Maximally edge-connected digraphs

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Abstract

In this paper we present some new sufficient conditions for equality of edge-connectivity and minimum degree of graphs and digraphs as well as of bipartite graphs and digraphs.

1. Introduction

We consider finite graphs and digraphs without loops and multiple edges. For a vertex $v \in V(D)$ of a digraph D, the *degree* of v, denoted by $d(v) = d(v, D)$, is defined as the minimum value of its out-degree $d^+(v) = d^+(v, D)$ and its in-degree $d^-(v) = d^-(v, D)$. The *degree sequence* of D is defined as the nonincreasing sequence of the degrees of the vertices of D. For two vertex sets X, Y of a digraph or graph let (X, Y) be the set of arcs or edges from X to Y. If D is a digraph (graph) and $X \subseteq V(D)$, then let $D[X]$ be the subdigraph (subgraph) induced by X. For other graph theory terminology we follow Chartrand and Lesniak [3].

Sufficient conditions for equality of edge-connectivity and minimum degree for graphs and digraphs were given by several authors, for example: Chartrand [2], Lesniak [11], Plesník [12], Goldsmith and White [10], Bollobás [1], Goldsmith and Entringer [9], Soneoka, Nakada, Imase, and Peyrat [14], Plesník and Znám [13], Volkmann [15], [16], Fàbrega and Fiol [7], [8], Xu [17], and Dankelmann and Volkmann [4], [5], [6].

In this paper we present degree sequence, distance, and neighborhood conditions for equality of edge-connectivity and minimum degree of graphs and digraphs as well as of bipartite graphs and digraphs, which extend and generalize several known results.

2. Degree sequence conditions

We start with a simple and well known, but useful lemma.

Lemma 2.1 *Let D be a digraph of edge-connectivity λ and minimum degree δ. If $\lambda < \delta$, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq \delta + 1$.*

Proof. Suppose, without loss of generality, that $|X| \leq \delta$. Then we obtain the contradiction

$$|X|\delta \leq \sum_{x \in X} d^+(x) \leq |X|(|X|-1) + \lambda \leq \delta(|X|-1) + \delta - 1.$$

□

Next we will improve the following result by Dankelmann and Volkmann [5].

Theorem 2.2 (Dankelmann, Volkmann [5] 1997) *Let D be a digraph of order n and edge-connectivity λ with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta$. If $\delta \geq \lfloor n/2 \rfloor$ or if $\delta \leq \lfloor n/2 \rfloor - 1$ and*

$$\sum_{i=1}^{2k} d_{n+1-i} \geq kn - 3$$

for some k with $2 \leq k \leq \delta$, then $\lambda = \delta$.

Theorem 2.3 *Let D be a digraph of order n and edge-connectivity λ with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta$. If $\delta \geq \lfloor n/2 \rfloor$ or if $\delta \leq \lfloor n/2 \rfloor - 1$ and*

$$\sum_{i=1}^{2k} d_{n+1-i} \geq \max\{k(n-1) - 1, (k-1)n + 2\delta - 1\}$$

for some k with $2 \leq k \leq \delta$, then $\lambda = \delta$.

Proof. Suppose to the contrary that $\lambda < \delta$. Then, according to Lemma 2.1, there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq \delta + 1$. This leads to $\delta \leq \lfloor n/2 \rfloor - 1$.

Now let $S \subseteq X$ and $T \subseteq Y$ be two k-sets with $2 \leq k \leq \delta$. If we choose S such that the number of arcs of (X,Y) incident with S is minimal, then we conclude

$$\sum_{v \in S} d^+(v) \leq k(|X|-1) + \delta - 1 - \min\{\delta - 1, |X| - k\}. \tag{1}$$

If we choose T such that the number of arcs of (X,Y) incident with T is minimal, then we conclude

$$\sum_{v \in T} d^-(v) \leq k(|Y|-1) + \delta - 1 - \min\{\delta - 1, |Y| - k\}. \tag{2}$$

Case 1. Let $\delta - 1 \leq |X| - k$ and $\delta - 1 \leq |Y| - k$. The inequalities (1) and (2) imply the following contradiction to the hypothesis:

$$\begin{aligned} \sum_{i=1}^{2k} d_{n+1-i} &\leq \sum_{v \in S \cup T} d(v) \leq k(|X|-1) + k(|Y|-1) \\ &= k(n-2) < k(n-1) - 1 \end{aligned}$$

