulant latin square of order 9 will always give the following sequence of exactly 8 additional elements:

$(r_3, c_2; e_2)$
$(r_3, c_3; e_1)$
$(r_4, c_3; e_2)$
$(r_4, c_4; e_1)$
$(r_5, c_4; e_2)$
$(r_5, c_5; e_1)$
$(r_6, c_5; e_2)$
$(r_6, c_1; e_1)$

There are 6 conjugations of each latin interchange generated in this way, forming a total of $6 \times 243 = 1458$ different (but equivalent) latin interchanges. An example generated starting from $(2, 8; 9)$ and $(6, 4; 9)$ is shown in Figure 7.

$$\begin{array}{ccccccccc}
1 & 2 & 3 & \underline{\mathbf{4}} & \underline{\mathbf{5}} & 6 & 7 & 8 & 9 \\
2 & 3 & 4 & \underline{\mathbf{5}} & 6 & 7 & 8 & \underline{\mathbf{9}} & 1 \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 \\
4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 \\
5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 & 4 \\
6 & 7 & 8 & \underline{\mathbf{9}} & 1 & 2 & 3 & \underline{\mathbf{4}} & 5 \\
7 & 8 & 9 & 1 & 2 & 3 & \underline{\mathbf{4}} & \underline{\mathbf{5}} & 6 \\
8 & 9 & 1 & 2 & 3 & \underline{\mathbf{4}} & \underline{\mathbf{5}} & 6 & 7 \\
9 & 1 & 2 & 3 & \underline{\mathbf{4}} & \underline{\mathbf{5}} & 6 & 7 & 8
\end{array}$$

Figure 7: A latin interchange

If for a potential critical set C and a latin interchange P we have $|C \cap P| \geq 1$, we will say that C *covers* P. By Lemma 2, we may reject any C which does not cover all of the 1458 latin interchanges constructed above. Each of the 567 inequivalent ways to place 10 elements in the first 3 rows and columns covers between 1136 and 1272 of these latin interchanges, leaving between 186 and 322 of them uncovered. The remaining 10 elements in the bottom right must be chosen so that not only will there be a uniquely completable set in each 3×3 subsquare, but also so that all of the remaining latin interchanges will be covered. A fairly short computer search (approximately 10 minutes of CPU time) reveals that this can be done in only 14 ways. Of these, 13 were found to have two completions (and so are not critical sets) and the remaining one is equivalent to the standard critical set, proving

Lemma 8 *Any critical set of size* $scs(9) = 20$ *in a latin square of order* 9 *is equivalent to the standard critical set.*

8 $\mathbf{n = 10}$

The back-circulant latin square of order 10 contains four disjoint back-circulant latin subsquares of order 5. By row and column rearrangement and renumbering, they can be placed in four separate quadrants as was done for the case $n = 8$.

We are looking for critical sets of size $\lfloor 10^2/4 \rfloor = 25$ (and also 26). By Lemma 4, each quadrant must contain a uniquely completable set. By Lemma 1, the minimum size of such a set is 6. Therefore, in a critical set of size 25 or 26, at least 2 quadrants must contain exactly 6 elements, forming critical sets of those subsquares. Since it has been shown that the critical set of size 6 in a 5×5 back-circulant latin square is unique (up to equivalence), we may choose the elements (1,1;1), (1,2;2), (2,1;2), (4,5;3), (5,4;3), and (5,5;4), without loss of generality, in the upper left quadrant.

Groups and Graphs was used to determine that, with these 6 elements selected, the latin square still has an automorphism group of order 12, and all three remaining quadrants are still equivalent. Therefore, without loss of generality, we may choose to place a second critical set of size 6 in the upper right quadrant. Of the 50 possible critical sets of this size (all equivalent), *Groups and Graphs* was again used to show that they produce only 14 inequivalent sets of 12 elements in the two upper subsquares.

The back-circulant latin square of order 10 also contains 25 disjoint latin subsquares of order 2. Since these are latin interchanges, by Lemma 3 any critical set must intersect each of them in at least one element. A critical set of size 25 must intersect each 2×2 subsquare in exactly one element. A critical set of size 26 must intersect one of them in two elements, and the rest in one element. Of the 14 inequivalent choices for the 12 elements in the two upper subsquares, only 3 of them intersect 12 distinct 2×2 subsquares, as required for the formation of a critical set of size 25. There are also 4 which intersect 11 distinct 2×2 subsquares, one of them twice and the rest once, which could yield a critical set of size 26. The other 7 choices may be eliminated from consideration, since they intersect 10 or fewer distinct 2×2 subsquares.

