# Minimum cycle bases of product graphs

Wilfried Imrich

Department of Mathematics and Applied Geometry
Montanuniversität Leoben
Franz Josef-Straße 18
A-8700 Leoben, Austria
imrich@unileoben.ac.at

Peter F. Stadler*

Institute for Theoretical Chemistry and Molecular Structural Biology
University of Vienna
Währingerstrasse 17
A-1090 Vienna, Austria
studla@tbi.univie.ac.at

**Abstract**

A construction for a minimum cycle basis for the Cartesian and the strong product of two graphs from the minimum length cycle bases of the factors is presented. Furthermore, we derive asymptotic expressions for the average length of the cycles in the minimum cycle bases of the powers (iterated products) of graphs. In the limit only triangles and squares play a role.

## 1 Introduction

Minimum length bases of the cycle space of a graph (MCBs) have a variety of applications in science and engineering, for example, in structural flexibility analysis [15], electrical networks [5], and in chemical structure storage and retrieval systems [6]. Brief surveys and extensive references can be found in [13, 12].

In general, minimum cycle bases are not very well behaved under graph operations. Neither the total length $\ell(G)$ nor the length of the longest cycle $\lambda(G)$ in a MCB of $G$ are minor monotone, see Fig. 1 for a counterexample. Hence, there does not seem to be a general way of extending MCBs of a certain collection of partial graphs of $G$ to an MCB of $G$. Consequently, not much is known about the length

* Current address: The Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA

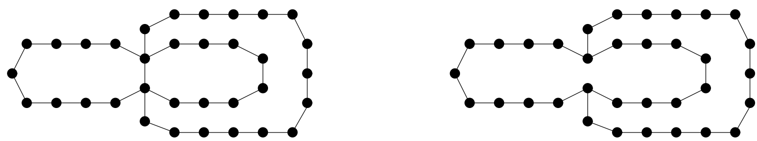


Figure 1: $G_1$ has $\nu(G_1) = 3$ and $\ell(G_1) = 38$. Deletion of a single edge leads to $G_2$ with $\nu(G_2) = 2$ but $\ell(G_2) = 44$.

$\ell(G)$ of the MCB. The sharp upper bound $\ell(G) \leq \ell(K_m) = 3(m-1)(m-2)/2$ for graphs with $m$ vertices is proved in [13, Thm.6]. For 2- connected outerplanar and planar graphs we have $\ell(G) \leq 3m-6$ and $\ell(G) \leq 6m-15$, respectively [16, Thm.11]. A global upper bound $\ell(G) \leq \nu(G) + \kappa(\mathsf{T}(G))$, where $\nu(G)$ is the cyclomatic number of $G$ and $\kappa(\mathsf{T}(G))$ the connectivity of the tree graph of $G$, is derived in [17].

Iterated Cartesian products of small graphs play a role in many fields [14]. The generalized hypercubes (homogeneous Hamming graphs), for instance, are ubiquitous in computer science, theoretical biology, and the physics of disordered systems. Cartesian powers of cycles appear as configuration spaces in certain spin glass models [9].

In this paper we construct MCBs for two classes of composite graphs: the Cartesian and the strong products. Moreover, we construct MCBs for iterated products of graphs with respect to these products, and compute the average cycle length of MCBs of such powers. Interestingly, in the limit only cycles of lengths three and four play a role. Corresponding results for the direct product will be published separately.

# 2 Preliminaries

Throughout this contribution we consider only simple, unweighted, undirected graphs. Let $G(V, E)$ be the graph with vertex set $V$ and edge set $E$. For the edge $e \in E$ joining the vertices $x, y \in V$ we write $e = xy$. The subgraph induced by $W \subset V$ is denoted by $G[W]$. The cardinality of a set $A$ is $|A|$.

## 2.1 Cycle Bases

Let $G(V, E)$ be a graph. The set $\mathcal{P}(E)$ of all subsets of $E$ forms an $|E|$-dimensional vector space over GF(2) with vector addition $X \oplus Y := (X \cup Y) \setminus (X \cap Y)$ and scalar multiplication $1 \cdot X = X$, $0 \cdot X = \emptyset$ for all $X, Y \in \mathcal{P}(E)$.

A (generalized) *cycle* is a subgraph such that every vertex has even degree. Such graphs are also known as Eulerian subgraphs. We represent a (generalized) cycle by its edge set $C$ and write $V_C$ for its vertex set. An *elementary cycle* is a connected minimal subgraph such that every vertex in $V_C$ has degree 2. The set $\mathcal{C}$ of all generalized cycles forms a subspace of $(\mathcal{P}(E), \oplus, \cdot)$ which is called the *cycle space* $\mathcal{C}(G)$ of $G$. A basis $\mathcal{B}$ of the cycle space $\mathcal{C}$ is called a *cycle basis* of $G(V, E)$ [3].

The dimension of the cycle space is the *cyclomatic number* or *first Betti number* $\nu(G) = |E| - |V| + 1$, see [8].