Case 2. Let $\delta - 1 \le |X| - k$ and $\delta - 1 \ge |Y| - k$. In view of Lemma 2.1, we have $-|Y| \le -\delta - 1$. Hence (1) and (2) lead to the following contradiction to the hypothesis:

$$\begin{aligned}\sum_{i=1}^{2k} d_{n+1-i} &\le \sum_{v \in S \cup T} d(v) \le k(|X|-1) + k(|Y|-1) + \delta - 1 - |Y| + k \\ &\le kn - k + \delta - 1 - \delta - 1 = k(n-1) - 2 < k(n-1) - 1\end{aligned}$$

Case 3. The case $\delta - 1 \ge |X| - k$ and $\delta - 1 \le |Y| - k$ can be proved analogously to Case 2.

Case 4. Let $\delta - 1 \ge |X| - k$ and $\delta - 1 \ge |Y| - k$. Then (1) and (2) yield to the following contradiction to the hypothesis:

$$\begin{aligned}\sum_{i=1}^{2k} d_{n+1-i} &\le \sum_{v \in S \cup T} d(v) \\ &\le k(|X|-1) + \delta - 1 - |X| + k + k(|Y|-1) + \delta - 1 - |Y| + k \\ &\le kn + 2\delta - n - 2 = (k-1)n + 2\delta - 2 \\ &< (k-1)n + 2\delta - 1\end{aligned}$$

□

Corollary 2.4 *Let G be a graph of order n and edge-connectivity λ with degree sequence $d_1 \ge d_2 \ge \ldots \ge d_n = \delta$. If $\delta \ge \lfloor n/2 \rfloor$ or if $\delta \le \lfloor n/2 \rfloor - 1$ and*

$$\sum_{i=1}^{2k} d_{n+1-i} \ge \max\{k(n-1) - 1, (k-1)n + 2\delta - 1\}$$

for some k with $2 \le k \le \delta$, then $\lambda = \delta$.

Proof. Define the digraph D on the vertex set $V(G)$ by replacing each edge of G by two arcs in opposite directions and apply Theorem 2.3. □

3. Distance maximal sets

Let u and v be two vertices of a graph (digraph) D. The *distance* $d_D(u, v) = d(u, v)$ from u to v is the length of a shortest (directed) path from u to v in D. The *distance* $d_D(X, Y)$ from a vertex set X to a vertex set Y in D is given by

$$d_D(X, Y) = d(X, Y) = \min_{x \in X,\ y \in Y} d(x, y).$$

A pair of vertex sets X and Y of D with distance $d_D(X, Y) = k$, $k \in \mathbf{N}$, is called *k-distance maximal*, if there exist no vertex sets $X_1 \supseteq X$ and $Y_1 \supseteq Y$ with $X_1 \ne X$ or $Y_1 \ne Y$ such that $d_D(X_1, Y_1) = k$.

Theorem 3.1 *Let D be a strong connected digraph of edge-connectivity λ and minimum degree δ. If for all 3-distance maximal pairs of vertex sets X and Y there exists an isolated vertex u in $D[X \cup Y]$, then $\lambda = \delta$.*

Proof. Suppose to the contrary that $\lambda < \delta$. Then, there exist two disjoint sets $S, T \subset V(D)$ with $S \cup T = V(D)$ and $|(S,T)| = \lambda$. Now let $A \subseteq S$ and $B \subseteq T$ be the sets of vertices incident with an arc of (S,T). Furthermore, we define $A_0 = S - A$ and $B_0 = T - B$. In view of our assumption, we see that $|A|, |B| \leq \lambda < \delta$. Now we shall investigate two cases.

