Intersection numbers of Latin squares with their own orthogonal mates*

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Abstract

Let $J^*(v)$ be the set of all integers k such that there is a pair of Latin squares L and L' with their own orthogonal mates on the same v-set, and with L and L' having k cells in common. In this article we completely determine the set $J^*(v)$ for integers $v \geq 24$ and $v = 1, 3, 4, 5, 8, 9$. For $v = 7$ and $10 \leq v \leq 23$, there are only a few cases left undecided for the set $J^*(v)$.

1 Introduction

A Latin square of order v is a $v \times v$ array in which each cell contains a single element from a v-set S, such that each element occurs exactly once in each row and exactly once in each column.

Let S and S' be v-sets. Two Latin squares $L = (a_{ij})$ on symbol set S and $L' = (b_{ij})$ on symbol set S' are orthogonal if every element in $S \times S'$ occurs exactly once among

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the v^2 pairs (a_{ij}, b_{ij}), $1 \le i, j \le v$. Bose, Parker and Shrikhande [1] proved that a pair of orthogonal Latin squares of order v exists if and only if $v \ne 2, 6$. A Latin square L of order v is said to possess an orthogonal mate if there exists a Latin square L' of the same order such that L and L' are orthogonal. A Latin square of order v with an orthogonal mate is equivalent to a resolvable $TD(3, v)$.

Denote by $J(v)$ the set of all integers k such that there is a pair of Latin squares L and L' on the same v-set having k cells in common. Let $S(t)$ denote the set of all non-negative integers less than or equal to t, with the exceptions of $t-5$, $t-3$, $t-2$ and $t-1$. Define $I(v) = S(v^2)$. Fu [5] determined completely the set $J(v)$ and proved that $J(v) = I(v)$ for integer $v \ge 1$, except $J(3) = I(3) \setminus \{1, 2, 5\}$ and $J(4) = I(4) \setminus \{5, 7, 10\}$. Similarly, let $J^*(v)$ be the set of all integers k such that there is a pair of Latin squares L and L' with their own orthogonal mates on the same v-set, and L and L' have k cells in common. By Fu's result [5] and [1], $J^*(v) \subseteq J(v)$ for $v \ne 2, 6$.

In this article we will study the intersection problem for Latin squares with their own orthogonal mates.

2 Recursive constructions

Let X be a v-set and $\mathcal{P} = \{S_1, S_2, \cdots, S_k\}$ a partition of a subset S of X. An incomplete Latin square with k disjoint empty subarrays on S_1, S_2, $\cdots$, S_k respectively, denoted by $LS(v, |S_1|, |S_2|, \cdots, |S_k|)$, is an $|X|$ by $|X|$ array L indexed by X satisfying the following properties:

1. A cell of L either contains an element of X or is empty.

2. The subarrays indexed by $S_i \times S_i$ are empty, for $1 \le i \le k$ (these subarrays are called holes).

3. The elements occurring in row (or column) $s \in S_i$ of L are precisely those in $X \setminus S_i$.

4. The elements occurring in row (or column) $s \in X \setminus (\cup_{i=1}^{k} S_i)$ of L are precisely those in X.

The type of L is the multiset $\{|S_1|, |S_2|, \cdots, |S_k|\}$. Suppose that L and M are two Latin squares with k common disjoint empty subarrays on S_1, S_2, $\cdots$, S_k. We say L and M are orthogonal if their superposition yields every ordered pair in $X^2 \setminus (\cup_{i=1}^{k} S_i^2)$. We also say M is an orthogonal mate of L. The pair L and M will be denoted by MOLS$(v, n_1, n_2, \cdots, n_k)$ where $|S_i| = n_i$ for $1 \le i \le k$. If $n_1 = n_2 = \cdots = n_k = n$, we write briefly MOLS$(v, n^k)$ for MOLS$(v, n_1, n_2, \cdots, n_k)$.

Denote by $J^*(v, n)$ the set of all integers k such that there is a pair of $LS(v, n)$ L and L' with their own orthogonal mates on the same set and with the same empty subarray, and with L and L' having k cells in common. It is useful to note that if $v > n_1 + n_2 + \cdots + n_k$, then a MOLS$(v, 1, n_1, n_2, \cdots, n_k)$ exists if and only if a MOLS$(v, n_1, n_2, \cdots, n_k)$ exists. If any n_i is zero we will simply ignore it. It is easy to see that $J^*(v+1,1) = \{k-1: \ k \in J^*(v+1) \setminus \{0\}\}$. Next we quote a result as follows.

Lemma 2.1 [6] *For any integers v and n, a MOLS(v,n) exists if and only if $v \geq 3n$ and $(v,n) \neq (6,1)$.*

Theorem 2.2 *If $s \in J^*(v,n)$ and $t \in J^*(n)$, then $s+t \in J^*(v)$.*

Proof. Let $I_{v-n} = \{1,2,\cdots,v-n\}$ and $Y = \{\infty_1,\infty_2,\cdots,\infty_n\}$. Let A and B be $LS(v,n)$ with their own orthogonal mates on the set $I_{v-n} \cup Y$ with the same empty subarray on Y such that $|A \cap B| = s$. Let C and D be a pair of orthogonal Latin squares of order n on the set Y, C' and D' a pair of orthogonal Latin squares of order n on the set Y such that C and C' have $t \in J^*(n)$ cells in common. By filling the Latin squares C and C' into the holes of A and B, the resulting Latin squares of order v possess their own orthogonal mates which are obtained by filling Latin squares D and D' into the holes of the orthogonal mates of A and B. It is readily checked that the two resulting Latin squares have $s+t$ cells in common. This completes the proof. □

Theorem 2.3 *If $v \geq 3n$ and $n \geq 3$ $(n \neq 6)$, then $av + b(v-n) + k \in J^*(v)$ for any integers $a \in [0, v-n] \setminus \{v-n-1\}$, $b \in [0,n] \setminus \{n-1\}$ and $k \in J^*(n)$.*

Proof. Let $I_{v-n} = \{1,2,\cdots,v-n\}$ and $Y = \{\infty_1,\infty_2,\cdots,\infty_n\}$. By Lemma 2.1 there is a MOLS(v,n) A and B on the set $I_{v-n} \cup Y$ with the same empty subarray on Y. Let π be the element permutation acting on A and B as follows:

$$\pi = (1\ 2 \cdots\ v-n-a)(\infty_1\ \infty_2\ \cdots\ \infty_{n-b})$$

where $a \in [0, v-n] \setminus \{v-n-1\}$ and $b \in [0,n] \setminus \{n-1\}$. Then πA and πB is also a MOLS(v,n) on $I_{v-n} \cup Y$ with the empty subarray on Y at the same location as A and B. It is readily checked that A and πA have $av + b(v-n)$ cells in common. The conclusion follows from Theorem 2.2. □

Theorem 2.4 *If v is an integer and $v \neq 2,6$, then $tv \in J^*(v)$ for any integer $t \in [0,v] \setminus \{v-1\}$.*

Proof. For $v \neq 2,6$, there exists a Latin square L with an orthogonal mate on $I_v = \{1,2,\cdots,v\}$. Let π be the element permutation acting on L: $\pi = (1\ 2 \cdots\ v-t)$ for $t \in [0,v] \setminus \{v-1\}$. Then πL is also a Latin square with an orthogonal mate. It is readily checked that L and πL have tv cells in common. □

Theorem 2.5 *Let m and n be integers greater than or equal to 3, but not equal to 6. Then $\sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} \in J^*(mn)$ where each $k_{ij} \in J^*(m)$.*

Proof. Let $A = (a_{ij})_{n \times n}$ be a Latin square of order n with an orthogonal mate $B = (b_{ij})_{n \times n}$. For $i,j = 1, 2 \cdots, n$, let C_{ij} and D_{ij} be a pair of orthogonal Latin squares of order m, and C'_{ij} and D'_{ij} a pair of orthogonal Latin squares of order m such that C_{ij} and C'_{ij} have $k_{ij} \in J^*(m)$ cells in common. Define four Latin squares

L_1, L_2, L_1' and L_2' of order mn as follows:

$$L_1 = \begin{array}{ccc} (a_{11}, C_{11}) & \cdots & (a_{1n}, C_{1n}) \\ (a_{21}, C_{21}) & \cdots & (a_{2n}, C_{2n}) \\ \cdots & \cdots & \cdots \\ (a_{n1}, C_{n1}) & \cdots & (a_{nn}, C_{nn}) \end{array} \qquad L_2 = \begin{array}{ccc} (b_{11}, D_{11}) & \cdots & (b_{1n}, D_{1n}) \\ (b_{21}, D_{21}) & \cdots & (b_{2n}, D_{2n}) \\ \cdots & \cdots & \cdots \\ (b_{n1}, D_{n1}) & \cdots & (b_{nn}, D_{nn}) \end{array}$$

$$L_1' = \begin{array}{ccc} (a_{11}, C_{11}') & \cdots & (a_{1n}, C_{1n}') \\ (a_{21}, C_{21}') & \cdots & (a_{2n}, C_{2n}') \\ \cdots & \cdots & \cdots \\ (a_{n1}, C_{n1}') & \cdots & (a_{nn}, C_{nn}') \end{array} \qquad L_2' = \begin{array}{ccc} (b_{11}, D_{11}') & \cdots & (b_{1n}, D_{1n}') \\ (b_{21}, D_{21}') & \cdots & (b_{2n}, D_{2n}') \\ \cdots & \cdots & \cdots \\ (b_{n1}, D_{n1}') & \cdots & (b_{nn}, D_{nn}') \end{array}$$

where $(a, L) = ((a, l_{ij}))$ if $L = (l_{ij})$ is a Latin square. Then L_1 and L_2, L_1' and L_2' are two pairs of orthogonal Latin squares of order mn. It is easy to check that L_1 and L_1' have $\sum_{i=1}^n \sum_{j=1}^n k_{ij}$ cells in common. The conclusion follows immediately. □

Let Y_1 and Y_2 be n-sets such that $|Y_1 \cap Y_2| = l \geq 1$. Let $\mathcal{A}$ denote the set of all Latin squares on Y_1 with an orthogonal mate, and $\mathcal{B}$ the set of all Latin squares on Y_2 with an orthogonal mate. Define $J_l(n) = \{k : \ |A \cap B| = k \text{ for } A \in \mathcal{A}, B \in \mathcal{B}\}$.