The length $|C|$ of a generalized cycle $C$ is the number of its edges. The length $\ell(\mathcal{B})$ of a cycle basis $\mathcal{B}$ is the sum of the lengths of its generalized cycles: $\ell(\mathcal{B}) = \sum_{C\in\mathcal{B}} |C|$. A *minimum cycle basis* (MCB) $\mathcal{M}$ is a cycle basis with minimum length. Since the cycle space $\mathcal{C}(G)$ is a matroid in which an element has weight $|C|$ the greedy algorithm can be used to extract a MCB, see e.g. [20]. The cycles in $\mathcal{M}$ are edge-connected, chordless, and isometric, see e.g. [13]. An MCB can be computed in polynomial time [13, 1].

A cycle is *relevant* if it is contained in some MCB [18].

**Proposition 1** [19] *A cycle $C$ is relevant if and only if it cannot be written as a $\oplus$-sum of shorter cycles.*

## 2.2 Products

Given two non-empty graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ the Cartesian product $G\Box H$ has vertex set $V_G \times V_H$ and $(x_1, x_2)(y_1, y_2)$ is an edge in $E_{G\Box H}$ iff either $x_2 = y_2$ and $x_1y_1 \in E_G$ or if $x_1 = y_1$ and $x_2y_2 \in E_H$, see e.g. [14]. In particular, the product of two edges $x_1y_1 = e_1 \in E_G$ and $x_2y_2 = e_2 \in E_H$ is the chordless 4-cycle

$$e_1\Box e_2 = \{(x_1, y_1)(x_1, y_2), (x_1, y_2)(x_2, y_2), (x_2, y_2)(x_2, y_1), (x_2, y_1)(x_1, y_1)\}. \quad (1)$$

Two edges $(x_1, x_2)(y_1, y_2)$ and $(x'_1, x'_2)(y'_1, y'_2)$ are *parallel* if $x_1 = y_1$, $x'_1 = y'_1$, $x_2 = x'_2$, and $y_2 = y'_2$ or if $x_2 = y_2$, $x'_2 = y'_2$, $x_1 = x'_1$, and $y_1 = y'_1$. Note that the square $e\Box f$ consists of two pairs of parallel edges. We write $\mathcal{C}_\Box = \{e\Box f | e \in E_G, f \in E_H\}$.

The direct product $G \times H$ has the same vertex set as the Cartesian product, and $(x_1, x_2)(y_1, y_2)$ is an edge if $x_1y_1 \in E_G$ and $x_2y_2 \in E_H$. The strong product $G \boxtimes H$ is also defined on $V_G \times V_H$. Its edge set is the union of the edge sets of the Cartesian and the direct product. All three products are commutative and associative, which implies that powers (iterated products) with respect to the three products are well defined.

The induced subgraphs $G^y = G\Box H[\{(x, y)|x \in V_G\}]$ and ${}^xH = G\Box H[\{(x, y)|y \in V_H\}]$ are the *fibers* of $G\Box H$; for the other products fibers are defined analogously. In the case of the Cartesian and the strong product the fibers are isomorphic to the corresponding factors. The fibers with respect to the direct product have no edges.

The edges of $G\Box H$ can be labeled as ${}^xf$ for $f \in E_H$ and $x \in V_G$ or $e^y$ for $e \in E_G$ and $y \in V_H$. In this notation we have $e\Box f = \{e^x, {}^vf, e^y, {}^uf\}$ where $e = uv \in E_G$ and $f = xy \in E_H$. Note that two edges of $G\Box H$ are parallel if and only if they are of the form $e^x$ and $e^y$ or ${}^uf$ and ${}^vf$, respectively. If $C$ and $D$ are cycles in $G$ and $H$, respectively, we write $C^y$ and ${}^xD$ for the corresponding cycles in the fibers $G^y$ and ${}^xH$.

## 2.3 Hammack's Cycle Basis for Cartesian Products

In his paper [11] on the cyclicity of graphs, Richard Hammack constructs a cycle basis for $G\Box H$ in the following way: Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two non-empty graphs, $T_G$ and $T_H$ spanning trees, and $\mathcal{B}_G$ and $\mathcal{B}_H$ cycle bases of $G$ and $H$, respectively. By [11, p.165] the union of the pairwise disjoint sets

$$\begin{aligned} \mathcal{H}_1 &= \{e\Box f | e \in T_G,\ f \in T_H\} \\ \mathcal{H}_2 &= \{C^y | C \in \mathcal{B}_G,\ y \in V_H\} \\ \mathcal{H}_3 &= \{{}^x C | x \in V_G,\ C \in \mathcal{B}_H\} \end{aligned} \tag{2}$$

is a cycle basis of $G\Box H$. We denote it by $\mathcal{H}$ and call it a *Hammack basis* of $G\Box H$.

Recall that a graph is *null-homotopic* if each of its cycles can be written as a sum of triangles [2, 7].