Case 1. Let $A_0, B_0 \neq \emptyset$. Then, clearly, the distance from A_0 to B_0 in D is finite and at least 3. Choose a 3-distance maximal pair X and Y with $A_0 \subseteq X$ and $B_0 \subseteq Y$. According to our assumption, there is an isolated vertex in $D[X \cup Y]$. If $u \in A_0$, then we obtain the contradiction $\delta \leq |N^+(u)| \leq |A| < \delta$. If $u \in B_0$, then we obtain the contradiction $\delta \leq |N^-(u)| \leq |B| < \delta$. If $u \in A$, then the definition of A and the fact that u has positive neighbors only in $A \cup B$ leads to the contradiction

$$\begin{aligned} \delta &\leq d^+(u) = |N^+(u) \cap B| + |N^+(u) \cap A| \\ &\leq |N^+(u) \cap B| + \sum_{x \in N^+(u) \cap A} |N^+(x) \cap B| \\ &\leq \sum_{x \in A} |N^+(x) \cap B| = \lambda < \delta. \end{aligned}$$

Analogously, $u \in B$ leads to the contradiction

$$\begin{aligned} \delta &\leq d^-(u) = |N^-(u) \cap A| + |N^-(u) \cap B| \\ &\leq |N^-(u) \cap A| + \sum_{x \in N^-(u) \cap B} |N^-(x) \cap A| \\ &\leq \sum_{x \in B} |N^-(x) \cap A| = \lambda < \delta. \end{aligned}$$

Case 2. Let $A_0 = \emptyset$ or $B_0 = \emptyset$. If $A_0 = \emptyset$, then we obtain the same contradiction for an arbitrary vertex $w \in A = S$ instead of $u \in A$. Finally, if $B_0 = \emptyset$, then we obtain the same contradiction for an arbitrary vertex $w \in B = T$ instead of $u \in B$.

Since we have discussed all possible cases, the proof is complete. $\square$

Corollary 3.2 (Dankelmann, Volkmann [4] 1995) *Let G be a connected graph of edge-connectivity λ and minimum degree δ. If for all 3-distance maximal pairs of vertex sets $X, Y \subset V(G)$ there exists an isolated vertex in $G[X \cup Y]$, then $\lambda = \delta$.*

Corollary 3.3 *If in a strong connected digraph D there exist no four vertices u_1, v_1, u_2, v_2 with*

$$d(u_1, u_2),\ d(u_1, v_2),\ d(v_1, u_2),\ d(v_1, v_2) \geq 3,$$

then $\lambda = \delta$.

Proof. If $X, Y \subseteq V(D)$ is a pair of 3-distance maximal sets, then the hypothesis yields $\min\{|X|, |Y|\} \leq 1$, and the desired result is immediate by Theorem 3.1. □

Corollary 3.4 (Plesník, Znám [13] 1989) *If in a connected graph G there exist no four vertices u_1, v_1, u_2, v_2 with*

$$d(u_1, u_2),\ d(u_1, v_2),\ d(v_1, u_2),\ d(v_1, v_2) \geq 3,$$

then $\lambda = \delta$.

Corollary 3.5 *If D is a digraph of diameter at most two, then $\lambda = \delta$.*

Corollary 3.6 (Plesník [12] 1975) *If G is a graph of diameter at most two, then $\lambda = \delta$.*

Corollary 3.7 *Let D be a digraph of order n. If $d^+(x) + d^-(y) \geq n - 1$ for all pairs of nonadjacent vertices x and y, then $\lambda = \delta$.*

Corollary 3.8 (Lesniak [11] 1974) *Let G be a graph of order n. If $d(x)+d(y) \geq n-1$ for all pairs of nonadjacent vertices x and y, then $\lambda = \delta$.*

Corollary 3.9 *Let D be a digraph of order n. If $n \leq 2\delta + 1$, then $\lambda = \delta$.*

Corollary 3.10 (Chartrand [2] 1966) *Let G be a graph of order n. If $n \leq 2\delta + 1$, then $\lambda = \delta$.*

4. Bipartite graphs and digraphs

In the sequel let D be a bipartite graph or a digraph with bipartition $V(D) = V' \cup V''$. We adopt the convention that for every subset X of $V(D)$, we denote the set $X \cap V'$ by X' and $X \cap V''$ by X''.

A pair of vertex sets X and Y of a bipartite graph or digraph D with $d_D(X', Y') = k$, and $d_D(X'', Y'') = k$, $k \in \mathbf{N}$, is called (k, k)-*distance maximal*, if there exist no vertex sets $X_1 \supseteq X$ and $Y_1 \supseteq Y$ with $X_1 \neq X$ or $Y_1 \neq Y$ such that $d_D(X_1', Y_1') = d_D(X_1'', Y_1'') = k$.

Analogously to Theorem 3.1, one can prove the following Theorems 4.1 and 4.6, which generalize the corresponding results in [4] for graphs.

