New domination conditions for tournaments

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Abstract

The definition for the domination graph of a tournament states that it has the same vertices as the tournament with an edge between two vertices if every other vertex is beaten by at least one of them. In this paper two generalisations of domination graphs are proposed by using different relaxations of the adjacency definition. The first type is formed by reducing the number of vertices which must be dominated by a pair of vertices and the second by increasing the number of steps allowable for domination. Properties of these new types of domination graphs are presented with comparison between them where appropriate. In particular, a full characterisation of both generalisations is given for rotational tournaments.

1 Introduction

A *tournament* $T = (V, A)$ is a complete, directed graph, with a vertex set V and an arc set A. For $x, y \in V$ we denote an arc from x to y by $(x, y) \in A$ and say x *beats* y. For any vertex x, let $out(x)$, the *out set* of x, denote the set of vertices that x beats. Similarly, let $in(x)$, the *in set* of x, denote the set of vertices that beat x.

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A tournament T on n vertices is said to be *regular* if n is odd and every vertex has out-degree $\frac{n-1}{2}$. Tournaments are well documented in Moon [11], Reid and Beineke [13], and Reid [14].

Given a tournament $T = (V, A)$, a pair of vertices x and y *dominate* T *(in one step)* if for all vertices $z \neq x, y$ either x beats z or y beats z. A pair of such vertices is called a *dominant pair*. The *domination graph* of a tournament T, denoted by $dom(T)$, is the graph with vertex set V and edges between those pairs of vertices which are dominant pairs. Numerically this condition states that xy is an edge in $dom(T)$ if and only if $|in(x) \cap in(y)| < 1$. Domination graphs of digraphs and tournaments have been completely characterised in a series of papers (see [2, 3, 4, 5, 6, 7]). For a tournament T, $dom(T)$ is either a cycle of odd length, with or without isolated and/or pendant vertices, or a forest of caterpillars.

A vertex in a tournament is called a *king* if it beats all other vertices in one or two steps; that is, vertex x is a king if for all $y \in V$, x beats y or x beats a vertex that beats y. Landau [9] proved that every vertex of maximum out degree in a tournament is a king; consequently, a tournament may have more than one king. A vertex which beats all other vertices in one step is called an *emperor*. A tournament has exactly one king if and only if that king is an emperor [10].

2 Relaxation of domination conditions

Other interesting types of domination graphs may be obtained by using different, more relaxed conditions for a dominating pair of vertices. We propose two generalisations of the standard domination graph of a tournament. The first possibility is to reduce the number of vertices which must be dominated; the second is to increase the number of allowable steps for domination. Pertinent definitions and results are presented in this section.

Let T be a tournament with n vertices. For $1 \leq k \leq n-2$, the *k-domination graph* of T is a graph with the same vertices as T and an edge between two distinct vertices if and only if that pair of vertices dominates in one step all of the other vertices of T with the possible exception of $k-1$ vertices. Thus two vertices x and y in T are adjacent in the k-domination graph provided they are not each dominated by some common set of k other vertices, that is $|in(x) \cap in(y)| < k$. We denote the k-domination graph of a tournament T by $dom_k(T)$. Clearly $dom_1(T) \subseteq \cdots dom_k(T) \subseteq dom_{k+1}(T) \cdots \subseteq dom_{n-2}(T)$ where $dom_1(T) = dom(T)$.

For the domination graphs described above, domination by a pair of vertices was required to occur in just one step. Now we allow a domination condition more equivalent to that of a king, that is, domination by a pair of vertices in one or two steps. We define a *royal pair* to be a pair of vertices in a tournament T which together act as a king. That is, vertices x and y form a royal pair if, for all $z \in V \backslash \{x, y\}$, either x or y beats z in at most two steps. The *royal graph*, denoted by $roy(T)$, is the graph on the vertices of V with an edge between each pair of vertices which form a royal pair. It follows from their respective definitions that $dom_1(T)$ is a subgraph of $roy(T)$.

