A new bound on the size of the largest 2-critical set in a latin square

Nicholas J. Cavenagh

Centre for Discrete Mathematics and Computing
Department of Mathematics
The University of Queensland
Queensland 4072
Australia
email: njc@maths.uq.edu.au

Abstract

A critical set is a partial latin square that has a unique completion to a latin square of the same order, and is minimal in this property. If P is a critical set in a latin square L, then each element of P must be contained in a latin trade Q in L such that $|P \cap Q| = 1$. In the case where each element of P is contained in an intercalate (latin trade of size 4) Q such that $|P \cap Q| = 1$ we say that P is 2-critical. In this paper we show that the size of a 2-critical set in a latin square L is no greater than $n^2 - O(n^{5/4})$.

1 Background information

In any combinatorial configuration it is possible to identify a subset which uniquely determines the structure of the configuration and in some cases is minimal with respect to this property. Examples of such subsets can be found by studying the literature on critical sets in latin squares (see Donovan and Howse [**8**]) and defining sets in block designs (see Street [**11**]). The recent research in these areas has focused on building a bank of knowledge which may be used to determine the spectrum of the prescribed subsets. With this current paper we restrict ourselves to a discussion of critical sets in latin squares.

We define $scs(n)$ and $lcs(n)$ to be the sizes of the smallest and largest critical sets in any latin square of order n. The problem of determining these values exactly for every n remains unsolved. However progress has been made in both cases.

Fu, Fu and Rodger ([9]) showed that if $n > 20$, $\mathrm{scs}(n) \geq \lfloor (7n-3)/6 \rfloor$. The smallest critical set so far constructed for any latin square of size n has size $\lfloor n^2/4 \rfloor$ ([5], [4]). A critical set of such size is known to exist in back circulant latin squares, namely those latin squares based on the addition table for the integers modulo n.

Donovan, Cooper, Nott and Seberry ([7]) examined lower bounds for critical sets of latin squares based on certain groups. Bate and van Rees ([1]) showed that the size of the smallest strong critical set (a critical set with a certain type of completion) is $\lfloor n^2/4 \rfloor$.

Stinson and van Rees ([10]) determined some lower bounds for $lcs(n)$ for small values of n. The largest critical sets that have been constructed in latin squares of order n have size $4^m - 3^m$, where $n = 2^m$. It is interesting to note that these critical sets are also 2-critical. This has lead to a conjecture made by Bean and Mahmoodian [3].

Conjecture 1 $lcs(n) \leq n^2 - n^{\log_2 3} \approx n^2 - n^{1.6}$.

A larger upper bound for $lcs(n)$ is conjectured by Brankovic, Horak, Miller and Rosa [2].

Conjecture 2 $lcs(n) \leq n^2 - n^{1.5}$.

Curran and van Rees [5] showed that $lcs(n) \leq n^2 - n$; this bound was improved by Bean and Mahmoodian [3]:

Theorem 3 $lcs(n) \leq n^2 - 3n + 3$.

This is currently the best known upper bound that applies to arbitrary critical sets. In Theorem 11 we find an improved upper bound for 2-critical sets.

2 Definitions

We start with basic definitions which allow us to state and prove our main results.

Let $N = \{1, 2, \ldots, n\}$. A *partial latin square* P of order n is a set of ordered triples of the form $(i, j; k)$, where $i, j, k \in N$ with the following properties:

- if $(i, j; k) \in P$ and $(i, j; k') \in P$ then $k = k'$,
- if $(i, j; k) \in P$ and $(i, j'; k) \in P$ then $j = j'$ and
- if $(i, j; k) \in P$ and $(i', j; k) \in P$ then $i = i'$.

We may also represent a partial latin square P as an $n \times n$ array with entries chosen from the set N such that if $(i, j; k) \in P$, the *entry* k occurs in cell (i, j). It is important for the reader to note that an *element* of P is a triple $(i, j; k) \in P$, whereas an *entry* in P is an element $k \in N$ such that $(i, j; k) \in P$, for some i and j. A partial latin square has the property that each entry occurs at most once in each row and at most once in each column.

Every partial latin square has six *conjugates* (five and itself). In this paper we will make use of the conjugates

$\{(k, j; i) \mid (i, j; k) \in P\}$, and

$$\{(i,k;j) \mid (i,j;k) \in P\}.$$

Because of the existence of these conjugates, statements made about the entries of a partial latin square may also be made about its rows, and in turn, columns. (See, for example, Corollary 5.)

If all the cells of the array are filled then the partial latin square is termed a latin square. That is, a *latin square* L of order n is an $n \times n$ array with entries chosen from the set $N = \{1, \ldots, n\}$ in such a way that each element of N occurs precisely once in each row and precisely once in each column of the array.