**Lemma 2** *Let $G$ and $H$ be two null-homotopic graphs and $\mathcal{M}_G$ and $\mathcal{M}_H$ be MCBs of $G$ and $H$, respectively. Then the resulting Hammack basis is a minimum cycle basis of $G\Box H$.*

*Proof.* By the above the cycle space $\mathcal{C}(G\Box H)$ has a basis that consists of triangles and squares only. If $C$ is a triangle then it must be contained in a fiber $G^y$ or a fiber ${}^xH$. Furthermore, a maximal set of linearly independent triangles is the union of maximal sets of independent triangles in each fiber. In the present case it is the union of the MCBs of all fibers. The remaining basis elements must therefore be squares. □

The Hammack basis is not a MCB in general. The smallest counterexample is the pentagonal prism $K_2\Box C_5$. Its Hammack basis consists of two pentagons ${}^xC_5$ and ${}^yC_5$ and four squares. Thus $\ell(\mathcal{B}) = 26$. However, a minimum cycle basis $\mathcal{M}$ of $K_2\Box C_5$ consists of all five squares and only one of the two pentagons. Thus $\ell(G) = 25$.

# 3 A Minimum Cycle Basis for the Cartesian Product

In this section we construct a minimal basis of $\mathcal{C}(G\Box H)$. The crucial observation is that corresponding cycles in different fibers can be transformed into each other by a series of $\oplus$-additions of squares taken from the set $\mathcal{C}_\Box$.

**Lemma 3** *Let $C \in \mathcal{C}(G)$ and $y, z \in V_H$. Then there is a collection of squares $\mathcal{Q}' \subseteq \mathcal{C}_\Box$ such that*

$$C^z = C^y \oplus \bigoplus \mathcal{Q}' \tag{3}$$

*Proof.* For $y, z \in V_H$ let $P$ be a path from $y$ to $z$ consisting of the edges $f_1, \dots, f_l$ and the vertices $q_0 = y, q_1, \dots, q_l = z$. Then ${}^xP$ is the corresponding path in the fiber ${}^xH$ with vertices $(x, q_j)$ and edges ${}^xf_j$. For each edge $g = x_1x_2 \in E_G$ and each path $P$ in $H$ we write

$$C(g; P) = \{{}^{x_1}f_1, {}^{x_1}f_2, \dots, {}^{x_1}f_l, g^z, {}^{x_2}f_l, {}^{x_2}f_{l-1}, \dots, {}^{x_2}f_1, g^y\} \tag{4}$$

for the cycle composed of the paths ${}^{x_1}P$ from $(x_1, y)$ to $(x_1, z)$ and ${}^{x_2}P$ from $(x_2, z)$ to $(x_2, y)$ together with the edges $(x_1, z)(x_2, z)$ and $(x_2, y)(x_1, y)$. In particular, if $P$

consists of a single edge $h$ we have $C(g;P) = g\Box h$. Write $P_k$ and $P^k$ for the subpaths from $y$ to $q_k$ and from $q_k$ to $z$, respectively. Then

$$C(g;P) = C(g,P_k) \oplus C(g,P^k) \tag{5}$$

since $C(g,P_k)$ and $C(g,P^k)$ have exactly the edge ${}^{q_k}g$ in common. Thus we can decompose any cycle of the form $C(g;P)$ into a $\oplus$-sum of 4-cycles:

$$C(g;P) = \bigoplus_{j=1}^{l} C(g;f_j) = \bigoplus_{j=1}^{l} (g\Box f_j) \tag{6}$$

Now consider a path $Q$ in $G$ from $u$ to $v$ with edges $g_i$. Set $C(Q;P) = {}^uP \cup Q^z \cup {}^vP \cup Q^y$. Then $C(Q;P) = \bigoplus_i C(g_i;P)$ because $C(g_i;P)$ and $C(g_{i-1};P)$ have exactly the edges of the path ${}^{x_i}P$ in common, where $x_i = g_{i-1} \cap g_i \in V_G$.

Finally, let $y, z \in H$, $C \in \mathcal{C}(G)$ and $P$ a path in $H$ connecting $y$ and $z$. Let $u$ and $v$ be adjacent vertices in $C$ and write $g = uv$. Then $C = Q \cup \{g\}$ where $Q$ is a path in $G$ connecting $u$ and $v$. Now $C^z \oplus C(Q;P) \oplus C(g,P) = C^y$. We already know that both $C(Q;P)$ and $C(g,P)$ can be written as $\oplus$-sums of squares from $\mathcal{C}_\Box$. $\Box$

The same argument can be used to show that for $D \in \mathcal{C}(H)$ and $u, v \in V_G$ there is a collection of squares $\mathcal{Q}'' \subseteq \mathcal{C}_\Box$ such that ${}^vD = {}^uD \oplus \bigoplus \mathcal{Q}''$.