Theorem 2.6 *Let v, n and l be integers such that $v \geq 3n$ and $n \geq 3$ $(n \neq 6)$ and $1 \leq l < n$. Then $av + b(v-n) + k \in J^*(v)$ for integers $a \in [0, v-2n+l]$, $b \in [0, l]$ and $k \in J_l(n)$.*

Proof. Let $I_{v-n} = \{1, 2, \cdots, v-n\}$ and $Y = \{\infty_1, \infty_2, \cdots, \infty_n\}$. By Lemma 2.1 there is a MOLS(v, n) A and B on the set $I_{v-n} \cup Y$ with the same empty subarray on Y. Let π be the element permutation acting on A and B as follows:

$$(\infty_1 \; 1 \; \infty_2 \; 2 \; \cdots \infty_{n-l-1} \; n-l-1 \; \infty_{n-l} \; \infty_{n-l+1} \; \cdots \; \infty_{n-b} \; n-l \; n-l+1 \; \cdots \; v-n-a)$$

where $1 \leq l < n$, $a \in [0, v-2n+l]$ and $b \in [0, l]$. Then πA and πB is also a MOLS(v, n) on $I_{v-n} \cup Y$ with the empty subarray on πY at the same location as A and B. Clearly, $|Y \cap \pi Y| = l$. Let C and D be a pair of orthogonal Latin squares of order n on the set Y, and C' and D' a pair of orthogonal Latin squares of order n on the set πY such that C and C' have $k \in J_l(n)$ cells in common. By filling the Latin squares C and C' into the holes of A and πA, the resulting two Latin squares of order v possess their own orthogonal mates which are obtained by filling Latin squares D and D' into the holes of B and πB. It is readily checked that the two resulting $LS(v)$ have $av + b(v-n) + k$ cells in common. This completes the proof. □

Theorem 2.7 *Let v, $n \geq 3$, $k \geq 2$ and l be integers such that $v \geq kn$ and $1 \leq l < n$. If there exists a MOLS(v, n^k), then $av + b(v-n) + \sum_{i=1}^k a_i \in J^*(v)$ where $a \in [0, v-kn]$, $b \in [0, kl]$ and $a_i \in J_l(n)$ for $i \in [1, k]$.*

Proof. Let $X = \{1, 2, \cdots, v-kn\} \cup (\cup_{i=1}^k Y_i)$ where $Y_i = \{x_1^{(i)}, x_2^{(i)}, \cdots, x_n^{(i)}\}$ for $i \in [1, k]$. Let A and B be a MOLS(v, n^k) on the set X with k common disjoint empty subarrays on Y_1, Y_2, $\cdots$, Y_k. For $1 \leq l < n$, $a \in [0, v-kn]$ and $b \in [0, kl]$, let

$b = sl + t$ where $0 \le t < l$. Then $0 \le i \le k$ and $n - t \ge 2$. Let $\pi = \pi_1 \cdot \pi_2$ be the element permutation acting on A and B as follows:

$$\pi_1 = (x_{t+1}^{(s+1)}\ x_{t+2}^{(s+1)}\ \cdots\ x_n^{(s+1)})(x_1^{(s+2)}\ x_2^{(s+2)}\ \cdots\ x_n^{(s+2)})\cdots(x_1^{(k)}\ x_2^{(k)}\ \cdots\ x_n^{(k)})$$

for $0 \le s \le k-1$ or $\pi_1 = (1)$ for $s = k$;

$$\pi_2 = \left[\prod_{i=l+1}^{n-1} (x_i^{(1)}\ x_i^{(2)}\ \cdots\ x_i^{(k)})\right](x_n^{(1)}\ x_n^{(2)}\ \cdots\ x_n^{(k)}\ a+1\ a+2\ \cdots\ v-kn).$$

Then πA and πB is also a MOLS(v, n^k) on X with k common disjoint empty subarrays on πY_1, πY_2, $\cdots$, πY_k at the same locations as A and B. It is easy to check that $|Y_i \cap \pi Y_i| = l$ for $i \in [1, k]$. For $i \in [1, k]$, let C_i and D_i be a pair of orthogonal Latin squares of order n on Y_i, and C_i' and D_i' a pair of orthogonal Latin squares of order n on πY_i such that C_i and C_i' have $a_i \in J_l(n)$ cells in common. By filling the Latin squares C_i, C_i' ($i \in [1, k]$) into the holes of A and πA respectively, the resulting Latin squares of order v possess their own orthogonal mates which are obtained by filling Latin squares D_i, D_i' ($i \in [1, k]$) into the holes of B and πB. It is readily checked that the two resulting $LS(v)$ have $av + b(v-n) + \sum_{i=1}^{k} a_i$ cells in common. This completes the proof. □

For $n \ge 4$ and $n \ne 6, 10$, it is well known that there are three mutually orthogonal Latin squares of order n. Now we assume that L_1, L_2 and $L_3 = (a_{ij})_{n\times n}$ are three mutually orthogonal Latin squares on $I_n = \{1, 2, \cdots, n\}$. Let $\mathcal{T}_k = \{(i, j) :\ a_{ij} = k\}$ for $k \in I_n$. Then L_1 and L_2 are orthogonal and have the same n disjoint transversals $\mathcal{T}_1$, $\mathcal{T}_2$, $\cdots$, $\mathcal{T}_n$. The following construction is to take the squares L_1 and L_2, and replace each cell of them by a $q \times q$ array; this array will in general either be a MOLS(q) or be combined with additional rows and columns to L_1 and L_2 to form a MOLS$(qn + x, x)$. For each cell in $\mathcal{T}_k$ ($k \in [1, n]$), we add x_k rows and columns to L_1 and L_2 using a MOLS$(q + x_k, x_k)$. The construction yields a MOLS$(qn + x, x)$ where $x = \sum_{k=1}^{n} x_k$.

Theorem 2.8 *Let q, n and x be integers and $n \ge 4$, $n \ne 6, 10$ and $1 \le x \le n$. Then $\sum_{i=1}^{xn} d_i + \sum_{i=xn+1}^{n^2} d_i \in J^*(qn + x, x)$ where all $d_i \in J^*(q+1, 1)$ for $1 \le i \le xn$ and $d_i \in J^*(q)$ for $xn + 1 \le i \le n^2$.*

Proof. Let $x_k = 1$ for $k \in [1, x]$ and 0 for $k \in [x+1, n]$. When $n \ge 4$ and $n \ne 6, 10$ and $1 \le x \le n$, let L_1, L_2 and $\mathcal{T}_k$ ($1 \le k \le n$) be as above. Then L_1 and L_2 are orthogonal and have the same n disjoint transversals $\mathcal{T}_1$, $\mathcal{T}_2$, $\cdots$, $\mathcal{T}_n$. For each cell $(i, j) \in \mathcal{T}_k$ ($k \in [1, n]$), let C_{ij} and D_{ij} be $LS(q + x_k, x_k)$ with their own orthogonal mates C_{ij}' and D_{ij}' such that C_{ij} and D_{ij} have $c_{ij} \in J^*(q + x_k, x_k)$ cells in common. For each cell in $\mathcal{T}_k$ ($k \in [1, n]$), we add x_k rows and columns to L_1 using C_{ij}. The resulting Latin square A is $LS(qn + x, x)$ with an orthogonal mate which is obtained by adding x_k rows and columns to L_2 using C_{ij}' for each cell in $\mathcal{T}_k$ ($k \in [1, n]$). Similarly, for each cell in $\mathcal{T}_k$ ($k \in [1, n]$), we add x_k rows and columns to L_1 using D_{ij}. The resulting Latin square A' is also $LS(qn + x, x)$ with an orthogonal mate which is obtained by adding x_k rows and columns to L_2 using D_{ij}' for each cell in $\mathcal{T}_k$

$(k \in [1,n])$. It is readily checked that A and A' have

$$\sum_{k=1}^{x} \sum_{(i,j)\in \mathcal{T}_k} c_{ij} + \sum_{k=x+1}^{n} \sum_{(i,j)\in \mathcal{T}_k} c_{ij}$$

cells in common. Hence $\sum_{i=1}^{xn} d_i + \sum_{i=xn+1}^{n^2} d_i \in J^*(qn+x, x)$ where all $d_i \in J^*(q+1,1)$ for $1 \le i \le xn$ and $d_i \in J^*(q)$ for $xn+1 \le j \le n^2$. □

3 The set $J^*(v)$ for $v = 3,4,5,7,8$

In this section we will consider the set $J^*(v)$ where $1 \le v \le 8$ and $v \ne 2,6$. Let L be a Latin square of order n on $I_n = \{1,2,\cdots,n\}$ with its own orthogonal mate L'. In what follows let π_r, π_c and π_e be row permutation, column permutation and element permutation. Then $\pi_r\pi_c\pi_e(L)$ is a Latin square with an orthogonal mate $\pi_r\pi_c\pi_e(L')$. Let $|L \cap \pi_r\pi_c\pi_e(L)| = k$ denote the fact that L and $\pi_r\pi_c\pi_e(L)$ have k cells in common.

Lemma 3.1 $J^*(1) = \{1\}$; $J^*(3) = \{0,3,9\}$; $J^*(4) = \{0,4,8,16\}$.

Proof. $J^*(1) = \{1\}$ is trivial. Apply Theorem 2.4 and $J^*(3) \subseteq J(3)$ to get $J^*(3) = \{0,3,9\}$.

Under row permutation and column permutation, there are only two $LS(4)$s A and its transpose $A^\top$ with their own orthogonal mates, where A is listed below:

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

1	3	4	5	2
4	2	5	3	1
5	1	3	2	4
2	5	1	4	3
3	4	2	1	5

1	3	4	5	2
3	2	5	1	4
4	5	3	2	1
5	1	2	4	3
2	4	1	3	5

It is easy to check that $J^*(4) = \{0,4,8,16\}$ by an exhausive search. □

Lemma 3.2 $J^*(5) = \{0\text{–}13, 15, 25\}$.

Proof. Under row permutation, column permutation and element permutation, there are only two $LS(5)$s with an orthogonal mate exhibited as above. The conclusion follows immediately by an exhaustive computer search. □

Lemma 3.3 $0, 7, 14, 21, 28, 35, 49 \subseteq J^*(7)$.

Proof. This follows immediately from Theorem 2.4. □

Lemma 3.4 17–20, 22–27, 29–33, 36, 37, 39–41, 45 $\subseteq J^*(7)$.

Proof. Let K_i $(i = 1,2,3,4,5)$ be Latin squares of order 7 with an orthogonal mate, as given in the Appendix. It is readily checked that:

$|K_1 \cap (2\ 5\ 6\ 7)_r(K_2)| = 17$;

$|K_2 \cap (2\ 3\ 4\ 5)_r(K_3)| = 18$;

$|K_1 \cap (2\ 3\ 4\ 5)_r(K_2)| = 19$;

$|K_2 \cap (1\ 4)_r(5\ 6\ 7)_r(K_3)| = 20;$
$|K_2 \cap (5\ 6\ 7)_r(K_3)| = 22;$
$|K_1 \cap (1\ 2\ 3\ 4)_r(K_2)| = 23;$
$|K_1 \cap (5\ 6\ 7)_r(K_2)| = 24;$
$|K_2 \cap (2\ 3\ 4)_r(K_3)| = 25;$
$|K_1 \cap (2\ 3\ 4)_r(K_2)| = 26;$
$|K_2 \cap (1\ 4)_r(6\ 7)_r(K_3)| = 27;$
$|K_2 \cap (6\ 7)_r(K_3)| = 29;$
$|K_1 \cap (1\ 2\ 4)_r(K_2)| = 30;$
$|K_1 \cap (6\ 7)_r(K_2)| = 31;$
$|K_2 \cap (1\ 2)_r(K_3)| = 32;$
$|K_1 \cap (1\ 2)_r(K_2)| = 33;$
$|K_1 \cap K_5| = 36;$
$|K_3 \cap K_4| = 37;$
$|K_1 \cap (1\ 4)_r(K_2)| = 39;$
$|K_2 \cap (1\ 4)_r(K_3)| = 41;$
$|K_2 \cap K_3| = 43;$
$|K_1 \cap K_2| = 45.$ □

Lemma 3.5 1–6, 8–13, 15, 16 $\subseteq J^*(7)$.