For a given partial latin square P the set of cells $\mathcal{S}_P = \{(i,j) \mid (i,j;k) \in P$, for some $k \in N\}$ is said to determine the *shape* of P and $|\mathcal{S}_P|$ is said to be the *size* of the partial latin square. That is, the size of P is the number of non-empty cells in the array. For each r, $1 \leq r \leq n$, let $\mathcal{R}_P^r$ denote the set of entries occurring in row r of P. Formally, $\mathcal{R}_P^r = \{k \mid (r,j;k) \in P\}$. Similarly, for each c, $1 \leq c \leq n$, we define $\mathcal{C}_P^c = \{k \mid (i,c;k) \in P\}$.

A partial latin square Q of order n is said to be a *latin trade* (or *latin interchange*) if $Q \neq \emptyset$ and there exists a partial latin square Q' (called a *disjoint mate* of Q) of order n, such that

$\mathcal{S}_Q = \mathcal{S}_{Q'}$,

if $(i,j;k) \in Q$ and $(i,j;k') \in Q'$, then $k \neq k'$,

for each r, $1 \leq r \leq n$, $\mathcal{R}_Q^r = \mathcal{R}_{Q'}^r$, and

for each c, $1 \leq c \leq n$, $\mathcal{C}_Q^c = \mathcal{C}_{Q'}^c$.

A *critical set* in a latin square L (of order n) is a partial latin square $\mathcal{C}$ in L, such that

(1) L is the only latin square of order n which has element k in cell (i,j) for each $(i,j;k) \in \mathcal{C}$; and

(2) no proper subset of $\mathcal{C}$ satisfies (1).

A *uniquely completable set* (UC) in a latin square L of order n is a partial latin square in L which satisfies condition (1) above. So a uniquely completable set P in a latin square L is a critical set if for each $(i,j;k) \in P$, there exists a latin trade Q in L such that $Q \cap P = \{(i,j;k)\}$. (This ensures that $P \setminus \{(i,j;k)\}$ has at least two completions: L and $(L \setminus Q) \cup Q'$.)

An *intercalate* is a latin trade of size 4. This is the smallest possible size for a latin trade, and each intercalate is isotopic to a latin square of order 2. Suppose that P is a critical set in a latin square L, with the property that for each $(i,j;k) \in P$, there exists an *intercalate* Q in L such that $Q \cap P = \{(i,j;k)\}$. In this case we say that P is 2-*critical*. (See [10] for a definition of $n_1, \ldots, n_l$-critical, for integers $n_1, \ldots n_l \geq 2$.)

If P is a critical set in a latin square L, the partial latin square given by $L \setminus P$ is called the *compliment* of P and is denoted by P^C.

3 Main Results

Lemma 4 *Let P be a 2-critical set in a latin square L of order n. Let the number of elements with entry i in P^C be e_i. Then,*

$$|P| \leq \sum_{i=1}^{n} \binom{e_i}{2}.$$

Proof. Let S be the set

$$\{Q \mid Q \text{ is an intercalate in } L \text{ and } |Q \cap P| = 1\}.$$

By the definition of a 2-critical set, $|P| \leq |S|$. Let $(i, j; k) \in P$. Then there exists an intercalate $Q = \{(i, j; k), (i, j'; k'), (i', j; k'), (i', j'; k)\} \in S$ for some integers $i' \neq i$, $j' \neq j$ and $k' \neq k$. No other intercalate in S can use both elements $(i, j'; k')$ and $(i', j; k')$ from P^C. Thus each intercalate in S uses a unique pair of elements of P^C that contain the same entry. The total number of such pairs is $\sum_{i=1}^{n} \binom{e_i}{2}$. The result follows. □

The following two corollaries are the conjugate arguments of the previous lemma. They may be proven similarly.

Corollary 5 *Let P be a 2-critical set in a latin square L of order n. Let $r_i = |\mathcal{R}^i_{PC}|$, the number of elements in row i in P^C. Then,*

$$|P| \leq \sum_{i=1}^{n} \binom{r_i}{2}.$$

Corollary 6 *Let P be a 2-critical set in a latin square L of order n. Let $c_i = |\mathcal{C}^i_{PC}|$, the number of elements in column i in P^C. Then,*

$$|P| \leq \sum_{i=1}^{n} \binom{c_i}{2}.$$

The next lemma applies to all critical sets, not just 2-critical sets.