8.1 Critical sets of size 25

There are three cases to consider, consisting of the previously chosen elements (1,1;1), (1,2;2), (2,1;2), (4,5;3), (5,4;3), and (5,5;4) from the upper left, and the following elements from the upper right:

Case 1: (1,6;6), (1,7;7), (2,6;7), (4,10;8), (5,9;8), (5,10;9)
Case 2: (1,6;6), (2,7;8), (2,9;10), (4,6;9), (4,8;6), (5,9;8)
Case 3: (1,7;7), (2,6;7), (2,7;8), (3,8;10), (3,9;6), (4,8;6)

In all three cases, *Groups and Graphs* shows that the two lower quadrants are still equivalent after the selection of the upper 12 elements. Therefore, without loss of generality, we may choose to place 6 elements in the lower left, leaving 7 for the lower right. These must not intersect any of the 2×2 subsquares that intersect the upper 12 elements. Of the 50 ways that a critical set of size 6 may be placed in the lower left, there are only 2 which have this property in each of Cases 1 and 3, and none in Case 2. The choice of the remaining 7 elements in the lower right is forced, and must consist of the elements from the 7 2×2 subsquares which do not yet intersect the critical set. Conveniently, in every case, these 7 elements do form a critical set of size 7 in the lower right quadrant. The four possible ways to complete a potential critical set of size 25 are:

Case 1A: LL: (6,1;6), (6,2;7), (7,1;7), (9,5;8), (10,4;8), (10,5;9),
LR: (6,6;2), (8,10;3), (9,9;3), (9,10;4), (8,8;3), (8,9;4), (8,10;5)
Case 1B: LL: (6,4;9), (8,3;10), (9,2;10), (9,3;6), (10,4;8), (10,5;9)
LR: (6,8;4), (6,9;5), (7,8;5), (9,7;1), (10,8;3), (10,9;4), (10,10;5)
Case 3A: LL: (6,2;7), (7,1;7), (8,4;6), (8,3;10), (7,2;8), (9,3;6)
LR: (6,6;2), (6,10;1), (7,10;2), (9,9;3), (10,6;1), (10,7;2), (10,10;5)
Case 3B: LL: (6,4;9), (8,3;10), (9,2;10), (9,3;6), (10,4;8), (10,5;9)
LR: (6,9;5), (6,10;1), (7,8;5), (7,9;1), (7,10;2), (8,6;4), (10,10;5)

Groups and Graphs shows that Cases 1B and 3A are equivalent, and that Case 1A is equivalent to the standard critical set. A very short computer search (milliseconds of CPU time) reveals that Cases 1B/3A and 3B are not uniquely completable sets, proving

Lemma 9 *Any critical set of size* 25 *in a back-circulant latin square of order* 10 *is equivalent to the standard critical set.*

8.2 Critical sets of size 26

Since there are now 26 elements to be distributed among 25 2×2 disjoint latin subsquares, we may now allow 2 elements to appear in the same subsquare. In addition to the 3 cases listed in the previous section (in which there is no overlap of this kind), we must allow for the following 4 additional inequivalent ways to place elements in the upper right quadrant, in which there is a single element from a subsquare already covered by an element from the upper left:

Case 4: (1,6;6), (2,6;7), (3,7;9), (3,8;10), (2,10;6), (4,7;10)
Case 5: (1,6;6), (3,7;9), (4,8;6), (5,10;9), (1,8;8), (3,10;7)
Case 6: (1,7;7), (3,6;8), (4,6;9), (5,7;6), (5,8;7), (4,10;8)
Case 7: (1,7;7), (3,6;8), (4,7;10), (5,9;8), (3,9;6), (1,10;10)

There are 14 elements to be placed in the lower half, which may be distributed into the quadrants as 6+8, 8+6, or 7+7.