**Theorem 4** *Let $x \in V_G$, $y \in V_H$, and $\mathcal{B}_G$ and $\mathcal{B}_H$ be cycle bases of $G$ and $H$. Furthermore, let $T_G$ and $T_H$ be spanning trees of $G$ and $H$. Set $\mathcal{B}_\Box = \{e\Box f | e \in T_G, f \in E_H\} \cup \{e\Box f | e \in E_G, f \in T_H\}$. Then*

$$\mathcal{B}^{xy}_{G\Box H} = \{{}^xC | C \in \mathcal{B}_H\} \cup \{C^y | C \in \mathcal{B}_G\} \cup \mathcal{B}_\Box \tag{7}$$

*is a cycle basis of $G\Box H$.*

*Proof.* Let $C$ be an arbitrary cycle of $G\Box H$. By Hammack's proposition it can be written as a superposition of cycles that are contained in the $G$ or $H$ fibers and squares from $\mathcal{C}_\Box$. By Lemma 3, however, we can replace $D^y$ by $D^{y'}$ and a suitable collection of squares. These squares contain a path $P$ connecting $y$ and $y'$. Since a spanning tree of $H$ contains such a path for any pair $y, y' \in V_H$ it is sufficient to use squares of the form $e\Box f$ with $e \in E_G$ and $f \in T_H$. Analogously, ${}^xD$ and ${}^{x'}D$ can be interconverted by means of squares of the form $e\Box f$ with $e \in T_G$ and $f \in E_H$. Obviously $\mathcal{H}_3 \subseteq \mathcal{B}_\Box$, hence $\mathcal{B}^{xy}_{G\Box H}$ generates the cycle space.

The cyclomatic number of $G\Box H$ is

$$\nu(G\Box H) = |E_G||V_H| + |E_H||V_G| - |V_G||V_H| + 1. \tag{8}$$

Since

$$\begin{aligned} |\mathcal{B}^{xy}_{G\Box H}| =& |E_G| - |V_G| + 1 + |E_H| - |V_H| + 1 + \\ & |E_G|(|V_H| - 1) + |E_H|(|V_G| - 1) - (|V_H| - 1)(|V_G| - 1) \\ =& |E_G||V_H| + |E_H||V_G| - |V_G||V_H| + 1 = \nu(G\Box H), \end{aligned}$$

we infer that $\mathcal{B}^{xy}_{G\Box H}$ is a cycle basis of $G\Box H$. $\Box$

By Proposition 1 a cycle basis $\mathcal{B}$ is minimal if each of its cycles in relevant, that is, if no cycle $C \in \mathcal{B}$ can be expressed as a $\oplus$-sum of strictly shorter cycles. In the following we will need a few technical results.

**Lemma 5** *If $C$ is a cycle that is contained in a single fiber with respect to a factor, say $G$, then it cannot be written as a $\oplus$-sum of squares from $\mathcal{C}_\Box$ and cycles that are in fibers with respect to $H$.*

*Proof.* Let $C'$ be a cycle in $G\Box H$. Suppose the projection $p_G(e)$ of $e \in C'$ into $G$ is an edge. Then we say the $G$-parity of $e$ is the parity of the number of edges in $C'$ that have the same projection into $G$ as $e$. The $G$-parity of edges in $H$-fibers is defined as 0. Analogously we define the $H$-parity of an edge with nontrivial projection into $H$.

For a square $S \in \mathcal{C}_\Box$ both the $G$- and the $H$-parity of every edge is 0. If both the $G$- and the $H$-parity of every edge in a cycle $C'$ is 0, then both parities are 0 for all edges of $C' \oplus S$. Moreover, the $G$-parity of any cycle $D$ in an $H$-fibers is 0, thus every edge of $C' \oplus D$ also has $G$-parity 0.

Since the $G$-parity of all edges of a cycle $C$ that is completely contained in a $G$-fiber is 1, $C$ cannot be represented as a $\oplus$-sum of 4-cycles from $\mathcal{C}_\Box$ and cycles in $H$-fibers. $\Box$

Lemma 5 is equivalent to the statement: $\mathcal{B}_\Box$ is a maximal independent subset of $\mathcal{C}_\Box$. For given $(x, y) \in V_{G\Box H}$, every cycle basis of $G\Box H$ contained in $\{{}^xC|C \in \mathcal{B}_H\} \cup \{C^y|C \in \mathcal{B}_G\} \cup \mathcal{C}_\Box$ must contain $\{C^y|C \in \mathcal{B}_G\}$ and $\{{}^xC|C \in \mathcal{B}_H\}$.

Next we argue that every cycle that is neither contained in a single fiber nor in $\mathcal{C}_\Box$ can be written as a $\oplus$-sum of (strictly) shorter cycles. We proceed in two steps. The first one is quite obvious:

**Lemma 6** *Let $C$ be a cycle that in neither contained in a single fiber nor in $\mathcal{C}_\Box$. Then $|C| \geq 5$.*

*Proof.* If $|C| = 3$ it is obviously contained in a single fiber. If $|C| = 4$ and $C$ is not contained in a single fiber then it must contain and edge $e$ from both a $G$-fiber and an edge $f$ from $H$-fiber that have a vertex, say $(x, y)$, in common. Thus $C$ contains the two edges ${}^xf$ and $e^y$, i.e., the coordinates of the two adjacent points are $(x, y')$ and $(x', y)$. The fourth point of $C$ is a neighbor of from both $(x, y')$ and $(x', y)$ hence its coordinates $(x', y')$ are uniquely determined. Thus the remaining two edges of $C$ are ${}^{x'}f$ and $e^{y'}$, i.e., $C = e\Box f \in \mathcal{C}_\Box$. $\Box$

The second step is less trivial.