Proof. Let $\pi_r = (1\ 4)(2\ 3\ 6\ 7\ 5)$ and $\pi_c = (1\ 4)(2\ 3\ 6\ 7\ 5)$ be the row permutation and column permutation acting on the Latin square K_1 which comes from the Appendix. Let $K_6 = \pi_r\pi_c(K_1)$. Then K_6 has an orthogonal mate. It is readily checked that:

$|K_1 \cap (1\ 3\ 7\ 4)_e(2\ 6\ 5)_e(K_6)| = 1;$
$|K_1 \cap (1\ 3\ 7\ 4\ 2\ 6\ 5)_e(K_6)| = 2;$
$|K_1 \cap (1\ 6\ 4\ 5)_e(2\ 7\ 3)_e(K_6)| = 3;$
$|K_1 \cap (1\ 3)_e(2\ 6\ 5\ 7\ 4)_e(K_6)| = 4;$
$|K_1 \cap (1\ 7\ 6\ 3\ 5)_e(K_6)| = 5;$
$|K_1 \cap (1\ 5\ 7)_e(2\ 4\ 6)_e(K_6)| = 6;$
$|K_1 \cap (1\ 4)_e(2\ 3\ 6\ 7\ 5)_e(K_6)| = 8;$
$|K_1 \cap (1\ 4\ 7\ 5\ 2\ 3\ 6)_e(K_6)| = 9;$
$|K_1 \cap (1\ 2\ 5)_e(3\ 4)_e(K_6)| = 10;$
$|K_1 \cap (1\ 2\ 5\ 3\ 4)_e(K_6)| = 11;$
$|K_1 \cap (1\ 5\ 6\ 7\ 2\ 4\ 3)_e(K_6)| = 12;$
$|K_1 \cap (3\ 4)_e(2\ 5\ 6\ 7)_e(K_6)| = 13;$
$|K_1 \cap (1\ 2)_e(3\ 4)_e(5\ 6\ 7)_e(K_6)| = 15;$

$|K_1 \cap (3\ 4)_e(1\ 5\ 6\ 7\ 2)_e(K_6)| = 16.$ □

Theorem 3.6 $I(7) \setminus \{34, 38, 40, 42\} \subseteq J^*(7)$.

Proof. This follows immediately from Lemma 3.3 to Lemma 3.5. □

Lemma 3.7 $0, 8, 16, 24, 32, 40, 48, 64 \in J^*(8)$.

Proof. This follows immediately from Theorem 2.4. □

Lemma 3.8 2, 4, 6, 10–12, 14, 17–23, 25–31, 33, 35–39, 41–47, 49, 52, 53, 56, 57, $60 \in J^*(8)$.

Proof. Let L_i $(i = 1, 2, 3, 4)$ be Latin squares of order 8 with an orthogonal mate in Appendix. It is readily checked that

$|L_1 \cap (1\ 8)_r(2\ 3\ \cdots\ 6-t)_r(L_2)| = 6 + 8t$ for $t = 0, 1, 2, 3$;

$|L_1 \cap (1\ 2\ \cdots\ 6-t)_r(L_2)| = 12 + 8t$ for $t = 0, 1, 2, 3, 4$;

$|L_1 \cap (1\ 6\ 3)_r(2\ 4\ 5\ 7\ 8)_r(L_4)| = 2$;

$|L_1 \cap (1\ 2\ 3\ 4\ 5\ 6)_r(7\ 8)_r(L_2)| = 4$;

$|L_1 \cap (1\ 6\ 3)_r(2\ 4\ 5\ 7)_r(L_4)| = 10$;

$|L_1 \cap (2\ 3)_r(4\ 5\ 7\ 8)_r(L_4)| = 11$;

$|L_1 \cap (3\ 4)_r(1\ 5\ 6)_r(L_3)| = 17$;

$|L_1 \cap (1\ 6\ 3)_r(2\ 4\ 5)_r(L_4)| = 18$;

$|L_1 \cap (2\ 3)_r(4\ 5\ 7)_r(L_4)| = 19$;

$|L_1 \cap (7\ 8)_r(1\ 5\ 6)_r(L_3)| = 21$;

$|L_1 \cap (1\ 2)_r(5\ 6)_r(L_3)| = 23$;

$|L_1 \cap (1\ 2)_r(3\ 5)_r(L_3)| = 25$;

$|L_1 \cap (1\ 6\ 3)_r(2\ 4)_r(L_4)| = 26$;

$|L_1 \cap (2\ 3)_r(4\ 5)_r(L_4)| = 27$;

$|L_1 \cap (1\ 5\ 6)_r(L_3)| = 29$;

$|L_2 \cap (2\ 3\ 4)_r(L_4)| = 31$;

$|L_1 \cap (2\ 4\ 5)_r(L_4)| = 33$;

$|L_1 \cap (2\ 3\ 4)_r(L_4)| = 35$;

$|L_1 \cap (1\ 5)_r(L_3)| = 37$;

$|L_1 \cap (5\ 6\ 7)_r(L_2)| = 38$;

$|L_1 \cap (1\ 2)_r(L_3)| = 39$;

$|L_1 \cap (3\ 4)_r(L_3)| = 41$;

$|L_1 \cap (1\ 6\ 3)_r(L_4)| = 42$;

$|L_1 \cap (2\ 3)_r(L_4)| = 43$;

$|L_1 \cap (7\ 8)_r(L_3)| = 45$;

$|L_1 \cap (1\ 8)_r(L_2)| = 46$;

$|L_3 \cap L_4| = 47$;

$|L_1 \cap (3\ 6)_r(L_4)| = 49$;

$|L_1 \cap (7\ 8)_r(L_2)| = 52$;

$|L_1 \cap L_3| = 53$;

$|L_2 \cap L_3| = 56$;

$|L_1 \cap L_4| = 57$;

$|L_1 \cap L_2| = 60$. □

Lemma 3.9 $15, 34, 50, 51, 54, 55, 58 \in J^*(8)$.

Proof. Let L_i $(i = 5, 6, 7, 8)$ be Latin squares of order 8 with an orthogonal mate in Appendix. It is checked that $|L_2 \cap L_5| = 50$; $|L_6 \cap L_8| = 51$; $|L_1 \cap L_5| = 54$; $|L_6 \cap L_7| = 55$; $|L_5 \cap L_6| = 58$; $|L_2 \cap (2\ 5)_r L_5)| = 34$; $|L_6 \cap (2\ 5\ 6\ 7\ 8)_r L_7)| = 15$. □

Lemma 3.10 $1, 3, 5, 7, 9, 13 \in J^*(8)$.

Proof. Let $\pi_r = (1\ 8)(2\ 7)(3\ 6)(4\ 5)$ be the row permutation acting on L_1 which comes from the Appendix. Let $\bar{L}_1 = \pi_r(L_1)$. It is readily checked that

$|L_2 \cap \pi_c\pi_e(\bar{L}_1)| = 1$ where $\pi_c = (1\ 4)(2\ 3)(5\ 8)(6\ 7)$ and $\pi_e = (1\ 7)(2\ 6)(3\ 5)$;

$|L_2 \cap \pi_c\pi_e(\bar{L}_1)| = 3$ where $\pi_c = (1\ 8)(2\ 7)(3\ 6)(4\ 5)$ and $\pi_e = (1\ 3)(4\ 8)(5\ 7)$;

$|L_2 \cap \pi_c\pi_e(\bar{L}_1)| = 5$ where $\pi_c = (1\ 8)(2\ 7)(3\ 6)(4\ 5)$ and $\pi_e = (1\ 4)(2\ 3)(5\ 8)(6\ 7)$;

$|L_2 \cap \pi_c\pi_e(\bar{L}_1)| = 7$ where $\pi_c = (1\ 8)(2\ 7)(3\ 6)(4\ 5)$ and $\pi_e = (1\ 8)(2\ 7)(3\ 6)(4\ 5)$;

$|L_2 \cap \pi_c\pi_e(\bar{L}_1)| = 9$ where $\pi_c = (1\ 7)(2\ 6)(3\ 5)$ and $\pi_e = (1\ 5)(2\ 4)(6\ 8)$;

$|L_2 \cap \pi_c\pi_e(\bar{L}_1)| = 13$ where $\pi_c = (1\ 7)(2\ 6)(3\ 5)$ and $\pi_e = (1\ 6)(2\ 5)(3\ 4)(7\ 8)$. □

Theorem 3.11 $J^*(8) = I(8)$.

Proof. This follows immediately from Lemma 3.7 to Lemma 3.10. □

4 The set $J^*(v)$ for $9 \leq v \leq 14$

In this section we will consider the set $J^*(v)$ where $9 \leq v \leq 14$.

Lemma 4.1 *$v^2 - 9$, $v^2 - 6 \in J^*(v)$ for any integer $v \geq 9$; $v^2 - 8 \in J^*(v)$ for any integer $v \geq 12$.*

Proof. It follows immediately by Theorem 2.3 with $n = 3$ or 4 and Lemma 3.1. □

Lemma 4.2 *$J_1(3) = \{0, 1, 3\}$; $J_2(3) = \{0, 2, 3, 6\}$.*

Proof. This follows from an exhaustive search. □

Lemma 4.3 *$I(9) \setminus \{52, 58, 61, 62, 64, 65, 67, 68, 70, 71, 73, 74, 77\} \subseteq J^*(9)$.*

Proof. Apply Theorem 2.5 with $m = n = 3$ to get $\sum_{i=1}^{3}\sum_{j=1}^{3} k_{ij} \in J^*(9)$ where each $k_{ij} \in J^*(3) = \{0, 3, 9\}$. Then $3t \in J^*(9)$ for $t \in [0, 27] \setminus \{26\}$. By Theorem 2.6 with $v = 9$, $n = 3$ and $l = 1$ or 2, we have $9a + 6b + k \in J^*(9)$ where $a \in [0, 3 + l]$,

$b \in [0, l]$ and $k \in J_l(3)$ which is taken from Lemma 4.2. It is readily checked that $1, 2, 7, 8, 10, 11, 14, 16, 17, 19, 20, 23, 25, 26, 28, 29, 32, 34, 35, 37, 38, 41, 43, 44, 47, 50, 53, 59 \in J^*(9)$.

By the proof of Theorem 2.5, there is a MOLS$(9, 3^2)$. Apply Theorem 2.7 with $v = 9$, $n = 3$ and $l = 1, 2$ to get $9a + 6b + s + t \in J^*(9)$ where $a \in [0, 3]$, $b \in [0, 2l]$ and $s, t \in J_l(3)$. The remaining cases are obtained by taking suitable integers a, b, l, s and t as follows:

a	b	l	s	t	$9a + 6b + s + t \in J^*(9)$
0	0	2	2	2	4
0	0	2	2	3	5
0	2	1	0	1	13
1	2	1	0	1	22
2	2	1	0	1	31
3	2	1	0	1	40
2	4	2	2	2	46
3	3	2	2	2	49
3	4	2	2	2	55
3	4	2	2	3	56

□

Lemma 4.4 $52, 58, 61, 62, 64, 65, 67, 68, 70, 71, 73, 74, 77 \in J^*(9)$.