The 6+8 or 8+6 distribution: In all but Case 5, *Groups and Graphs* shows that the lower left and lower right quadrants are still equivalent, and so we may assume without loss of generality that there is a critical set of size 6 in the lower left. There are 50 such critical sets. Once one has been selected, the 18 elements chosen so far must intersect 17 or 18 distinct 2×2 subsquares. The three inequivalent ways of choosing elements which intersect 18 distinct subsquares were found in the previous section. A short search reveals that none of the possible choices yields elements which intersect exactly 17 distinct subsquares. In Case 5, we must also check all possible ways to choose a critical set of size 6 in the lower right quadrant, but again there are no solutions which intersect 17 distinct subsquares.

An exhaustive search was performed to determine whether or not an additional 8 elements could be added to any of the 3 inequivalent ways of choosing the initial 18 elements in the upper half and lower left. In order to provide a cross-check, these 8 elements were not restricted to the lower right quadrant. As expected, 75 solutions were produced, consisting of the unique standard critical set of size 25, with each of the other 75 elements added in turn. Since these are all supersets of the standard critical set, they are not critical sets themselves.

The 7+7 distribution: A uniquely completable set of size 7 must be placed in each of the two lower quadrants. There are 1,950 possible uniquely completable sets of size 7 in a back-circulant latin square of order 5, consisting of the 50 critical sets of size 6, each with one of the other 19 elements added to it, producing $50 \times 19 = 950$ sets, and 1000 critical sets of size 7 (4 inequivalent sets, each with 250 different ways of placing it).

Each of the 1,950 uniquely completable sets of size 7 in the lower left quadrant were added to each possible way to choose 12 elements in the upper half (Cases 1–7). The resulting sets of 19 elements may be rejected unless they intersect 18 or 19 distinct 2×2 subsquares. There are a total of 240 admissible sets (of the $7 \times 1,950 = 13,650$ possibilities). Adding a uniquely completable set of size 7 in the lower right must yield a set of 26 elements which intersects all 25 2×2 subsquares. This may be done in 152 ways (of the $240 \times 1,950 = 468,000$ possibilities). *Groups and Graphs* was used to reduce these to 60 inequivalent cases, of which 11 were found to be uniquely completable sets, and 49 were not.

Each other element was added in turn to the unique standard critical set of size 25, and *Groups and Graphs* was used to determine that, of the resulting 75 sets of size 26, 11 are inequivalent, and that they are the same 11 sets produced above. Therefore every uniquely completable set of size 26 is a superset of the standard critical set of size 25, and therefore none are critical sets. This proves

Lemma 10 *There are no critical sets of size* 26 *in a back-circulant latin square of order* 10*.*

9 n = 12

The back-circulant latin square of order 12 contains four disjoint back-circulant latin subsquares of order 6. By row and column rearrangement and renumbering, they can be placed in four separate quadrants, as was done in the previous sections. There are also 9 disjoint back-circulant latin subsquares of order 4, and each of these intersects each of the subsquares of order 6 in a subsquare of order 2. Figure 8 shows the four quadrants, and one of the subsquares of order 4 has been identified.

We are looking for critical sets of size $\lfloor 12^2/4 \rfloor = 36$ (and also 37). By Lemma 4, each quadrant must contain a uniquely completable set. By Lemma 1, the minimum size of such a set is 9. Therefore, in a critical set of size 36, each quadrant must contain exactly 9 elements, forming critical sets of those subsquares. In a critical set of size 37, one quadrant will contain a uniquely completable set of size 10. However, since there are no critical sets of size 10 in a back-circulant latin square of order 6,

1	2	**3**	4	5	**6**	**7**	8	9	**10**	11	12
2	3	4	5	6	1	8	9	10	11	12	7
3	4	5	6	1	2	9	10	11	12	7	8
4	5	**6**	1	2	**3**	**10**	11	12	**7**	8	9
5	6	1	2	3	4	11	12	7	8	9	10
6	1	2	3	4	5	12	7	8	9	10	11
7	8	9	10	11	12	2	3	4	5	6	1
8	9	**10**	11	12	**7**	**3**	4	5	**6**	1	2
9	10	11	12	7	8	4	5	6	1	2	3
10	11	12	7	8	9	5	6	1	2	3	4
11	12	**7**	8	9	**10**	**6**	1	2	**3**	4	5
12	7	8	9	10	11	1	2	3	4	5	6

Figure 8: The 12×12 back-ciculant latin square written as four quadrants, with a 4×4 subsquare identified

it must consist of a critical set of size 9 with one additional element. Therefore, all four quadrants must contain critical sets of size 9 in this case as well.