Lemma 7 *Let P be a critical set in a latin square L of order n. Let the number of elements with entry i in P^C be e_i. Then,*

$$|P| \leq 2\sum_{i=1}^{n} \binom{e_i}{2}.$$

Proof. Let $(i,j;k) \in P$. Then there exists a latin trade $Q \in L$ with $(i,j;k) \in Q$. Let Q' be the disjoint mate of Q, and let $k' \neq k$ be the entry that occurs in cell (i,j) of Q'. Then there must exist elements $(i,j';k')$ and $(i',j;k')$ in Q with $i' \neq i$ and $j' \neq j$. We define a pair of elements derived in this fashion to be *special* relative to the element $(i,j;k) \in P$. Let $(\alpha,\beta;\gamma)$ and $(\alpha',\beta';\gamma)$ be a pair of elements in P^C. This pair may be special relative to at most two elements of P: those occurring in cells (α,β') and (α',β). The result follows. $\square$

Once again, we have two conjugate corollaries.

Corollary 8 *Let P be a critical set in a latin square L of order n. Let $r_i = |\mathcal{R}^i_{P^C}|$, the number of elements in row i in P^C. Then,*

$$|P| \leq 2\sum_{i=1}^{n} \binom{r_i}{2}.$$

Corollary 9 *Let P be a critical set in a latin square L of order n. Let $c_i = |\mathcal{C}^i_{P^C}|$, the number of elements in column i in P^C. Then,*

$$|P| \leq 2\sum_{i=1}^{n} \binom{c_i}{2}.$$

Corollary 10 *Let P be a critical set in a latin square L of order n. Let the number of elements with entry i in P^C be e_i. Then there exists i such that $e_i \geq \sqrt{n}$.*

Proof. From the previous lemma,

$$\begin{aligned} |P| &\leq 2\sum_{i=1}^{n} \binom{e_i}{2} \quad \text{or} \\ n^2 - |P^C| &\leq \sum_{i=1}^{n} e_i^2 - \sum_{i=1}^{n} e_i = \sum_{i=1}^{n} e_i^2 - |P^C| \quad \text{and so} \\ \sum_{i=1}^{n} e_i^2 &\geq n^2. \end{aligned}$$

Suppose that each $e_i < \sqrt{n}$. Then $\sum_{i=1}^{n} e_i^2 < n^2$, a contradiction. $\square$

Theorem 11 *Let P be a 2-critical set in a latin square L of order n. Then*

$$|P| \leq n^2 - O(n^{5/4}).$$

Proof. Let the number of elements with entry i in P^C be e_i. Assume that there exists i such that $e_i < \sqrt{n}$. (If not, $|P^C| \geq n^{3/2}$ and our theorem is true.) Without loss of generality, assume that $e_1 < \sqrt{n}$, and that the cells in L containing the entry 1 are the diagonal cells (i, i), $1 \le i \le n$. (If not, rows and columns may be permuted.) Let a_i be the number of elements in row i of P^C plus the number of elements in column i of P^C, excluding the element $(i, i; 1)$. Without loss of generality, assume that $a_1 \ge a_2 \ge \cdots \ge a_n$. (If not, once again, we may permute rows and columns to make this so.)

Let S be the set

$$\{Q \mid Q \text{ is an intercalate in } L,\ |Q \cap P| = 1 \text{ and if } Q \cap P = \{(i, j; k)\}, k \ne 1\}.$$

For each $(i, j; k) \in P$, either there exists an intercalate $Q \in S$ such that $Q \cap P = \{(i, j; k)\}$, or $k = 1$. Therefore $|P| \le |S| + n$. We will obtain an upper bound on the size of S, and thus an upper bound for the size of P.

First consider the case when $Q \in S$ and Q contains elements with the entry 1. The intercalate Q must be of the following form: $Q = \{(i, j; k), (i, i; 1), (j, j; 1), (j, i; k)\}$, where $(i, j; k) \in P$, and $\{(i, i; 1), (j, j; 1), (j, i; k)\} \subseteq P^C$. The elements $(i, i; 1)$ and $(j, j; 1)$ can occur as a pair in at most one such intercalate Q. Therefore the number of intercalates $Q \in S$ containing elements with entry 1 is bounded by $\binom{e_1}{2}$, which is no more than $n/2$.

Let T be the following subset of S:

$$T = \{Q \mid Q \in S \text{ and } Q \text{ contains no elements with entry } 1\}.$$

From the previous observation, $|P| \le |S| + n \le |T| + 3n/2$. Next we partition the set T of intercalates into disjoint subsets $T_1, T_2, \ldots, T_n$. Consider an intercalate Q. It has a row j and a column k $(j \ne k)$ that do not contain an element of the critical set. Let i be the minimum of j and k. Then Q is considered to be an element of T_i.