Proof. Let M_i and M_i' $(i = 1, 2)$ be Latin squares of order 9 as follows:

$$M_1 = \begin{array}{ccccccccc} & & & 4 & 5 & 6 & 7 & 8 & 9 \\ & A_1 & & 9 & 8 & 7 & 5 & 4 & 6 \\ & & & 8 & 7 & 9 & 6 & 5 & 4 \\ 7 & 6 & 5 & & & & 4 & 9 & 8 \\ 9 & 4 & 6 & & A_2 & & 8 & 7 & 5 \\ 8 & 5 & 4 & & & & 9 & 6 & 7 \\ 4 & 8 & 9 & 7 & 6 & 5 & & & \\ 6 & 9 & 7 & 5 & 4 & 8 & & A_3 & \\ 5 & 7 & 8 & 6 & 9 & 4 & & & \end{array} \qquad M_1' = \begin{array}{ccccccccc} & & & 4 & 5 & 6 & 7 & 8 & 9 \\ & A_1' & & 6 & 4 & 5 & 8 & 9 & 7 \\ & & & 5 & 6 & 4 & 9 & 7 & 8 \\ 4 & 5 & 6 & & & & 1 & 3 & 2 \\ 5 & 6 & 4 & & A_2' & & 3 & 2 & 1 \\ 6 & 4 & 5 & & & & 2 & 1 & 3 \\ 7 & 9 & 8 & 1 & 2 & 3 & & & \\ 8 & 7 & 9 & 2 & 3 & 1 & & A_3' & \\ 9 & 8 & 7 & 3 & 1 & 2 & & & \end{array}$$

where A_1, A_2, A_3 are any Latin squares on $\{1, 2, 3\}$, and A_i' are an orthogonal mate of A_i on $\{3i-2, 3i-1, 3i\}$ for $i = 1, 2, 3$.

$$M_2 = \begin{array}{ccccccccc} & & & 4 & 5 & 6 & 7 & 9 & 8 \\ & B_1 & & 9 & 8 & 7 & 5 & 4 & 6 \\ & & & 8 & 7 & 9 & 6 & 5 & 4 \\ 7 & 6 & 5 & & & & 4 & 8 & 9 \\ 9 & 4 & 6 & & B_2 & & 8 & 7 & 5 \\ 8 & 5 & 4 & & & & 9 & 6 & 7 \\ 4 & 8 & 9 & 7 & 6 & 5 & & & \\ 6 & 9 & 7 & 5 & 4 & 8 & & B_3 & \\ 5 & 7 & 8 & 6 & 9 & 4 & & & \end{array} \qquad M_2' = \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 9 & 1 & 6 & 4 & 3 & 8 & 5 & 7 \\ 3 & 1 & 8 & 5 & 2 & 4 & 9 & 7 & 6 \\ 4 & 5 & 6 & 7 & 8 & 9 & 1 & 3 & 2 \\ 5 & 3 & 4 & 8 & 6 & 7 & 2 & 9 & 1 \\ 6 & 4 & 2 & 9 & 7 & 5 & 3 & 1 & 8 \\ 7 & 8 & 9 & 1 & 3 & 2 & 4 & 6 & 5 \\ 8 & 7 & 5 & 3 & 9 & 1 & 6 & 2 & 4 \\ 9 & 6 & 7 & 2 & 1 & 8 & 5 & 4 & 3 \end{array}$$

where $B_1 = \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{matrix}$ and $B_2 = B_3 = \begin{matrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{matrix}$. It is readily checked that M_1 and M_1', M_2 and M_2' are mutually orthogonal, M_1 and M_2 have $(81-27-4)+(r_1+r_2+r_3)$ cells in common where $r_1, r_2, r_3 \in J^*(3)$. So $62, 65, 68, 71, 77 \in J^*(9)$.

Take $A_1 = B_1$, $A_2 = B_2$ and A_3 to be any Latin squares on $\{1,2,3\}$ in M_1. Let $\pi = (1\,4)$ be the row permutation acting on M_1. Then $\pi(M_1)$ and M_2 have $(67-9)+r$ cells in common where $r \in J^*(3)$. Hence $61, 67 \in J^*(9)$.

Let M_3 and M_3' be as follows:

$$M_3 = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & 9 & 8 & 7 & 5 & 4 & 6 \\ 2 & 3 & 1 & 8 & 7 & 9 & 6 & 5 & 4 \\ 8 & 6 & 5 & & & & 4 & 9 & 7 \\ 9 & 4 & 6 & & C & & 8 & 7 & 5 \\ 7 & 5 & 4 & & & & 9 & 6 & 8 \\ 4 & 8 & 9 & 7 & 6 & 5 & 1 & 3 & 2 \\ 6 & 9 & 7 & 5 & 4 & 8 & 2 & 1 & 3 \\ 5 & 7 & 8 & 6 & 9 & 4 & 3 & 2 & 1 \end{matrix} \qquad M_3' = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 1 & 6 & 3 & 5 & 8 & 9 & 7 \\ 3 & 1 & 6 & 5 & 2 & 4 & 9 & 7 & 8 \\ 6 & 5 & 4 & & & & 1 & 2 & 3 \\ 5 & 3 & 2 & & C' & & 4 & 6 & 1 \\ 4 & 6 & 5 & & & & 3 & 1 & 2 \\ 7 & 9 & 8 & 1 & 4 & 3 & 2 & 5 & 6 \\ 8 & 7 & 9 & 2 & 6 & 1 & 5 & 3 & 4 \\ 9 & 8 & 7 & 3 & 1 & 2 & 6 & 4 & 5 \end{matrix}$$

where C is any Latin square on $\{1,2,3\}$ and C' is an orthogonal mate of C on $\{7,8,9\}$. It is readily checked that M_3 and M_3' are mutually orthogonal. Then M_2 and M_3 have $(74-9)+r$ cells in common where $r \in J^*(3)$ and hence $74 \in J^*(9)$.

Let U_1 be obtained from M_1 by taking $A_1 = (1\ 2)_r(B_1)$, $A_2 = (2\ 3)_r(B_2)$ and A_3 any Latin square on $\{1,2,3\}$. Let $\pi = (1\ 4)$ be the row permutation acting on U_1. Then $\pi(U_1)$ and M_2 have $49+r$ cells in common where $r \in J^*(3)$. So, $52, 58 \in J^*(9)$.

Let U_2 be obtained from M_3 by taking $C = (1\ 3)_r(B_2)$. Let $\pi_1 = (4\ 6)$ be the row permutation acting on U_2. Then $\pi_1(U_2)$ and M_2 have 70 cells in common.

Let M_4 and M_4' be as follows:

$$M_4 = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & 9 & 8 & 7 & 5 & 4 & 6 \\ 2 & 3 & 1 & 8 & 7 & 9 & 6 & 5 & 4 \\ 7 & 6 & 5 & 1 & 3 & 2 & 4 & 9 & 8 \\ 9 & 4 & 6 & 2 & 1 & 3 & 8 & 7 & 5 \\ 8 & 5 & 4 & 3 & 2 & 1 & 9 & 6 & 7 \\ 4 & 9 & 7 & 5 & 6 & 8 & 1 & 3 & 2 \\ 6 & 8 & 9 & 7 & 4 & 5 & 2 & 1 & 3 \\ 5 & 7 & 8 & 6 & 9 & 4 & 3 & 2 & 1 \end{matrix} \qquad M_4' = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 1 & 8 & 6 & 5 & 3 & 9 & 7 \\ 3 & 1 & 6 & 5 & 4 & 7 & 9 & 2 & 8 \\ 6 & 5 & 8 & 2 & 7 & 9 & 1 & 4 & 3 \\ 5 & 6 & 4 & 7 & 9 & 8 & 2 & 3 & 1 \\ 4 & 7 & 5 & 9 & 8 & 3 & 6 & 1 & 2 \\ 7 & 3 & 9 & 6 & 2 & 1 & 8 & 5 & 4 \\ 8 & 9 & 2 & 1 & 3 & 4 & 5 & 7 & 6 \\ 9 & 8 & 7 & 3 & 1 & 2 & 4 & 6 & 5 \end{matrix}$$

It is readily checked that M_4 and M_4' are mutually orthogonal; M_4 and M_1 with $A_i = B_i$ $(i = 1,2,3)$ have 73 cells in common.

Let U_3 be obtained from M_1 by taking $A_1 = (1\ 2)_r(B_1)$, $A_2 = B_2$ and $A_3 = B_3$. Let $\pi = (1\ 4)$ be the row permutation acting on U_3. Then $\pi(U_3)$ and M_2 have 64 cells in common. □

Theorem 4.5 $J^*(9) = I(9)$.

Proof. This follows immediately from Lemma 4.3 and Lemma 4.4. □

Lemma 4.6 $I(10)\setminus\{4, 5, 15, 25, 35, 45, 55, 65, 68, 72, 75, 78, 81, 82, 83–85, 87–89, 92, 93, 96\} \subseteq J^*(10)$.

Proof. Apply Theorem 2.3 with $v = 10$ and $n = 3$ to get $10a + 7b + J^*(3) \in J^*(10)$ where $a \in [0, 7]\setminus\{6\}$ and $b \in [0, 3]\setminus\{2\}$. Direct computation shows that 0, 3, 7, 9, 10, 13, 16, 17, 19–21, 23, 24, 26, 27, 29–31, 33, 34, 36, 37, 39–41, 43, 44, 46, 47, 49–51, 53, 54, 56, 57, 59–61, 64, 66, 70, 71, 73, 74, 77, 79, 80, 86, 91, 94, 100 $\in J^*(10)$.

By Theorem 2.6 with $v = 10$ and $n = 3$, $10a + 7b + k \in J^*(10)$ where $l = 1, 2$, $a \in [0, 3 + l]$, $b \in [0, l]$ and $k \in J_1(3)$ which is taken from Lemma 4.2. The other cases follow by taking suitable integers l, a, b and k as follows:

l	1	2	2	1	1	2	2	1	2	1	2
a	0	0	0	0	1	1	0	1	2	2	3
b	0	10	0	1	0	0	2	1	0	1	0
k	1	2	6	1	1	2	0	1	2	1	0
$10a + 7b + k$	1	2	6	8	11	12	14	18	22	28	32

l	1	2	1	2	1	2	2	2	2	2
a	3	4	4	5	5	6	6	6	6	6
b	1	0	1	0	1	0	0	1	1	2
k	1	2	1	2	1	2	3	0	2	2
$10a + 7b + k$	38	42	48	52	58	62	63	67	69	76

□

Lemma 4.7 $4, 5, 15, 25, 35, 45, 55, 68, 72, 78, 84, 88, 92, 96 \in J^*(10)$.