Theorem 4.1 *Let D be a strong connected bipartite digraph of edge-connectivity λ and minimum degree δ. If for all (4,4)-distance maximal pairs of vertex sets X and Y there exists an isolated vertex in $D[X \cup Y]$, then $\lambda = \delta$.*

Corollary 4.2 *Let D be a bipartite digraph with bipartition $V(D) = V' \cup V''$. If $d(x, y) = 2$ for all different $x, y \in V'$, then $\lambda = \delta$.*

Corollary 4.3 (Dankelmann, Volkmann [4]) *Let G be a bipartite graph with bipartition $V(G) = V' \cup V''$. If $d(x, y) = 2$ for all different $x, y \in V'$, then $\lambda = \delta$.*

Corollary 4.4 *If D is a bipartite digraph of diameter at most three, then $\lambda = \delta$.*

Corollary 4.5 (Plesník, Znám [13] 1989) *If G is a bipartite graph of diameter at most three, then $\lambda = \delta$.*

Theorem 4.6 *Let D be a bipartite digraph of edge-connectivity λ, minimum degree δ, and diameter at most 4. If for all $(4, 4)$-distance maximal pairs of vertex sets X and Y with $|X'|, |Y'|, |X''|, |Y''| \geq 2$, there exists a vertex $u \in X \cup Y$ such that $d^+(u, D[X \cup Y]), d^-(u, D[X \cup Y]) \leq 1$, then $\lambda = \delta$.*

Also the next lemma, an analogue to Lemma 2.1 for bipartite digraphs, is well known but useful.

Lemma 4.7. *Let D be a bipartite digraph of edge-connectivity λ and minimum degree δ. If $\lambda < \delta$, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X, Y)| = \lambda$ such that $|X'|, |X''|, |Y'|, |Y''| \geq \delta$.*

Proof. In view of Lemma 2.1, $|X| \geq \delta + 1$. Hence, there exists a vertex $u \in X$ such that $N^+(u) \subseteq X$. If, without loss of generality, $u \in X''$, then it follows $|X'| \geq \delta$. Now there exists a vertex $v \in X'$ such that $N^+(v) \subseteq X$, and hence $|X''| \geq \delta$. Similarly one can show that $|Y'|, |Y''| \geq \delta$. □

As a generalization of a result of Goldsmith and White [10] for graphs, Xu [17] has given in 1994 the following sufficient condition for equality of edge-connectivity and minimum degree of a digraph.

Theorem 4.8 (Xu [17] 1994) *Let D be a digraph of order n. If there are $\lfloor n/2 \rfloor$ disjoint pairs of vertices (v_i, w_i) with $d(v_i) + d(w_i) \geq n$ for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$, then $\lambda = \delta$.*

In [5], Dankelmann and Volkmann gave a short proof of Xu's theorem. Applying the next theorem, we present two analogue results to Theorem 4.8 for bipartite digraphs.

Theorem 4.9 (Dankelmann, Volkmann [5] 1997) *Let D be a bipartite digraph of order n, edge-connectivity λ, and degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta$. If $\delta \geq \lceil (n + 1)/4 \rceil$ or if $\delta \leq \lfloor n/4 \rfloor$ and*

$$\sum_{i=1}^{k} (d_i + d_{n+1-2\delta+k-i}) \geq k(n - 2\delta) + 2\delta - 1$$

for some k with $1 \leq k \leq 2\delta$, then $\lambda = \delta$.

Theorem 4.10 *Let D be a bipartite (di)-graph of order $n \geq 2$, minimum degree δ, and edge-connectivity λ. If there are $\lfloor n/2 \rfloor$ disjoint pairs of vertices (v_i, w_i) with $d(v_i) + d(w_i) \geq n - 2\delta + 1$ for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$, then $\lambda = \delta$.*

Proof. If $\delta \geq \lceil (n+1)/4 \rceil$, then $\lambda = \delta$ by Lemma 4.7 or Theorem 4.9.