**Lemma 7** *Let $C^*$ be a cycle that in neither contained in a single fiber nor in $\mathcal{C}_\Box$. Then $C^*$ is not relevant.*

*Proof.* Let $(g, h)$ be an arbitrary vertex of $C^*$ and let $C^*_H$ be the projection of $C^*$ into ${}^gH$ and $C^*_G$ the projection into $G^h$. For each vertex $x \in G\Box H$ we define $\delta(x)$ as the sum of the distances of $x$ from the fibers $G^h$ and ${}^gH$. Furthermore, we write

$$\delta(C^*) = \sum_{x \in V_{C^*}} \frac{\deg_{C^*}(x)}{2} \delta(x), \tag{9}$$

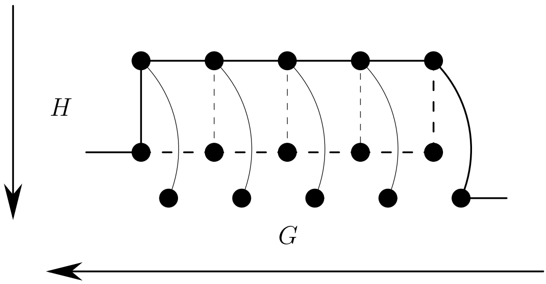


Figure 2: Reduction of $C$ in the proof of Lemma 7.

where $\deg_{C^*}(x)/2$ is the number of elementary subcycles of $C^*$ that contain $x$.

We wish to transform $C^*$ into a (possibly shorter) cycle $C^{**}$ with $\delta(C^{**}) < \delta(C^*)$. To this end we consider the vertices of $C^*$ of maximum distance from $G^h$, and among them the vertices of maximum distance from ${}^gH$. Let $M$ be this set and $x \in M$. The set $M$ spans a subgraph of $C^*$ whose components in the elementary subcycles $C$ of $C^*$ can only be single vertices, edges, paths or elementary cycles.

Let us consider these cases one by one, compare Figure 2. It suffices to show that $C$ can be transformed into a possibly shorter cycle $C'$ with $\delta(C') < \delta(C)$.

(1) The component of $x$ is a single vertex. Let $y, z$ be two vertices of $C$ incident with $x$. They are closer to either $G^h$ or ${}^gH$ than $x$. Let $Q = xy \Box xz$, and $w$ the vertex of distance 2 from $x$ in $Q$. Then $C' = C \oplus Q$ is a cycle with $|C'| \leq |C|$. Possibly $C'$ has $w$ as a new vertex, clearly $\delta(w) < \delta(x)$ and, if $x \in V(C')$, then $\deg'_C(x) = \deg_C(x) - 2$. Hence, $\delta(C') < \delta(C)$.

(2) The component of $x$ is an edge, say $xy$. Every neighbor $z \neq x$ of $x$ in $C$ is closer to one of the fibers $G^h$ and ${}^gH$ than $x$. We form $Q$ and $C'$ as above. Clearly $|C'| \leq |C|$ and $\delta(C') < \delta(C)$.

(3) The component of $x$ is a path $x = x_0, x_1, \ldots, x_k$ of length $k$. Again $x$ must have a neighbor $z = z_0$ that is closer to either $G^h$ or ${}^gH$ than $x$. Let $z_1, z_2, \ldots, z_k$ be chosen such that $x_i z_i$ are parallel to $xz$. The squares $Q_i$ are now defined as the cycles $x_{i-1}x_i z_i z_{i-1}$, where $1 \leq i \leq k$. Then

$$C' = C \oplus \bigoplus_{i=1}^{k} Q_i$$

satisfies our requirements $|C'| \leq |C|$ and $\delta(C') < \delta(C)$.

(4) The component of $x$ is a cycle $x = x_0, x_1, \ldots, x_{k-1}, x_k = x_0$ of length $k$. We proceed by the same construction as above.

Since $C^*$ is finite this procedure terminates after a finite number of iterations with a cycle $C^{**}$ that is contained in $G^h \cup {}^gH$. Since $G^h \cap^g H$ is a single vertex, $C^{**}$ is an edge-disjoint union of elementary cycles each of which is shorter than $C^*$.