Proof. Let N_i $(i = 1, 2, 3)$ be Latin squares of order 10 with an orthogonal mate in Appendix. It is readily checked that

$|N_1 \cap (1\ 2\ 3\ 4)_r(6\ 7\ 8\ 9)_r(5\ 10)_r(N_3)| = 4$;

$|N_1 \cap (3\ 8)_r(5\ 10)_r(N_2)| = 68$;

$|N_1 \cap (1\ 2)_r(N_2)| = 72$;

$|N_1 \cap (9\ 10)_r(N_3)| = 78$;

$|N_1 \cap (5\ 10)_r(N_3)| = 84$;

$|N_1 \cap (9\ 10)_c(N_2)| = 88$;

$|N_1 \cap N_2| = 92$;

$|N_1 \cap N_3| = 96$.

Here P is a (3,1,2)-conjugate orthogonal Latin square of order 10 with an empty subarray on $\{8, 9, 10\}$ exhibited in the Appendix, which actually comes from [2]. It is readily checked that:

$|P \cap \pi_r\pi_c(P)| = 5$ where $\pi_r = (1\ 7\ 6\ 5\ 4\ 3\ 2)$ and $\pi_c = (1\ 7\ 2\ 5\ 3\ 4\ 6)$;

$|P \cap \pi_r\pi_c(P)| = 15$ where $\pi_r = (1\ 7\ 6\ 5\ 4\ 3\ 2)$ and $\pi_c = (1\ 7\ 2)(3\ 6\ 5\ 4)$;

$|P \cap \pi_r\pi_c(P)| = 25$ where $\pi_r = (1\ 5\ 4\ 3\ 2)$ and $\pi_c = (2\ 3\ 4\ 5)$;

$|P \cap \pi_r\pi_c(P)| = 35$ where $\pi_r = (2\ 5\ 4)$ and $\pi_c = (1\ 5\ 4\ 3)$;

$|P \cap \pi_r\pi_c(P)| = 45$ where $\pi_r = (3\ 5)$ and $\pi_c = (1\ 3)(2\ 4)$;

$|P \cap \pi_r\pi_c(P)| = 55$ where $\pi_r = (3\ 5)$ and $\pi_c = (3\ 5)$. Hence $5, 15, 25, 35, 45, 55 \in J^*(10, 3)$. By Lemma 3.1 and Theorem 2.2, we have $5, 15, 25, 35, 45, 55 \in J^*(10)$. □

Theorem 4.8 $I(10) \setminus \{65, 75, 81, 82, 83, 85, 87, 89, 93\} \subseteq J^*(10)$.

Proof. This follows from Lemmas 4.6 and 4.7. □

Lemma 4.9 $I(11) \setminus \{4, 5, 7, 15, 26, 37, 48, 59, 70, 78, 81, 86, 89, 92, 94, 98, 100, 101, 102, 103, 104, 106–111, 113, 114, 117\} \subseteq J^*(11)$.

Proof. Apply Theorem 2.3 with $v = 11$ and $n = 3$ to get $11a + 8b + k \in J^*(11)$ where $a \in [0, 8] \setminus \{7\}$, $b \in [0, 3] \setminus \{2\}$ and $k \in J^*(3)$. Then $0, 3, 8, 9, 11, 14, 17, 19, 20, 22, 24, 25, 27, 28, 30, 31, 33, 35, 36, 38, 39, 41, 42, 44, 46, 47, 49, 50, 52, 53, 55, 57, 58, 60, 61, 63, 64, 66, 68, 71, 72, 74, 77, 79, 82, 83, 88, 90, 91, 93, 96, 97, 99, 105, 112, 115, 121 \in J^*(11)$.

By Theorem 2.6 with $v = 11$, $n = 3$ and $l = 1$ or 2, $11a+8b+k \in J^*(11)$ where $a \in [0, 5+l]$, $b \in [0, l]$ and $k \in J_l(3)$ which is taken from Lemma 4.2. It is readily checked that $1, 2, 6, 10, 12, 13, 16, 18, 21, 23, 29, 32, 34, 40, 43, 45, 51, 54, 56, 62, 65, 67, 69, 73, 75, 76, 80, 84, 85, 87, 95 \in J^*(11)$ by taking suitable integers l, a, b and k. □

Lemma 4.10 $4, 5, 7, 15, 26, 37, 48, 59, 81, 89 \in J^*(11)$.

Proof. Let $L = (a_{ij})$ be a Latin square of order 11 as follows: $a_{ij} = 6(i + j) \pmod{11}$. Then L has an orthogonal mate. It is readily checked that

$|L \cap \pi_r\pi_c(L)| = 4$ where $\pi_r = (10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1)$ and $\pi_c = (1\ 7\ 4\ 6\ 10\ 3\ 9\ 2)$;

$|L \cap \pi_r\pi_c(L)| = 5$ where $\pi_r = (10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1)$ and $\pi_c = (1\ 7\ 2\ 4\ 9)(3\ 6\ 8\ 10)$;

$|L \cap \pi_r\pi_c(L)| = 7$ where $\pi_r = (10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1)$ and $\pi_c = (2\ 4)(3\ 5\ 7)(8\ 10\ 9)$;

$|L \cap \pi_r\pi_c(L)| = 15$ where $\pi_r = (10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1)$ and $\pi_c = (1\ 7\ 5\ 6\ 8\ 10\ 2)$;

$|L \cap \pi_r\pi_c(L)| = 26$ where $\pi_r = (1\ 8\ 6\ 4\ 2\ 10\ 9\ 7\ 5\ 3)$ and $\pi_c = (1\ 7\ 2\ 4\ 9)(3\ 6\ 8\ 10)$;

$|L \cap \pi_r\pi_c(L)| = 37$ where $\pi_r = (1\ 5\ 4\ 3\ 2)$ and $\pi_c = (1\ 4\ 3\ 2\ 5)$;

$|L \cap \pi_r\pi_c(L)| = 48$ where $\pi_r = (1\ 5\ 4\ 3\ 2)$ and $\pi_c = (2\ 5\ 4)$;

$|L \cap \pi_r\pi_c(L)| = 59$ where $\pi_r = (2\ 5\ 4)$ and $\pi_c = (1\ 3\ 5\ 4)$;

$|L \cap \pi_r\pi_c(L)| = 81$ where $\pi_r = (3\ 5)$ and $\pi_c = (3\ 5)$.

Let A and B be $LS(11, 3)$, $\pi_r = (10\ 11)$ and $\pi_c = (10\ 11)$ be row permutation and column permutation acting on A. Then A and $\pi_r\pi_c(A)$ have their own orthogonal mates. It is checked that A and $\pi_r\pi_c(A)$ have 80 cells in common and hence $80 \in J^*(11, 3)$. By Lemma 3.1 and Theorem 2.2, $89 \in J^*(11)$. □

Theorem 4.11 $I(11) \setminus \{70, 78, 86, 92, 94, 98, 100, 101, 102, 103, 104, 106–111, 113,$

114, 117} $\subseteq J^*(11)$.

Proof. This follows from Lemma 4.9 and Lemma 4.10 □

Lemma 4.12 $I(12) \setminus \{103, 106, 107, 109, 115, 118, 119, 121, 122, 125, 127, 130, 131, 134, 137, 140\} \subseteq J^*(12)$.

Proof. Apply Theorem 2.5 with $n = 4$ and $m = 3$ to get $\sum_{i=1}^{4} \sum_{j=1}^{4} k_{ij} \in J^*(12)$ where each $k_{ij} \in J^*(3) = \{0, 3, 9\}$. Then $3t \in J^*(12)$ for any integers $t \in [0, 48] \setminus \{47\}$. Similarly, $\sum_{i=1}^{3} \sum_{j=1}^{3} k_{ij} \in J^*(12)$ where each $k_{ij} \in J^*(4) = \{0, 4, 8, 16\}$. Then $4t \in J^*(12)$ for any integer $t \in [0, 36] \setminus \{35\}$.

By the proof of Theorem 2.5, there is a MOLS$(12, 3^k)$ for $k = 2, 3$. Apply Theorem 2.7 with $k = 3$ and $l = 2$ to get $12a + 9b + \sum_{i=1}^{3} a_i \in J^*(12)$ where $a \in [0, 3]$, $b \in [0, 6]$ and $a_i \in J_2(3)$ for $i \in [1, 3]$. Clearly, $\sum_{i=1}^{3} a_i \in \{0, 2\text{–}12, 14, 15, 18\}$. Hence, $\{0, 2\text{–}102, 104, 105, 108\} \subseteq J^*(12)$. Apply Theorem 2.7 with $n = 3$, $k = 2$ and $l = 1, 2$ to get $12a + 9b + s + t \in J^*(12)$ where $a \in [0, 6]$, $b \in [0, 2l]$ and $s, t \in J_l(3)$. Then $1, 110, 113 \in J^*(12)$ by taking suitable l, s and t. □

Lemma 4.13 $I(12) \setminus \{115, 118, 119, 121, 122, 125, 127, 130, 131, 134, 137, 140\} \subseteq J^*(12)$.

Proof. Let $L(A_1, \cdots, A_4)$ and $L'(A'_1, \cdots, A'_4)$ be Latin squares on $I_4 \times I_3$ (where $I_t = \{1, 2, \cdots, t\}$ for $t = 3, 4$) as follows.

$$L = \begin{matrix} (1, A_1) & (2, B) & (4, B) & (3, B) \\ (2, B) & (1, A_2) & (3, B) & (4, B) \\ (4, B) & (3, B) & (1, A_3) & (2, B) \\ (3, B) & (4, B) & (2, B) & (1, A_4) \end{matrix} \qquad L' = \begin{matrix} (1, A'_1) & (2, B') & (4, B') & (3, B') \\ (2, B') & (1, A'_2) & (3, B') & (4, B') \\ (4, B') & (3, B') & (1, A'_3) & (2, B') \\ (3, B') & (4, B') & (2, B') & (1, A'_4) \end{matrix}$$

where A_i $(i = 1, 2, 3, 4)$ are any Latin squares on I_3 and B is fixed Latin square on I_3. A'_i $(i = 1, 2, 3, 4)$ is an orthogonal mate of A_i on I_3 and B' is an orthogonal mate of B on I_3. It is easy to see that $L(A_1, \cdots, A_4)$ and $L'(A'_1, \cdots, A'_4)$ are mutually orthogonal.

Let $\pi = ((1, 1)\ (2, 1))$ be the element permutation on $L(B_1, \cdots, B_4)$. It is readily checked that $L(A_1, \cdots, A_4)$ and $\pi(L(B_1, \cdots, B_4))$ have $96 + \sum_{i=1}^{4} r_i$ cells in common where each $r_i \in J_2(3) = \{0, 2, 3, 6\}$. Hence $103, 106, 107, 109 \in J^*(12)$. The conclusion follows from Lemma 4.12. □

Lemma 4.14 *Let a, b be integers such that* $\min\{a, b\} \geq 6$. *For any integer* $n \in [0, 3a + 4b] \setminus \{1, 2, 5, 3a + 4b - 19, 3a + 4b - 13, 3a + 4b - 11, 3a + 4b - 10, 3a + 4b - 7, 3a + 4b - 4\}$, *$n$ can be written as $3s + 4t$ where $s \in [0, a] \setminus \{a - 1\}$ and $t \in [0, b] \setminus \{b - 1\}$.*

Proof. This follows immediately. □

Lemma 4.15 $I(13) \setminus \{150, 156, 158, 159, 162, 165\} \subseteq J^*(13)$.