If $\delta \leq \lfloor n/4 \rfloor$, then from the $\lfloor n/2 \rfloor$ disjoint pairs of vertices choose $2\delta - 1$ pairs $(v'_1, w'_1), (v'_2, w'_2), \ldots, (v'_{2\delta-1}, w'_{2\delta-1})$ containing the $2\delta - 1$ vertices of lowest degree of v_i and w_i. Then we deduce for $k = 2\delta - 1$ that

$$\begin{aligned}
\sum_{i=1}^{k} (d_i + d_{n+1-2\delta+k-i}) &= \sum_{i=1}^{2\delta-1} (d_i + d_{n+1-2\delta+2\delta-1-i}) \\
&= \sum_{i=1}^{2\delta-1} (d_i + d_{n-i}) \\
&\geq \sum_{i=1}^{2\delta-1} (d(v'_i) + d(w'_i)) \\
&\geq (2\delta - 1)(n - 2\delta + 1) \\
&= (2\delta - 1)(n - 2\delta) + 2\delta - 1 \\
&= k(n - 2\delta) + 2\delta - 1.
\end{aligned}$$

Now Theorem 4.9 with $k = 2\delta - 1$ leads to $\lambda = \delta$. $\square$

For even n we can prove a slightly better result.

Theorem 4.11 *Let D be a bipartite (di)-graph of even order $n \geq 2$, minimum degree δ, and edge-connectivity λ. If there are $n/2 - 1$ disjoint pairs of vertices (v_i, w_i) with $d(v_i) + d(w_i) \geq n - 2\delta + 1$ for $i = 1, 2, \ldots, n/2 - 1$ and one further pair (v_j, w_j) with $d(v_j) + d(w_j) \geq n - 2\delta$, then $\lambda = \delta$.*

Proof. If $\delta \geq \lceil (n+1)/4 \rceil$, then $\lambda = \delta$ by Lemma 4.7 or Theorem 4.9. If $\delta \leq \lfloor n/4 \rfloor$, then from the $n/2$ disjoint pairs of vertices choose 2δ pairs $(v'_1, w'_1), (v'_2, w'_2), \ldots, (v'_{2\delta}, w'_{2\delta})$ containing the 2δ vertices of lowest degree of v_i and w_i and therefore containing the 2δ vertices of lowest degree in D. Then we deduce for $k = 2\delta$ that

$$\begin{aligned}
\sum_{i=1}^{k} (d_i + d_{n+1-2\delta+k-i}) &= \sum_{i=1}^{2\delta} (d_i + d_{n+1-2\delta+2\delta-i}) \\
&= \sum_{i=1}^{2\delta} (d_i + d_{n+1-i}) \\
&\geq \sum_{i=1}^{2\delta} (d(v'_i) + d(w'_i)) \\
&\geq (2\delta - 1)(n - 2\delta + 1) + n - 2\delta \\
&= 2\delta(n - 2\delta) + 2\delta - 1 \\
&= k(n - 2\delta) + 2\delta - 1.
\end{aligned}$$

Now Theorem 4.9 with $k = 2\delta$ leads to $\lambda = \delta$. $\square$

Example 4.12 Let H_1 and H_2 be two copies of the complete bipartite graph $K_{p,p}$ with $V(H_1) = \{x_1, x_2, \ldots, x_p\} \cup \{x'_1, x'_2, \ldots, x'_p\}$ and $V(H_2) = \{y_1, y_2, \ldots, y_p\} \cup \{y'_1, y'_2, \ldots, y'_p\}$. We define the bipartite graph G as the union of H_1 and H_1 together with the new edges $x_1y_1, x_2y_2, \ldots, x_{p-1}y_{p-1}$. Then $n(G) = n = 4p$, $\delta(G) = \delta = p$, and $\lambda(G) = \lambda = p-1 = \delta - 1$. Furthermore, $d(x_p) + d(x'_p) = d(y_p) + d(y'_p) = 2p = n - 2\delta$ and $d(x_i) + d(x'_i) = d(y_i) + d(y'_i) = 2p + 1 = n - 2\delta + 1$ for $i = 1, 2, \ldots, p-1$.

This example shows that Theorem 4.11 is best possible in the sense that the condition that there are $n/2-2$ disjoint pairs of vertices (v_i, w_i) with $d(v_i)+d(w_i) \geq n - 2\delta + 1$ for $i = 1, 2, \ldots, n/2 - 2$ and two further pairs with degree sum exactly $n - 2\delta$ does not guarantee $\lambda = \delta$.