By Lemma 6 we have $|C^*| > 4$, hence the squares added in the above procedure as well as the elementary cycles left at the end are shorter than $C^*$, and the lemma follows. $\square$

**Theorem 8** *Let $G$ and $H$ be triangle-free graphs and let $\mathcal{B}_G$ and $\mathcal{B}_H$ be minimum cycle bases of $G$ and $H$, respectively. Then $\mathcal{B}^{xy}$ is a minimum cycle basis of $G\Box H$.*

*Proof.* Recall that a MCB is obtained by a greedy algorithm, that is, an algorithm that selects independent cycles starting with the shortest ones from the set of all cycles. Since $G$ and $H$ are triangle-free and since the shortest cycles in $G\Box H$ that are not contained in a fiber are 4-cycles, a minimum cycle basis of $\mathcal{C}(G\Box H)$ can be constructed that contains a maximal collection of linearly independent 4-cycles from $\mathcal{C}_\Box$. Clearly, $\mathcal{B}_\Box$ is such a collection. By Lemma 7 all other relevant cycles are contained in fibers. Finally, Lemma 5 and Theorem 4 imply that $\mathcal{B}^{xy}$ is minimum cycle basis. $\square$

**Corollary 9** *If $G$ and $H$ are triangle free then*

$$\ell(G\Box H) = \ell(G) + \ell(H) + 4\big[|E_G|(|V_H|-1) + |E_H|(|V_G|-1) - (|V_H|-1)(|V_G|-1)\big].$$

The case of graphs with triangles can be treated analogously:

**Theorem 10** *For two graphs $G$ and $H$ let $\mathcal{B}_G$ and $\mathcal{B}_H$ be minimum cycle bases of $G$ and $H$. Let $\mathcal{T}_G \subseteq \mathcal{B}_G$ and $\mathcal{T}_H \subseteq \mathcal{B}_H$ be sets of triangles in the bases. Then, for each $u \in V_G$ and $v \in V_H$ there is a minimum cycle basis $\mathcal{B}^*$ of $G\Box H$ containing*

$$\{T^y | T \in \mathcal{T}_G, y \in V_H\} \cup \{{}^xT | T \in \mathcal{T}_H, x \in V_G\} \cup \{C^v | C \in \mathcal{B}_G \setminus \mathcal{T}_G\} \cup \{{}^uC | C \in \mathcal{B}_H \setminus \mathcal{T}_H\}$$

*and a suitable subset of squares $\mathcal{Q} \subseteq \mathcal{B}_\Box$.*

*Proof.* Using the greedy algorithm we first have to find a maximal set $\mathcal{T}$ of linearly independent triangles, then we have to extract a set $\mathcal{Q}$ of squares such that $\mathcal{T} \cup \mathcal{Q}$ is linearly independent, and finally we have to add the longer cycles from one of the $G$- and $H$-fibers. Since there are no triangles outside the fibers, the MCB must contain $\{T^y | T \in \mathcal{T}_G, y \in V_H\} \cup \{{}^xT | T \in \mathcal{T}_H, x \in V_G\}$. Since the longer cycles in the fibers cannot be obtained from the triangles and $\mathcal{C}_\Box$ we must have at least one copy of them in the basis. By Theorem 4 one copy is sufficient and the theorem follows. $\square$

Let $t_G$ and $t_H$ be the number of triangles in a minimum cycle basis (and hence in every minimum cycle basis [4, 10]) of $G$ and $H$, respectively. Then

$$\begin{aligned}\ell(G\Box H) &= \ell(G) + \ell(H) + 3\big[t_G(|V_H|-1) + 3t_H(|V_G|-1)\big] \\ &\quad + 4\big[(|E_G|-t_G)(|V_H|-1) + |E_H| - t_H)(|V_G|-1) - (|V_H|-1)(|V_G|-1)\big]\end{aligned} \tag{10}$$

Let $\lambda(G)$ be the minimum length of the longest cycle in an arbitrary cycle basis of $G$. Chickering [4] showed that $\lambda(G)$ is the length of the longest cycle in a MCB. It follows immediately from our results above that

$$\lambda(G\Box H) = \max\{4, \lambda(G), \lambda(H)\}\,. \tag{11}$$

The upper bound given in [11] is therefore an equality for all graphs $G$ and $H$.

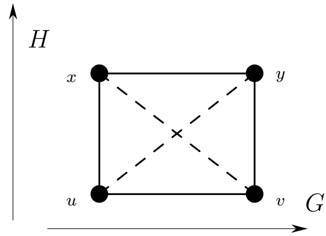


Figure 3: For each square from $\mathcal{C}_\Box$ the strong product $G \boxtimes H$ contains two non-Cartesian edges (dashed lines). Each of these edges is contained in a triangle containing two Cartesian edges.

# 4 Strong Products

The strong product $G \boxtimes H$ has $|V_G|\,|V_H|$ vertices and $2|E_G|\,|E_H|+|E_G|\,|V_H|+|V_G|\,|E_H|$ edges. The Cartesian product $G\Box H$ is a subgraph of $G \boxtimes H$; hence a cycle basis of $G \boxtimes H$ can be constructed starting from a cycle basis of $G\Box H$ by adding $2|E_G|\,|E_H|$ cycles, each of which contains exactly one of the "non-Cartesian edges", see Fig. 3, provided such a set of cycles exists. The resulting collection of cycles is linearly independent since no subset that contains a cycle with a non-Cartesian edge can have a vanishing $\oplus$-sum. The cycles without non-Cartesian edges, on the other hand, are linearly independent by construction. The resulting collection is a basis since

$$\nu(G \boxtimes H) = 2|E_G|\,|E_H| + \nu(G\Box H) \tag{12}$$

and since there are $2|E_G|\,|E_H|$ non-Cartesian edges.