Proof. Apply Theorem 2.8 with $n = 4$, $q = 3$ and $x = 1$ to get $\sum_{i=1}^{4} d_i + \sum_{i=5}^{16} d_i \in J^*(13, 1)$ where $d_i \in J^*(4, 1) = \{3, 7, 15\}$ for $i \in [1, 4]$ and $d_i \in J^*(3)$ for $i \in [5, 16]$.

It is easy to see that

$$\sum_{i=1}^{4} d_i \in \{4t+12:\ t \in [0,12] \setminus \{11\}\},$$

$$\sum_{i=5}^{16} d_i \in \{3s:\ s \in [0,36] \setminus \{35\}\}.$$

Then $3s+4t+13 \in J^*(13)$ where $s \in [0,36] \setminus \{35\}$ and $t \in [0,12] \setminus \{11\}$. When $k \in I(13) \setminus \{\ 0-12, 14, 15, 18, 150, 156, 158, 159, 162,\ 165\}$, $k \in J^*(13)$ by Lemma 4.14.

By the proof of Theorem 2.8, there is a MOLS$(13, 3^4)$. Apply Theorem 2.7 with $l=2$ to get $13a+10b+\sum_{i=1}^{4} a_i \in J^*(13)$ where $a \in [0,1]$, $b \in [0,8]$ and $a_i \in J_2(3)$ for $i \in [1,3]$. It is easy to see that $\sum_{i=1}^{4} a_i \in \{0, 2-18, 20, 21, 24\}$. Hence, $\{0, 2-12, 14, 15, 18\} \subseteq J^*(13)$. Similarly, $1 \in J^*(13)$ by Theorem 2.7 with $l=1$. □

Lemma 4.16 $I(14) \setminus \{5, 7, 19, 21, 35, 49, 63, 77, 91, 105, 119, 133, 141, 147, 149, 155, 161, 167, 169–173, 175, 177–179, 181–183, 185, 186, 189, 192\} \subseteq J^*(14)$.

Proof. Apply Theorem 2.3 with $v=14$ and $n=3$ or 4 to get $14a+(14-n)b+k \in J^*(14)$ where $a \in [0, 14-n] \setminus \{13-n\}$, $b \in [0,n] \setminus \{n-1\}$ and $k \in J^*(n)$ where $n=3,4$. Then $I(14) \setminus \{$1, 2, 5–7, 12, 13, 15, 19, 21, 27, 29, 35, 41, 43, 49, 55, 57, 63, 69, 71, 77, 83, 85, 91, 97, 99, 105, 111, 113, 119, 125, 127, 133, 139, 141, 143, 147, 149, 151, 153, 155, 161, 167, 169–173, 175, 177–179, 181–183, 185, 186, 189, 192$\} \subseteq J^*(14)$ by taking suitable n, a and b.

By Theorem 2.6 with $v=14$, $n=3$ and $l=1,2$, $14a+11b+k \in J^*(14)$ where $a \in [0, 8+l]$, $b \in [0,l]$ and $k \in J_l(3)$. The remaining cases follow immediately by taking suitable k, a and b. □

Lemma 4.17 $5, 7, 19, 21, 35, 49, 63, 77, 91, 105, 133 \in J^*(14)$.

Proof. Here Q is a (3,2,1)-conjugate orthogonal Latin square of order 14 with an empty subarray on $\{A, B, C, D\}$ exhibited in the Appendix which comes from [3]. It is readily checked that:

$|Q \cap \pi_r\pi_c(Q)| = 5$ where $\pi_r = (10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1)$ and $\pi_c = (1\ 6\ 5)(2\ 10\ 3\ 7\ 8\ 4\ 9)$;

$|Q \cap \pi_r\pi_c(Q)| = 7$ where $\pi_r = (10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1)$ and $\pi_c = (1\ 4\ 8\ 5)(2\ 10\ 9\ 7\ 3\ 6)$;

$|Q \cap \pi_r\pi_c(Q)| = 11$ where $\pi_r = (10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1)$ and $\pi_c = (1\ 8\ 5\ 10\ 9\ 7\ 3\ 4)(2\ 6)$;

$|Q \cap \pi_r\pi_c(Q)| = 35$ where $\pi_r = (1\ 8\ 5\ 10\ 9\ 7\ 3\ 6)$ and $\pi_c = (1\ 9\ 10\ 4)(6\ 7)$;

$|Q \cap \pi_r\pi_c(Q)| = 45$ where $\pi_r = (1\ 5\ 9\ 3\ 6\ 2\ 7)$ and $\pi_c = (1\ 5\ 9\ 3\ 6\ 2\ 7)$;

$|Q \cap \pi_r\pi_c(Q)| = 47$ where $\pi_r = (1\ 7\ 5\ 6\ 8\ 10\ 2)$ and $\pi_c = (1\ 7\ 8\ 4\ 6\ 10)$;

$|Q \cap \pi_r\pi_c(Q)| = 69$ where $\pi_r = (1\ 5\ 4\ 3\ 2)$ and $\pi_c = (1\ 5\ 4\ 3\ 2)$;

$|Q \cap \pi_r\pi_c(Q)| = 83$ where $\pi_r = (1\ 5\ 4\ 3\ 2)$ and $\pi_c = (1\ 4\ 5)$;

$|Q \cap \pi_r\pi_c(Q)| = 105$ where $\pi_r = (2\ 5\ 4)$ and $\pi_c = (2\ 5\ 4)$;

$|Q \cap \pi_r\pi_c(Q)| = 117$ where $\pi_r = (1\ 2\ 4)$ and $\pi_c = (2\ 5)$.

Hence, $5, 7, 11, 35, 45, 47, 69, 83, 105, 117 \in J^*(14, 4)$. The conclusion follows from Lemma 3.1 and Theorem 2.2. □

Theorem 4.18 *$I(14) \setminus \{119, 141, 147, 149, 155, 161, 167, 169–173, 175, 177–179, 181–183, 185, 186, 189, 192\} \subseteq J^*(14)$.*

Proof. This follows from Lemma 4.16 and Lemma 4.17. □

5 Conclusions

Lemma 5.1 *$I(v) \setminus \{v^2 - 11, v^2 - 7, v^2 - 4\} \subseteq J^*(v)$ for integer $v = 15, 20$.*

Proof. Apply Theorem 2.5 with $n = \frac{v}{5}$ and $m = 5$ to get $\sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} \in J^*(v)$ where each $k_{ij} \in J^*(5)$. By Lemma 3.2, $J^*(5) = \{0 - 13, 15, 25\}$. For any integer $k \in I(v) \setminus \{v^2 - 11, v^2 - 9, v^2 - 8, v^2 - 7, v^2 - 6, v^2 - 4\}$, it is easy to check that there exist $k_{ij} \in J^*(5)$ such that $k = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij}$. Then $I(v) \setminus \{v^2 - 11, v^2 - 9, v^2 - 8, v^2 - 7, v^2 - 6, v^2 - 4\} \subseteq J^*(v)$. The other three cases follow by Lemma 4.1. □

Lemma 5.2 *$I(v) \setminus \{v^2 - 11, v^2 - 7, v^2 - 4\} \subseteq J^*(v)$ for integers $v = 16, 18, 22$.*

Proof. Let $v = 3n + x$ where v, n and x $(1 \le x < n)$ are taken as follows: $(v, n, x) = (16, 5, 1)$, $(18, 5, 3)$, $(22, 7, 1)$. Apply Theorem 2.8 with $q = 3$ to get $\sum_{i=1}^{xn} d_i + \sum_{i=xn+1}^{n^2} d_i \in J^*(v, x)$ where $d_i \in J^*(4, 1) = \{3, 7, 15\}$ for $i \in [1, xn]$ and $d_i \in J^*(3) = \{0, 3, 9\}$ for $i \in [xn + 1, n^2]$. It is easy to see that

$$\sum_{i=1}^{xn} d_i \in \{4t + 3xn : \ t \in [0, 3xn] \setminus \{3xn - 1\} \},$$

$$\sum_{i=xn+1}^{n^2} d_i \in \{3s : \ s \in [0, 3n(n - x)] \setminus \{3n(n - x) - 1\} \}.$$

Then $3s + 4t + 3xn + k \in J^*(v)$ where $s \in [0, 3n(n - x)] \setminus \{3n(n - x) - 1\}$, $t \in [0, 3xn] \setminus \{3xn - 1\}$ and $k \in J^*(x)$. By Lemma 4.14 and $\{0, x^2\} \subseteq J^*(x)$, it is not difficult to check that $I(v) \setminus ([0, 3xn - 1] \cup \{3xn + 1,\ 3xn + 2,\ 3xn + 5,\ v^2 - 19,\ v^2 - 13,\ v^2 - 11,\ v^2 - 10,\ v^2 - 7,\ v^2 - 4\}) \subseteq J^*(v)$.

By the proof of Theorem 2.8, there is a MOLS$(v, 3^n)$. Apply Theorem 2.7 with $l = 2$ to get $av + b(v - 3) + \sum_{i=1}^{n} a_i \in J^*(v)$ where $a \in [0, x]$, $b \in [0, 2n]$ and $a_i \in J_2(3) = \{0, 2, 3, 6\}$ for $i \in [1, n]$. It is easy to see that $6(n - 1) > v - 3$ by the choices of v, n as above, and

$$\sum_{i=1}^{n} a_i \in [2, 6(n - 1)] \cup \{0, 6n - 4, 6n - 3, 6n\}.$$

Hence, $[2, 3xn - 1] \cup \{0, 3xn + 1,\ 3xn + 2,\ 3xn + 5\} \subseteq J^*(v)$. Similarly, $1 \in J^*(v)$ by Theorem 2.7 with $l = 1$. By Lemma 2.1 there is a MOLS$(v, 5)$ and hence $v^2 - 25 \in J^*(v, 5)$. Then $v^2 - 19,\ v^2 - 13,\ v^2 - 10 \in J^*(v)$ by Theorem 2.2 and Lemma 3.2. This completes the proof. □

Lemma 5.3 *$I(v) \setminus \{v^2-11, v^2-7, v^2-4\} \subseteq J^*(v)$ for integers $v = 17, 19, 21, 23$.*

Proof. Let $v = 4n + x$ where v, n and x ($1 \le x < n$) are taken as follows: $(v, n, x) = (17, 4, 1)$, $(19, 4, 3)$, $(21, 5, 1)$ and $(23, 5, 3)$. Apply Theorem 2.8 with $q = 4$ to get $\sum_{i=1}^{xn} d_i + \sum_{i=xn+1}^{n^2} d_i \in J^*(v, x)$ where $d_i \in J^*(5,1) = \{0 - 12, 14, 24\}$ for $i \in [1, xn]$ and $d_i \in J^*(4) = \{0, 4, 8, 16\}$ for $i \in [xn+1, n^2]$. It is easy to see that

$$\sum_{i=1}^{xn} d_i \in S(24xn) \setminus \{24xn-11, 24xn-9, 24xn-8, 24xn-7, 24xn-6, 24xn-4\},$$

$$\sum_{i=xn+1}^{n^2} d_i \in \{4t :\ t \in [0, 4n(n-x)] \setminus \{4n(n-x)-1\}\ \}.$$

Then $s+4t+k \in J^*(v)$ where $s \in S(24xn) \setminus \{24xn-11, 24xn-9, 24xn-8, 24xn-7, 24xn-6, 24xn-4\}$, $t \in [0, 4n(n-x)] \setminus \{4n(n-x)-1\}$ and $\{0, x^2\} \subseteq J^*(x)$. Hence $I(v) \setminus \{v^2-11, v^2-9, v^2-8, v^2-7, v^2-6, v^2-4\} \subseteq J^*(v)$. The other cases follow from Lemma 4.1. □

Theorem 5.4 *$I(v) \setminus \{v^2-11, v^2-7, v^2-4\} \subseteq J^*(v)$ for any integers $15 \le v \le 20$; $I(v) \setminus \{v^2-11, v^2-7\} \subseteq J^*(v)$ for $v = 21, 22, 23$.*

Proof. By Lemmas 5.1 to 5.3, $I(v) \setminus \{v^2-11, v^2-7, v^2-4\} \subseteq J^*(v)$ for any integer $15 \le v \le 23$. Apply Theorem 2.3 with $n = 7$ and Theorem 3.6 to get $v^2 - 4 \in J^*(v)$ for $v = 21, 22, 23$. □

Now we are in position to present the main result.