For a non-complete digraph D let

$$NC2(D) = \min\{|N^+(x) \cup N^+(y)|, |N^-(x) \cup N^-(y)| \,:\, x, y \in V(D), d(x, y) = 2\}.$$

Theorem 4.13 *Let D be a strong connected bipartite digraph of order $n \geq 3$, edge-connectivity λ, and minimum degree δ. If $NC2(D) \geq \lceil (n+1)/4 \rceil$, then $\lambda = \delta$.*

Proof. Suppose to the contrary that $\lambda < \delta$ and thus $\delta \geq 2$. Then, by Lemma 4.7, there exist two disjoint sets $S, T \subset V(D)$ with $S \cup T = V(D)$, $|(S, T)| = \lambda$, and $|S'|, |S''|, |T'|, |T''| \geq \delta$. Now let $A \subseteq S$ and $B \subseteq T$ be the set of vertices incident with an arc of (S, T). Furthermore, we define $A_0 = S - A$ and $B_0 = T - B$. If $V' \cup V''$ is the bipartition of D, then let $A'_0 = A_0 \cap V'$, $A' = A \cap V'$, $A''_0 = A_0 \cap V''$, $A'' = A \cap V''$, $B'_0 = B_0 \cap V'$, $B' = B \cap V'$, $B''_0 = B_0 \cap V''$, and $B'' = B \cap V''$. In view of our assumption, we see that $|A|, |B| \leq \lambda < \delta$.

Firstly, we show that $A'_0, A''_0, B'_0, B''_0 \neq \emptyset$. Suppose that $A'_0 = \emptyset$. If there is a vertex $v \in A''_0$, then $\delta \leq |N^+(v)| = |N^+(v) \cap A'| \leq \lambda < \delta$, a contradiction. Consequently, $A''_0 = \emptyset$. If there is a vertex $v \in A'$, then

$$\begin{aligned} \delta &\leq |N^+(v)| = |N^+(v) \cap A''| + |N^+(v) \cap B''| \\ &\leq |N^+(v) \cap B''| + \sum_{x \in N^+(v) \cap A''} |N^+(x) \cap B'| \\ &\leq \sum_{x \in A} |N^+(x) \cap B| = \lambda < \delta. \end{aligned}$$

a contradiction and hence $A' = \emptyset$. Analogously, one can show that $A'' = \emptyset$. This leads to the contradiction $S = \emptyset$ and so $A'_0 \neq \emptyset$. Similar to this proof A''_0, B'_0, and B''_0 are nonempty.

Secondly, we show that $|A'_0|, |A''_0|, |B'_0|, |B''_0| \geq 2$. Without loss of generality, suppose that A'_0 consists of a single vertex u. Then it follows for each $x \in A''_0$

$$\delta \leq |N^+(x)| = |N^+(x) \cap A'_0| + |N^+(x) \cap A'| \leq 1 + \lambda \leq \delta$$

and thus $|A'| = \delta - 1$ and $|A''| = 0$. Therefore, $|A_0''| \geq 2$ and $N^+(x) = A' \cup \{u\}$ for each vertex $x \in A_0''$. Since $A'' = \emptyset$ and $\delta \geq 2$, the vertex u has at least two positive neighbors y, z in A_0''. Consequently, $d(y, z) = 2$ and hence the hypothesis implies

$$\frac{n+1}{4} \leq NC2(D) \leq |N^+(y) \cup N^+(z)| = |A'| + 1 = \delta.$$

This yields $n \leq 4\delta - 1$, a contradiction to Lemma 4.7 or Theorem 4.9. Finally, we distinguish two cases.

Case 1. Assume that the four sets A_0', A_0'', B_0', B_0'' contain vertices of distance two. If $x, y \in A_0'$ with $d(x, y) = 2$, then it follows from $N^+(x) \cup N^+(y) \subseteq A_0'' \cup A''$ and $NC2(D) \geq (n+1)/4$ that $|A_0'' \cup A''| \geq (n+1)/4$. Analogously, we obtain

$$|A_0' \cup A'|,\ |B_0' \cup B'|,\ |B_0'' \cup B''| \geq \frac{n+1}{4}.$$

These inequalities lead to the contradiction

$$n = |A_0' \cup A'| + |A_0'' \cup A''| + |B_0' \cup B'| + |B_0'' \cup B''| \geq 4\,\frac{n+1}{4} = n + 1.$$

Case 2. Assume that at least one of the sets A_0', A_0'', B_0', B_0'', say A_0', does not contain two vertices of distance two. Because of $|A''|, |A'| < \delta$, each vertex $u \in A_0'$ has at least one positive neighbor u' in A_0'', and u' has at least one positive neighbor v in A_0'. Since A_0' does not contain two vertices of distance two, it follows $u = v$ and hence $|A'| = \delta - 1$ and $|A''| = 0$.