Each of the non-Cartesian edges is contained in a triangle the two other edges of which are Cartesian. Consequently, we obtain a basis $\mathcal{B}$ of $G \boxtimes H$ by expanding a basis of $G\Box H$ with these non-Cartesian triangles.

Recall that the cycle basis $\mathcal{B}^{xy}_{G\Box H}$ of $G\Box H$ defined in Theorem 4 contains a maximal independent set of squares from $\mathcal{C}_\Box$, which, in particular, generates $\mathcal{C}_\Box$. Each of these squares is contained in a $K_4$, see Fig. 3, of which $\mathcal{B}$ contains the Cartesian square $Q = (x, y, v, u)$, and two non-Cartesian triangles $N_1 = (x, y, u)$ and $N_2 = (x, y, v)$. For each of the Cartesian squares in $\mathcal{B}$ we can therefore choose a third triangle, say, $T = (u, v, x)$, of the corresponding $K_4$. We have $Q = T \oplus N_2$. Thus we obtain a shorter basis $\mathcal{B}'$ by replacing each Cartesian square by this third triangle from the same $K_4$.

The basis $\mathcal{B}'$ consists of a MCB of ${}^{x}H$ and $G^y$ for single vertices $x \in V_G$, $y \in V_H$, and triangles. We can argue as above that none of the cycles from ${}^{x}H$ or $G^y$ can be written as a $\oplus$-sum of the triangles (or the Cartesian squares), hence $\mathcal{B}'$ must be minimal.

As an immediate consequence we have

$$\begin{aligned}\ell(G \boxtimes H) &= \ell(G) + \ell(H) + 3\left(\nu(G \boxtimes H) - \nu(G) - \nu(H)\right)\,, \\ \lambda(G \boxtimes H) &= \max\left\{3, \lambda(G), \lambda(H)\right\}\,. \end{aligned} \tag{13}$$

# 5 Average MCB Cycle Lengths of Powers of Graphs

The $n$-th Cartesian power $G^n$ of a non-empty graph $G$ is defined recursively by

$$G^n = G \Box G^{n-1} \simeq G^{n-1} \Box G, \qquad G^1 = G \tag{14}$$

The Cartesian product of finitely many graphs is connected if and only if each factor is connected. Recall that $|V_{G^n}| = |V|^n$ and $|E_{G^n}| = n|E||V|^{n-1} = an|V|^n$ with $a = |E|/|V|$. Consequently, we have $\nu(G^n) = (na-1)|V|^n + 1$. Furthermore, let $t_G = \tau|V|$ be the number of triangles in an MCB of $G$. Note that the distribution of cycle lengths is the same in all MCBs of $G$, see [4]. It is easy to verify $t_{G^n} = n\tau|V|^n$.

Instead of the longest cycles it is also interesting to consider the average length of a basis cycle, which we denote by $L(G) = \ell(G)/\nu(G)$. We write $L_n = L(G^n)$. For Cartesian powers equation (10) becomes

$$\begin{aligned}\ell(G^{n+1}) &= \ell(G) + \ell(G^n) + 3\tau\big[|V|(|V^n| - 1) + n|V|^n(|V| - 1)\big] \\ &\quad + 4(a-\tau)\big[|V|(|V|^n - 1) + n|V|^n(|V| - 1) - (|V|^n - 1)(|V| - 1)\big].\end{aligned} \tag{15}$$

Dividing by $\nu(G^{n+1})$, setting $\xi = 1/V$, and retaining only the leading order in $\mathcal{O}\left\{(n+1)^{-1}\right\}$ we obtain

$$L_{n+1} = \xi L_n + 3\frac{\tau}{a}(1-\xi) + 4\frac{a-\tau}{a}(1-\xi) + \mathcal{O}(1/n). \tag{16}$$

Thus $L_n$ approaches the limit

$$L_\infty = \lim_{n\to\infty} L_n = 3\frac{\tau}{a} + 4\frac{a-\tau}{a}. \tag{17}$$

It is interesting to compare this with the average cycle length $x_n$ in Hammack's basis for $G^n$. The recursion for $x_n$ can be written, after multiplication of both numerator and denominator by $\xi^{n+1}$, as

$$x_{n+1} = \frac{na - 1 + \xi^n}{(n+1)a - 1 + \xi^{n+1}} x_n + 4\frac{(1-\xi)(1-\xi^n)}{(n+1)a - 1 + \xi^{n+1}} + L_1 \frac{a - 1 + \xi}{(n+1)a - 1 + \xi^{n+1}}.$$