Main Theorem *$J^*(v) = I(v)$ for any integer $v \ge 24$.*

Proof. When $24 \le v \le 37$, apply Theorem 2.3 with $n = 8$ to get $av + b(v-8) + k \in J^*(v)$ for any integers $a \in [0, v-8] \setminus \{v-9\}$, $b \in [0, 8] \setminus \{7\}$ and $k \in J^*(8)$. Note that $2v < 6(v-8)$ and $2(v-8) \le 58$. Then $J^*(v) = I(v)$.

When $38 \le v \le 44$, similarly apply Theorem 2.3 with $n = 9$ to get $J^*(v) = I(v)$.

When $v \ge 45$, let $n = [\frac{v}{3}]$ where $[*]$ denotes the integer part of a real number "$*$". Then $n \ge 15$. By the induction and Theorem 5.4, $I(n) \setminus \{n^2-11, n^2-7, n^2-4\} \subseteq J^*(n)$. Apply Theorem 2.3 to get $av + b(v-n) + k \in J^*(v)$ for any integers $a \in [0, v-n] \setminus \{v-n-1\}$, $b \in [0, n] \setminus \{n-1\}$ and $k \in J^*(n)$. For any integer $i \in I(v) \setminus \{v^2-11, v^2-7, v^2-4\}$, it is easy to check that there exist $a \in [0, v-n] \setminus \{v-n-1\}$, $b \in [0, n] \setminus \{n-1\}$ and $k \in J^*(n)$ such that $i = av + b(v-n) + k$. Then $I(v) \setminus \{v^2-11, v^2-7, v^2-4\} \subseteq J^*(v)$. By Theorem 3.11, $53, 57, 60 \in J^*(8)$. Apply Theorem 2.3 with $n = 8$ to get $v^2-11, v^2-7, v^2-4 \in J^*(v)$. This completes the proof. □

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Appendix

K_i $(i = 1, 2, 3, 4, 5)$ are Latin squares of order 7 with an orthogonal mate K'_i as follows:

$$K_1 = \begin{matrix} 1&2&3&4&5&6&7\\ 2&3&1&5&6&7&4\\ 6&7&4&1&3&5&2\\ 3&5&7&6&2&4&1\\ 5&4&6&2&7&1&3\\ 7&1&5&3&4&2&6\\ 4&6&2&7&1&3&5 \end{matrix} \qquad K'_1 = \begin{matrix} 1&2&3&4&5&6&7\\ 7&1&4&2&3&5&6\\ 5&6&1&7&2&4&3\\ 6&7&2&1&4&3&5\\ 3&5&7&6&1&2&4\\ 4&3&6&5&7&1&2\\ 2&4&5&3&6&7&1 \end{matrix}$$

$$K_2 = \begin{matrix} 1&5&3&4&2&6&7\\ 2&3&1&5&6&7&4\\ 6&7&4&1&3&5&2\\ 3&2&7&6&5&4&1\\ 5&4&6&2&7&1&3\\ 7&1&5&3&4&2&6\\ 4&6&2&7&1&3&5 \end{matrix} \qquad K_2' = \begin{matrix} 1&2&3&4&5&6&7\\ 6&5&7&1&2&4&3\\ 4&3&1&5&6&7&2\\ 7&1&2&3&4&5&6\\ 3&6&5&7&1&2&4\\ 5&4&6&2&7&3&1\\ 2&7&4&6&3&1&5 \end{matrix}$$

$$K_3 = \begin{matrix} 3&5&7&4&2&6&1\\ 2&3&1&5&6&7&4\\ 6&7&4&1&3&5&2\\ 1&2&3&6&5&4&7\\ 5&4&6&2&7&1&3\\ 7&1&5&3&4&2&6\\ 4&6&2&7&1&3&5 \end{matrix} \qquad K_3' = \begin{matrix} 1&2&3&4&5&6&7\\ 3&5&1&6&7&4&2\\ 4&7&5&3&2&1&6\\ 2&4&6&5&3&7&1\\ 7&3&2&1&6&5&4\\ 5&6&4&7&1&2&3\\ 6&1&7&2&4&3&5 \end{matrix}$$

$$K_4 = \begin{matrix} 1&5&3&4&2&6&7\\ 2&3&6&5&7&1&4\\ 6&7&4&1&3&5&2\\ 3&2&7&6&5&4&1\\ 5&4&1&2&6&7&3\\ 7&1&5&3&4&2&6\\ 4&6&2&7&1&3&5 \end{matrix} \qquad K_4' = \begin{matrix} 1&2&3&4&5&6&7\\ 4&7&1&6&3&2&5\\ 3&4&7&5&6&1&2\\ 5&1&2&7&4&3&6\\ 7&6&4&3&2&5&1\\ 6&3&5&2&1&7&4\\ 2&5&6&1&7&4&3 \end{matrix}$$

$$K_5 = \begin{matrix} 1&2&3&4&5&6&7\\ 2&3&1&6&7&5&4\\ 6&5&4&7&3&1&2\\ 3&6&7&5&2&4&1\\ 5&4&6&2&1&7&3\\ 7&1&5&3&4&2&6\\ 4&7&2&1&6&3&5 \end{matrix} \qquad K_5' = \begin{matrix} 1&2&3&4&5&6&7\\ 4&6&5&3&2&7&1\\ 2&4&7&6&1&3&5\\ 7&5&4&1&3&2&6\\ 6&3&1&7&4&5&2\\ 3&7&2&5&6&1&4\\ 5&1&6&2&7&4&3 \end{matrix}$$

L_i $(1 \le i \le 8)$ are Latin squares of order 8 with an orthogonal mate L_i' as follows:

$$L_1 = \begin{matrix} 1&2&3&4&5&6&7&8\\ 6&3&2&1&8&5&4&7\\ 4&6&7&8&1&3&2&5\\ 7&8&5&6&4&2&1&3\\ 3&5&6&7&2&4&8&1\\ 2&4&1&5&7&8&3&6\\ 8&1&4&3&6&7&5&2\\ 5&7&8&2&3&1&6&4 \end{matrix} \qquad L_1' = \begin{matrix} 1&2&3&4&5&6&7&8\\ 5&1&8&2&4&7&3&6\\ 2&8&1&5&7&4&6&3\\ 4&3&2&1&6&5&8&7\\ 6&4&7&3&1&8&2&5\\ 3&7&4&6&8&1&5&2\\ 7&6&5&8&3&2&1&4\\ 8&5&6&7&2&3&4&1 \end{matrix}$$

$$L_2 = \begin{matrix} 1&2&3&4&5&6&7&8\\ 6&3&2&1&8&5&4&7\\ 4&6&7&8&1&3&2&5\\ 7&8&5&6&4&2&1&3\\ 3&5&6&7&2&4&8&1\\ 2&4&1&5&7&8&3&6\\ 8&7&4&3&6&1&5&2\\ 5&1&8&2&3&7&6&4 \end{matrix} \qquad L'_2 = \begin{matrix} 1&2&3&4&5&6&7&8\\ 2&1&4&5&6&7&8&3\\ 3&4&6&1&7&8&5&2\\ 4&5&1&8&2&3&6&7\\ 5&6&7&2&8&1&3&4\\ 6&7&8&3&1&4&2&5\\ 7&8&5&6&3&2&4&1\\ 8&3&2&7&4&5&1&6 \end{matrix}$$

$$L_3 = \begin{matrix} 1&2&3&4&5&6&7&8\\ 6&3&2&8&1&5&4&7\\ 4&6&7&1&8&3&2&5\\ 8&7&5&6&4&2&1&3\\ 3&5&6&7&2&4&8&1\\ 2&4&1&5&7&8&3&6\\ 7&8&4&3&6&1&5&2\\ 5&1&8&2&3&7&6&4 \end{matrix} \qquad L'_3 = \begin{matrix} 1&2&3&4&5&6&7&8\\ 2&1&8&5&4&7&6&3\\ 3&8&1&7&6&5&4&2\\ 7&5&4&1&2&3&8&6\\ 4&6&7&2&1&8&3&5\\ 5&7&6&3&8&1&2&4\\ 6&4&5&8&3&2&1&7\\ 8&3&2&6&7&4&5&1 \end{matrix}$$

$$L_4 = \begin{matrix} 1&2&7&4&5&6&3&8\\ 6&3&2&1&8&5&4&7\\ 4&6&1&8&7&3&2&5\\ 7&8&5&6&4&2&1&3\\ 3&5&6&7&2&4&8&1\\ 2&4&3&5&1&8&7&6\\ 8&1&4&3&6&7&5&2\\ 5&7&8&2&3&1&6&4 \end{matrix} \qquad L'_4 = \begin{matrix} 1&2&3&4&5&6&7&8\\ 5&4&7&2&1&8&3&6\\ 2&1&8&3&4&5&6&7\\ 8&5&2&7&6&1&4&3\\ 6&3&4&1&8&7&2&5\\ 3&8&1&6&7&4&5&2\\ 7&6&5&8&3&2&1&4\\ 4&7&6&5&2&3&8&1 \end{matrix}$$

$$L_5 = \begin{matrix} 1&2&7&4&5&6&3&8\\ 6&3&2&1&8&5&4&7\\ 7&6&1&8&4&3&2&5\\ 4&8&5&6&7&2&1&3\\ 3&5&6&7&2&4&8&1\\ 2&4&3&5&1&8&7&6\\ 8&1&4&3&6&7&5&2\\ 5&7&8&2&3&1&6&4 \end{matrix} \qquad L'_5 = \begin{matrix} 1&2&3&4&5&6&7&8\\ 3&1&4&2&7&8&6&5\\ 6&5&7&3&1&2&8&4\\ 8&4&1&7&2&3&5&6\\ 4&7&8&1&6&5&2&3\\ 7&3&5&6&8&1&4&2\\ 5&6&2&8&4&7&3&1\\ 2&8&6&5&3&4&1&7 \end{matrix}$$