Since no two vertices of A_0' have distance two, we deduce $(N^+(u) \cap A_0'') \cap (N^+(v) \cap A_0'') = \emptyset$ for all $u, v \in A_0'$ with $u \neq v$. Because of $|A_0'| \geq 2$ and $|A''| = 0$, we therefore obtain $|A_0''| \geq 2\delta$ and so $|S| \geq 3\delta + 1$. Hence, the bound $|T| \geq 2\delta$ implies $n \geq 5\delta + 1$.

In addition, let $x, y \in N^+(u)$ with $x \neq y$ for an $u \in A_0'$. Then, we have seen above that $N^+(x) = N^+(y) = A' \cup \{u\}$. Hence, $d(x, y) = 2$ and we arrive finally at the contradiction

$$\frac{5\delta + 2}{4} \leq \frac{n+1}{4} \leq NC2(D) \leq |N^+(x) \cup N^+(y)| = \delta.$$

□

Corollary 4.14 (Dankelmann, Volkmann [4] 1995) *Let G be a connected bipartite graph of order $n \geq 3$, edge-connectivity λ, and minimum degree δ. If $NC2(G) \geq (n+1)/4$, then $\lambda = \delta$, where* $NC2(G) = \min\{|N(x) \cup N(y)| : x, y \in V(G), d(x, y) = 2\}$.

References

[1] B. Bollobás, On graphs with equal edge-connectivity and minimum degree, *Discrete Math.* **28** (1979), 321–323.

[2] G. Chartrand, A graph-theoretic approach to a communication problem, *SIAM J. Appl. Math.* **14** (1966), 778–781.

[3] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, 3rd Edition, Wadsworth, Belmont, CA, 1996.

[4] P. Dankelmann and L. Volkmann, New sufficient conditions for equality of minimum degree and edge-connectivity, *Ars Combin.* **40** (1995), 270–278.

[5] P. Dankelmann and L. Volkmann, Degree sequence conditions for maximally edge-connected graphs and digraphs, *J. Graph Theory* **26** (1997), 27–34.

[6] P. Dankelmann and L. Volkmann, Degree sequence conditions for maximally edge-connected graphs depending on the clique number, *Discrete Math.* **211** (2000), 217–223.

[7] J. Fàbrega and M.A. Fiol, Maximally connected digraphs, *J. Graph Theory* **13** (1989), 657–668.

[8] J. Fàbrega and M.A. Fiol, Bipartite graphs and digraphs with maximum connectivity. *Discrete Appl. Math.* **69** (1996), 271–279.

[9] D.L. Goldsmith and R.C. Entringer, A sufficient condition for equality of edge-connectivity and minimum degree of a graph, *J. Graph Theory* **3** (1979), 251–255.

[10] D.L. Goldsmith and A.T. White, On graphs with equal edge-connectivity and minimum degree, *Discrete Math.* **23** (1978), 31–36.

[11] L. Lesniak, Results on the edge-connectivity of graphs, *Discrete Math.* **8** (1974), 351–354.

[12] J. Plesník, Critical graphs of given diameter, *Acta Fac. Rerum Natur. Univ. Comenian Math.* **30** (1975), 71–93.

[13] J. Plesník and S. Znám, On equality of edge-connectivity and minimum degree of a graph, *Arch. Math.* (Brno) **25** (1989), 19–25.

[14] T. Soneoka, H. Nakada, M. Imase, and C. Peyrat, Sufficient conditions for maximally connected dense graphs, *Discrete Math.* **63** (1978), 53–66.

[15] L. Volkmann, Bemerkungen zum p-fachen Kantenzusammenhang in Graphen, *An. Univ. Buccuresti Mat.* **37** (1988), 75–79.

[16] L. Volkmann, Edge-connectivity in p-partite graphs, *J. Graph Theory* **13** (1989), 1–6.

[17] J.-M. Xu, A sufficient condition for equality of arc-connectivity and minimum degree of a digraph, *Discrete Math.* **133** (1994), 315–318.

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