Now, each intercalate in T_i uses a unique pair of elements from either row or column i of P^C, so $|T_i| \le \binom{a_i}{2}$.

Another bound for $|T_i|$ can be obtained using a function $f_i : T_i \to P^C$, defined as follows. Let Q be an intercalate in T_i. Then Q contains two elements in either row i or column i, one element in P and a third element of P^C, say $(\alpha, \beta; \gamma)$, where $\alpha \ne i$ and $\beta \ne i$. We define $f_i(Q) = (\alpha, \beta; \gamma)$.

Given i and an element $(i', j'; k') \in P^C$ we wish to give an upper bound to the size of the set

$$f_i^{-1}((i', j'; k')) = \{Q \mid Q \in T_i \text{ and } f_i(Q) = (i', j'; k')\}.$$

The size of this set depends on whether $i' > i$ or $i' < i$, and on whether $j' > i$ or $j' < i$.

Case 1: $i' > i$ and $j' > i$. Here $(i', j'; k')$ may be contained in at most two intercalates Q such that $f_i(Q) = (i'j'; k')$, one intersecting row i and the other column i. Thus $|f_i^{-1}((i', j'; k'))| \le 2$. This is illustrated in Figure 1. The entries in bold indicate elements of P.

Case 2: $i' > i$ and $j' < i$. Here $|f_i^{-1}((i', j'; k'))| \le 1$. The element $(i', j'; k')$ may be used by at most one intercalate that intersects row i. If there is an intercalate intersecting $(i', j'; k')$ and column i, this is an element of $T_{j'}$.

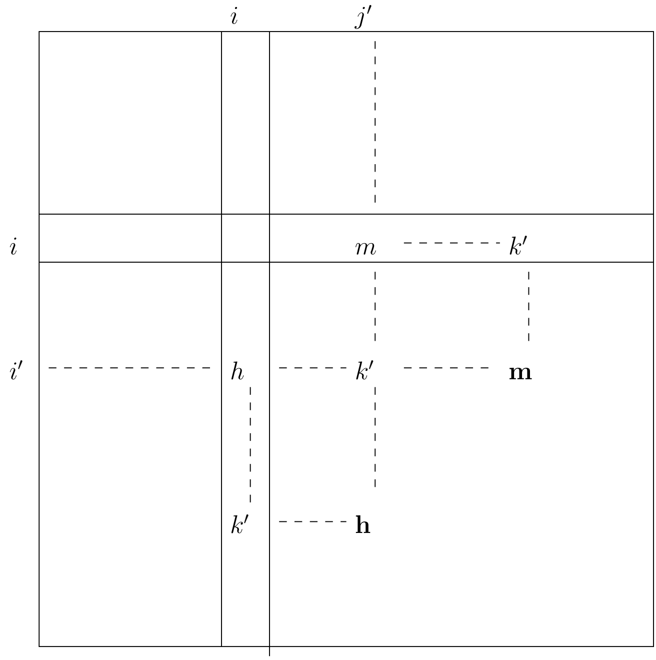

Figure 1: $i' > i$ and $j' > j$

Case 3: $i' < i$ and $j' > i$. This is similar to Case 2 and we get at most one intercalate.

Case 4: $i' < i$ and $j' < i$. Here $|f_i^{-1}((i', j'; k'))| = 0$. Any intercalates that intersect row or column i and $(i', j'; k')$ are elements of $T_{i'}$ and $T_{j'}$ respectively.

So depending on the location of each element of P^C, it may give rise to either 0, 1 or 2 intercalates in T_i. Thus $|T_i|$ is no more than

$$\begin{gathered}2|\{(i', j'; k') \mid (i', j'; k') \in P^C, i' > i, j' > i\}|+\\ |\{(i', j'; k') \mid (i', j'; k') \in P^C, i' > i, j' < i\}|+\\ |\{(i', j'; k') \mid (i', j'; k') \in P^C, i' < i, j' > i\}|.\end{gathered}$$

But this value is no greater than $\sum_{j=i+1}^{n} a_j$. Noting the fact that $\sum_{j=1}^{n} a_j < 2|P^C|$, we have that $|T_i| < 2|P^C| - \sum_{j=1}^{i} a_j$.