This is of the form $x_{n+1} = (1-\alpha_n)x_n + \beta_n$. One easily verifies $n\alpha_n = 1 + \mathcal{O}(1/n)$ and $n\beta_n = x_\infty + \mathcal{O}(1/n)$, where

$$x_\infty = \frac{4 + (a-1)L_1}{a} + \frac{\xi}{a}(L_1 - 4). \tag{18}$$

The substitution $y_n = x_n - x_\infty$ yields $y_{n+1} = (1-\alpha_n)y_n + e_n$ where $e_n = \beta_n - x_\infty\alpha_n = \mathcal{O}(\xi^n)$. It is not hard to check that $y_n$ cannot converge exponentially to 0. Thus we can suppress the $\xi^n$ terms and obtain $|y_n| \sim \prod_{k=0}^{n-1}[1 - 1/n + \mathcal{O}(1/n^2)]|y_0|$. The product converges to 0 for $n \to \infty$, hence $y_n \to 0$ and $x_n \to x_\infty$.

Of course $x_\infty \geq L_\infty$, equality holding if and only if $G$ is null-homotopic. In this case we have

$$L_\infty = x_\infty = 3 + \frac{1-\xi}{a} \tag{19}$$

since $\tau = \nu(G)/|E| = a - 1 + \xi$. For the iterated products of the $m$-cycle $C_m$, on the other hand, we have

$$L_\infty(C_m) = 4 \qquad \text{and} \qquad x_\infty(C_m) = 5 - \frac{4}{m}. \tag{20}$$

In general Hammack's basis has a larger average cycle length than the MCB even though the longest cycles have the same size.

For the strong product we have $L_\infty^{\boxtimes} = 3$ for all graphs $G$ as an immediate consequence of equ.(13).

# References


[1] R. Balducci and R.S. Pearlman. Efficient exact solution of the ring perception problem. *J. Chem. Inf. Comput. Sci.*, 34:822–831, 1994.

[2] C. Champetier. On the null-homotopy of graphs. *Discr. Math.*, 64:97–98, 1987.

[3] W.-K. Chen. On vector spaces associated with a graph. *SIAM J. Appl. Math.*, 20:525–529, 1971.

[4] D.M. Chickering, D. Geiger and D. Heckerman. On finding a cycle basis with a shortest maximal cycle. *Inform. Processing Let.*, 54:55–58, 1994.

[5] L.O. Chua and L. Chen. On optimally sparse cycle and coboundary basis for a linear graph. *IEEE Trans. Circuit Theory*, 20:54–76, 1973.

[6] G.M. Downs, V.J. Gillet, J.D. Holliday and M.F. Lynch. Review of ring perception algorithms for chemical graphs. *J. Chem. Inf. Comput. Sci.*, 29:172–187, 1989.

[7] P. Duchet, M. Las Vergnas and H. Meyniel. Connected cutsets of a graph and a triangle basis of the cycle space. *Discr. Math.*, 62:145–154, 1986.

[8] L. Euler. Elementa doctrinae solidorum. *Novi Comm. Acad. Sci. Imp. Petropol.*, 4:109–140, 1752. (Latin).

[9] R. García-Pelayo and P.F. Stadler. Correlation length, isotropy, and meta-stable states. *Physica D*, 107:240–254, 1997.

[10] P.M. Gleiss, J. Leydold and P.F. Stadler. Interchangeability of relevant cycles in graphs. *Elec. J. Comb.*, 7:R16 [16pages], 2000.

[11] R. Hammack. Cyclicity of graphs. *J. Graph Theory*, 32:160–170, 1999.

[12] D. Hartvigsen and R. Mardon. The all-pair min-cut problem and the minimum cycle basis problem on planar graphs. *SIAM J. Discr. Math.*, 7:403–418, 1994.

[13] J.D. Horton. A polynomial-time algorithm to find the shortest cycle basis of a graph. *SIAM J. Comput.*, 16:359–366, 1987.

[14] W. Imrich and S. Klavžar. *Product Graphs: Structure and Recognition.* Wiley, New York, 2000.

[15] A. Kaveh. *Structural Mechanics: Graph and Matrix Methods.* Research Studies Press, Exeter, UK, 1992.

[16] J. Leydold and P.F. Stadler. Minimal cycle basis of outerplanar graphs. *Elec. J. Comb.*, 5:209–222 [R16: 14 p.], 1998. `http://www.combinatorics.org/`.

[17] G. Liu. On connectivities of tree graphs. *J. Graph Theory*, 12:435–459, 1988.

[18] M. Plotkin. Mathematical basis of ring-finding algorithms in CIDS. *J. Chem. Doc.*, 11:60–63, 1971.

[19] P. Vismara. Union of all the minimum cycle bases of a graph. *Electr. J. Comb.*, 4:73–87, 1997. Paper No. #R9 (15 pages).

[20] D.J.A. Welsh. Kruskal's theorem for matroids. *Proc. Cambridge Phil. Soc.*, 64:3–4, 1968.

*(Received 12/8/2001)*