The unique critical set of size 9 may be placed in the upper left quadrant without loss of generality, giving the elements (1,1;1), (1,2;2), (1,3;3), (2,1;2), (2,2;3), (3,1;3), (5,6;4), (6,5;4), (6,6;5). There are 72 possible critical sets of size 9 in the upper right, and *Groups and Graphs* shows that 20 of these yield inequivalent sets of 18 elements when combined with the 9 elements already chosen in the upper left.

9.1 Critical sets of size 36

There are now 72 possible critical sets of size 9 that could be placed in the lower left quadrant, giving a total of $20 \times 72 = 1,440$ sets of 27 elements (from the upper half and lower left). This may be reduced by noting that any such set S will intersect any 4×4 latin subsquare P in exactly 3 elements, and those 3 elements must be a subset of some critical set of size 4 of P (the remaining element will come from the lower right). Of the 1,440 candidates, only 32 have this property, and *Groups and Graphs* shows that 14 of these are inequivalent. That is, there are 14 inequivalent ways to choose 27 elements which form a critical set of size 9 in each of the upper left, upper right, and lower left quadrants, and which intersect each of the 9 4×4 subsquares P in 3 elements which are a subset of some critical set of size 4 of P.

Once 3 elements from a critical set of size 4 in a 4×4 back-circulant latin square are chosen, the fourth element is forced. (It is simple to check that no two of the 32 such critical sets intersect in 3 elements.) This means that the choice of the 9 elements from the lower right quadrant is now forced in every case. In 5 of the 14 cases under consideration, these 9 elements do not form a critical set of the lower right quadrant, and thus these cases may be eliminated. Of the 9 that remain, *Groups and Graphs* shows that 4 of them are inequivalent. A short search (less than 1 second of CPU time) reveals that 3 of these 4 are not uniquely completable sets for the 12×12 latin square as a whole, and the remaining one is equivalent to the standard critical set. This proves

Lemma 11 *Any critical set of size* 36 *in a back-circulant latin square of order* 12 *is equivalent to the standard critical set.*

9.2 Critical sets of size 37

The first 18 elements may be placed in the upper half in 20 inequivalent ways, exactly as in the last section, without loss of generality, and the same 1,440 sets of 27 elements arise when a critical set of size 9 is added in the lower left. However, since there is now one extra element, only 8 of the 9 4×4 subsquares P must intersect these sets in 3 elements which are a subset of some critical set of size 4 of P. Relaxing this requirement yields the same 32 solutions as before, of which again 14 are inequivalent. No new possibilities arise.

Similarly, although we need now complete only 8 of the 9 critical sets of size 4 in the 4×4 subsquares, each of the 72 possible critical sets of size 9 in the lower right completes either all 9 of them (in the same 4 ways as before), or else 7 or fewer of them, again yielding no new possibilities. A short search reveals that adding a 37th element to any of the three final sets of 36 elements (not including the one that is equivalent to the standard critical set) does not yield a uniquely completable set. This completes

Lemma 12 *There are no critical sets of size* 37 *in a back-circulant latin square of order* 12.

10 Conclusions

Combining the results listed in Table 4 and Lemmas 5, 6, 8, 9, and 11, gives the following result.

Theorem 1 *For all $n \leq 12$ ($n \neq 11$), the size of the smallest critical set in a back-circulant latin square of order n is $\lfloor n^2/4 \rfloor$, and there exists only one inequivalent critical set of this size (the standard critical set).*

Since in all cases where $scs(n)$ is currently known ($n \leq 7$), only the back-circulant latin square of order n has a critical set of size $\lfloor n^2/4 \rfloor$, we feel there is enough evidence to make the following conjecture.

Conjecture 1 *For all n, the standard critical set in the back-circulant latin square of order n is the unique inequivalent minimal critical set of size $scs(n) = \lfloor n^2/4 \rfloor$.*

Combining the results of Table 4 and Lemmas 7, 10, and 12 we obtain

Theorem 2 *For all even n, $4 < n \leq 12$, there exist no critical sets of size $\lfloor n^2/4 \rfloor + 1$ in the back-circulant latin square of order n.*

We conjecture that the above theorem is true for all even $n > 4$. We note that Cavenagh, Donovan, and Khodkar have recently shown in [3] that for all odd values of n there do exist critical sets of size $\lfloor n^2/4 \rfloor + 1$.

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