$$L_6 = \begin{matrix} 1&6&7&4&5&3&2&8\\ 6&3&2&1&8&5&4&7\\ 7&2&1&8&4&6&3&5\\ 4&8&5&6&7&2&1&3\\ 3&5&6&7&2&4&8&1\\ 2&4&3&5&1&8&7&6\\ 8&1&4&3&6&7&5&2\\ 5&7&8&2&3&1&6&4 \end{matrix} \qquad L'_6 = \begin{matrix} 1&2&3&4&5&6&7&8\\ 3&1&4&7&6&2&8&5\\ 8&6&5&3&1&7&2&4\\ 2&5&1&6&7&8&4&3\\ 4&7&8&2&3&5&1&6\\ 5&3&7&8&2&4&6&1\\ 7&8&6&5&4&1&3&2\\ 6&4&2&1&8&3&5&7 \end{matrix}$$

$$L_7 = \begin{array}{cccccccc} 1 & 2 & 7 & 4 & 5 & 6 & 3 & 8 \\ 6 & 3 & 2 & 1 & 8 & 5 & 4 & 7 \\ 7 & 8 & 1 & 6 & 4 & 3 & 2 & 5 \\ 4 & 6 & 5 & 8 & 7 & 2 & 1 & 3 \\ 3 & 5 & 6 & 7 & 2 & 4 & 8 & 1 \\ 2 & 4 & 3 & 5 & 1 & 8 & 7 & 6 \\ 8 & 1 & 4 & 3 & 6 & 7 & 5 & 2 \\ 5 & 7 & 8 & 2 & 3 & 1 & 6 & 4 \end{array} \qquad L_7' = \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 6 & 8 & 4 & 2 & 5 & 7 \\ 5 & 3 & 2 & 7 & 1 & 8 & 4 & 6 \\ 6 & 4 & 7 & 2 & 8 & 1 & 3 & 5 \\ 2 & 8 & 5 & 1 & 7 & 3 & 6 & 4 \\ 8 & 7 & 4 & 3 & 6 & 5 & 2 & 1 \\ 7 & 5 & 8 & 6 & 2 & 4 & 1 & 3 \\ 4 & 6 & 1 & 5 & 3 & 7 & 8 & 2 \end{array}$$

$$L_8 = \begin{array}{cccccccc} 1 & 2 & 7 & 4 & 5 & 6 & 3 & 8 \\ 6 & 3 & 2 & 1 & 8 & 5 & 4 & 7 \\ 7 & 8 & 1 & 6 & 4 & 3 & 2 & 5 \\ 4 & 6 & 5 & 8 & 7 & 2 & 1 & 3 \\ 3 & 5 & 6 & 7 & 2 & 1 & 8 & 4 \\ 2 & 4 & 3 & 5 & 1 & 8 & 7 & 6 \\ 8 & 1 & 4 & 3 & 6 & 7 & 5 & 2 \\ 5 & 7 & 8 & 2 & 3 & 4 & 6 & 1 \end{array} \qquad L_8' = \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 4 & 2 & 3 & 8 & 6 & 7 \\ 6 & 5 & 7 & 3 & 1 & 2 & 8 & 4 \\ 8 & 4 & 1 & 7 & 2 & 3 & 5 & 6 \\ 3 & 7 & 8 & 1 & 6 & 4 & 2 & 5 \\ 7 & 3 & 5 & 6 & 8 & 1 & 4 & 2 \\ 4 & 6 & 2 & 8 & 7 & 5 & 3 & 1 \\ 2 & 8 & 6 & 5 & 4 & 7 & 1 & 3 \end{array}$$

N_i $(i = 1, 2, 3)$ are Latin squares of order 10 with an orthogonal mate N_i' as follows:

$$N_1 = \begin{array}{cccccccccc} 9 & 4 & 1 & 6 & 3 & 8 & 2 & 7 & 5 & 0 \\ 3 & 8 & 0 & 5 & 2 & 7 & 1 & 6 & 4 & 9 \\ 2 & 7 & 4 & 9 & 1 & 6 & 0 & 5 & 8 & 3 \\ 1 & 6 & 3 & 8 & 0 & 5 & 4 & 9 & 2 & 7 \\ 0 & 5 & 2 & 7 & 4 & 9 & 3 & 8 & 1 & 6 \\ 4 & 9 & 6 & 1 & 8 & 3 & 7 & 2 & 0 & 5 \\ 8 & 3 & 5 & 0 & 7 & 2 & 6 & 1 & 9 & 4 \\ 7 & 2 & 9 & 4 & 6 & 1 & 5 & 0 & 3 & 8 \\ 6 & 1 & 8 & 3 & 5 & 0 & 9 & 4 & 7 & 2 \\ 5 & 0 & 7 & 2 & 9 & 4 & 8 & 3 & 6 & 1 \end{array} \qquad N_1' = \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\ 9 & 3 & 4 & 8 & 6 & 2 & 5 & 0 & 1 & 7 \\ 0 & 1 & 7 & 6 & 4 & 3 & 9 & 5 & 2 & 8 \\ 2 & 9 & 6 & 0 & 1 & 7 & 8 & 3 & 4 & 5 \\ 5 & 6 & 2 & 7 & 0 & 4 & 3 & 9 & 8 & 1 \\ 6 & 5 & 8 & 9 & 7 & 0 & 4 & 1 & 3 & 2 \\ 8 & 4 & 1 & 2 & 9 & 5 & 6 & 7 & 0 & 3 \\ 3 & 8 & 9 & 5 & 2 & 1 & 0 & 6 & 7 & 4 \\ 7 & 0 & 5 & 1 & 3 & 8 & 2 & 4 & 6 & 9 \\ 4 & 7 & 0 & 3 & 8 & 9 & 1 & 2 & 5 & 6 \end{array}$$

$$N_2 = \begin{array}{cccccccccc} 9 & 4 & 1 & 6 & 3 & 8 & 2 & 7 & 5 & 0 \\ 3 & 8 & 0 & 5 & 2 & 7 & 1 & 6 & 4 & 9 \\ 2 & 7 & 4 & 9 & 1 & 6 & 0 & 5 & 3 & 8 \\ 1 & 6 & 3 & 8 & 0 & 5 & 4 & 9 & 2 & 7 \\ 0 & 5 & 2 & 7 & 4 & 9 & 3 & 8 & 6 & 1 \\ 4 & 9 & 6 & 1 & 8 & 3 & 7 & 2 & 0 & 5 \\ 8 & 3 & 5 & 0 & 7 & 2 & 6 & 1 & 9 & 4 \\ 7 & 2 & 9 & 4 & 6 & 1 & 5 & 0 & 8 & 3 \\ 6 & 1 & 8 & 3 & 5 & 0 & 9 & 4 & 7 & 2 \\ 5 & 0 & 7 & 2 & 9 & 4 & 8 & 3 & 1 & 6 \end{array} \qquad N_2' = \begin{array}{cccccccccc} 1 & 2 & 7 & 8 & 5 & 6 & 3 & 4 & 9 & 0 \\ 2 & 3 & 9 & 6 & 0 & 1 & 8 & 7 & 4 & 5 \\ 6 & 8 & 0 & 4 & 1 & 9 & 5 & 3 & 7 & 2 \\ 4 & 5 & 6 & 7 & 3 & 0 & 1 & 8 & 2 & 9 \\ 7 & 1 & 4 & 3 & 8 & 2 & 9 & 5 & 0 & 6 \\ 5 & 0 & 1 & 2 & 4 & 3 & 6 & 9 & 8 & 7 \\ 9 & 4 & 5 & 1 & 7 & 8 & 2 & 0 & 6 & 3 \\ 0 & 7 & 3 & 9 & 6 & 5 & 4 & 2 & 1 & 8 \\ 3 & 9 & 8 & 0 & 2 & 4 & 7 & 6 & 5 & 1 \\ 8 & 6 & 2 & 5 & 9 & 7 & 0 & 1 & 3 & 4 \end{array}$$

$$N_3 = \begin{array}{cccccccccc}
9&4&1&6&3&8&2&7&5&0\\
3&8&0&5&2&7&1&6&4&9\\
2&7&4&9&1&6&0&5&8&3\\
1&6&3&8&0&5&4&9&2&7\\
0&5&2&7&4&9&3&8&6&1\\
4&9&6&1&8&3&7&2&0&5\\
8&3&5&0&7&2&6&1&9&4\\
7&2&9&4&6&1&5&0&3&8\\
6&1&8&3&5&0&9&4&7&2\\
5&0&7&2&9&4&8&3&1&6
\end{array}
\qquad
N_3' = \begin{array}{cccccccccc}
1&2&7&8&5&6&3&4&9&0\\
2&1&6&5&7&8&4&3&0&9\\
0&3&5&7&8&9&1&2&4&6\\
9&4&8&3&2&1&6&0&5&7\\
8&0&1&2&4&5&9&7&6&3\\
3&6&2&1&0&4&5&9&7&8\\
5&7&3&4&9&2&0&6&8&1\\
6&8&4&9&1&0&7&5&3&2\\
7&5&9&0&6&3&2&8&1&4\\
4&9&0&6&3&7&8&1&2&5
\end{array}$$

P and Q are exhibited as follows (Note that P is a (3,1,2)-conjugate orthogonal Latin square of order 10 with an empty subarray on $\{8, 9, 10\}$, which comes from [2]; Q is a (3,2,1)-conjugate orthogonal Latin square of order 14 with an empty subarray on $\{A, B, C, D\}$, which comes from [3]):

$$\begin{array}{cccccccccc}
1&5&2&8&3&10&9&4&7&6\\
9&2&6&3&8&4&10&5&1&7\\
10&9&3&7&4&8&5&6&2&1\\
6&10&9&4&1&5&8&7&3&2\\
8&7&10&9&5&2&6&1&4&3\\
7&8&1&10&9&6&3&2&5&4\\
4&1&8&2&10&9&7&3&6&5\\
2&3&4&5&6&7&1&&&\\
3&4&5&6&7&1&2&&&\\
5&6&7&1&2&3&4&&&
\end{array}$$

P

$$\begin{array}{cccccccccccccc}
0&6&A&5&B&9&C&3&D&7&8&4&2&1\\
8&1&7&A&6&B&0&C&4&D&9&5&3&2\\
D&9&2&8&A&7&B&1&C&5&0&6&4&3\\
6&D&0&3&9&A&8&B&2&C&1&7&5&4\\
C&7&D&1&4&0&A&9&B&3&2&8&6&5\\
4&C&8&D&2&5&1&A&0&B&3&9&7&6\\
B&5&C&9&D&3&6&2&A&1&4&0&8&7\\
2&B&6&C&0&D&4&7&3&A&5&1&9&8\\
A&3&B&7&C&1&D&5&8&4&6&2&0&9\\
5&A&4&B&8&C&2&D&6&9&7&3&1&0\\
1&2&3&4&5&6&7&8&9&0&&&&\\
7&8&9&0&1&2&3&4&5&6&&&&\\
3&4&5&6&7&8&9&0&1&2&&&&\\
9&0&1&2&3&4&5&6&7&8&&&&
\end{array}$$

Q

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