Now, choose m to be the greatest integer between 1 and n so that $a_m \geq \sqrt{n}$. (The justification for our choice of $\sqrt{n}$ is that it gives a significantly low upper bound for $|P|$ at the end of our proof.) We obtain an upper bound for $|T| = \sum_{i=1}^{n} |T_i|$, using the bound $2|P^C| - \sum_{j=1}^{i} a_j$ for $|T_i|$ when $1 \leq i \leq m$, and using the bound $\binom{a_i}{2}$ for $|T_i|$ when $m+1 \leq i \leq n$. As previously observed, there are less than n elements of P with entry 1, and at most $n/2$ intercalates in S containing 1 as an entry. So we have:

$$|P| \;<\; 3n/2 + \sum_{i=1}^{m}(2|P^C| - \sum_{j=1}^{i} a_i) + \sum_{i=m+1}^{n} \binom{a_i}{2}.$$

Next use the fact that $|P| + |P^C| = n^2$.

$$\begin{aligned} n^2 - |P^C| &< 3n/2 + 2m|P^C| - \sum_{i=1}^{m}\sum_{j=1}^{i} a_i + \sum_{i=m+1}^{n} \binom{a_i}{2}, \text{ or} \\ (2m+1)|P^C| &> n^2 - 3n/2 + \sum_{i=1}^{m}\sum_{j=1}^{i} a_i - \sum_{i=m+1}^{n} \binom{a_i}{2}. \end{aligned}$$

If $i \le m$ then $a_i \ge \sqrt{n}$ and if $i \ge m+1$, $-a_i > -\sqrt{n}$. Therefore

$$(2m+1)|P^C| \;>\; n^2 - 3n/2 + m\sqrt{n}(m+1)/2 - (n-m)n/2.$$

We add $|P^C|$ to the left hand side, and rearrange the right hand side to obtain:

$$(2m+2)|P^C| \;>\; n^2/2 + m\sqrt{n}(m+1)/2 + (m-3)n/2.$$

If $m = 1$, we have

$$|P^C| \;>\; (n^2 - 2n + 2\sqrt{n})/8,$$

and if $m = 2$, we have

$$|P^C| \;>\; (n^2 - n + 6\sqrt{n})/12.$$

Certainly in both cases $|P^C| > O(n^{5/4})$. Otherwise $m \ge 3$ and we can let the term $(m-3)n/2$ disappear from our inequality:

$$\begin{aligned} (2m+2)|P^C| &> n^2/2 + m\sqrt{n}(m+1)/2, \quad \text{or equivalently} \\ |P^C| &> n^2/(4(m+1)) + m\sqrt{n}/4. \end{aligned}$$

By differentiating the right hand side with respect to m, and letting this expression equal zero, we have $m + 1 = n^{3/4}$. The second derivative is always positive, so $m = n^{3/4} - 1$ yields a minimum value for the right hand side. It follows that $|P^C| > n^{5/4}/2 - n^{1/2}/4 \ge n^{5/4}/4$. And so $|P^C| > O(n^{5/4})$, or equivalently $|P| < n^2 - O(n^{5/4})$. $\square$

References

[1] J.A. Bate and G.H.J. van Rees, The size of the smallest strong critical set in a Latin square, Ars Combin. 53 (1999), 73–83.

[2] L. Brankovic, P. Horak, M. Miller and A. Rosa, Premature partial latin squares, submitted to Ars Combin.

[3] R. Bean and E.S. Mahmoodian, A new bound on the size of the largest critical set in a latin square, Discrete Math. (to appear).

[4] J. Cooper, D. Donovan and J. Seberry, Latin squares and critical sets of minimal size, Australas. J. Combin. 4 (1991), 113–120.

[5] D. Curran and G.H.J. van Rees, Critical sets in latin squares, Proc. 8th Manitoba Conference on Numerical Mathematics and Computing, (Congressus Numerantium XXII), Utilitas Math. Pub., Winnipeg, 1978, pp. 165–168.

[6] D. Donovan and J. Cooper, Critical sets in back circulant latin squares, Aequationes Math. 52, No. 1-2 (1996), 157–179.

[7] D. Donovan, J. Cooper, D.J. Nott and J. Seberry, Latin squares: critical sets and their lower bounds, Ars Combin. 39 (1995), 33–48.

[8] D. Donovan and A. Howse, Towards the spectrum of critical sets, Australas. J. Combin. 21 (2000), 107–130.

[9] C-M. Fu, H-L. Fu and C.A. Rodger, The minimum size of critical sets in latin squares, J. Statist. Plann. Inference. 62, No. 2 (1997), 333–337.

[10] D.R. Stinson and G.H.J. van Rees, Some large critical sets, Congressus Numerantium 34 (1982), 441–456.

[11] A.P. Street, "Trades and defining sets," CRC Handbook of Combinatorial Designs, C.J. Colbourn and J.H. Dinitz (Editors), CRC Publishing Co., 1996, pp. 474–478.

(Received 27/8/2001)