Lemma 1 *Let T be a tournament, then $roy(T)$ is connected.*

PROOF. Since every pair of vertices containing a king must be a royal pair, and every tournament must have at least one king, namely every vertex of highest out-degree, $roy(T)$ must be connected. ■

If all the n vertices in a tournament T are kings, then $roy(T)$ is the complete graph K_n. Rotational tournaments satisfy this condition. Maurer [10] showed that the probability of every vertex being a king in a random tournament on n vertices approaches 1 as $n \to \infty$, and so $roy(T)$ fails as a good representative of dominating pairs in this situation.

The structures of $roy(T)$ and $dom_2(T)$ are strongly related. The next result shows that, depending on the presence of an emperor, one of these types of domination graphs is always a subgraph of the other.

Theorem 2 *Let T be a tournament.*

(i) If T has an emperor, then $roy(T) \subseteq dom_2(T)$.

(ii) If T has no emperor, then $dom_2(T) \subseteq roy(T)$.

PROOF.

(i) Suppose a vertex z is an emperor in T, then the only edges in $roy(T)$ are those in the star graph with centre at z. Since an emperor beats every other vertex in one step, every edge in $roy(T)$ is also in $dom_1(T)$.

(ii) Suppose that T has no emperor and x and y are adjacent vertices in $dom_2(T)$. If xy is also an edge in $dom_1(T)$ then it is an edge in $roy(T)$. However, if xy is not an edge in $dom_1(T)$, then there is a vertex w which beats both x and y. Consequently, if there is an arc from $V \setminus \{w, x, y\}$ to w, then x and y must form a royal pair, otherwise w is an emperor. ■

Kings play a key role in determining when the graphs $dom_2(T)$ and $roy(T)$ are the same.

Theorem 3 *Let T be a tournament, if $dom_2(T) = roy(T)$ then every royal pair contains a king.*

PROOF. Assume there is a royal pair x and y where y beats x and neither x nor y is a king. Then there exists a vertex u which prevents x from being a king. This means u beats x and u beats every vertex in $out(x)$. Similarly, there is a vertex v which prevents y from being a king. (Note that $v \neq u$, since x and y form a royal pair.)

Case 1. Assume $u \notin out(y)$, then u beats y. Then xy is not an edge in $dom_2(T)$, since both u and v beat x and y. Thus $roy(T) \neq dom_2(T)$.

Case 2. Assume y beats u, and there is some vertex $t \neq v$ which beats both x and y. Then xy is not an edge in $dom_2(T)$, since both t and v beat x and y. Thus $roy(T) \neq dom_2(T)$.

Case 3. Assume y beats u, and v is the only vertex which beats both x and y. Then $V = \{v, y\} \cup out(x) \cup out(y)$.

Let A be the set of all vertices which beat x and all of the vertices in $out(x)$. Note that $u \in A$, hence A is not empty. Since $\{x, y\}$ is a royal pair there must exist some $w \in out(x)$ which beats v. Moreover, w beats y, otherwise y would beat v in two steps (and therefore be a king).
<u>Case 3a.</u> Assume $|A| = 1$, and so $A = \{u\}$. Assume that there is some other vertex s which prevents $\{u, x\}$ from being a royal pair. That is, s beats u, s beats x, and s beats all the vertices in both $out(u)$ and $out(x)$. This means that $s \in A$, a contradiction. Hence $\{u, x\}$ is a royal pair. Moreover, since v and y dominate both of u and x, ux is not an edge in $dom_2(T)$. Thus $roy(T) \neq dom_2(T)$.
<u>Case 3b.</u> Assume $|A| > 1$. Then the edge wx is not in $dom_2(T)$, since $|A| > 1$ and every vertex in A beats both w and x. But wx is an edge in $roy(T)$, since w beats y, w beats v, and w beats all vertices in $out(y)$ in at most two steps. Thus $roy(T) \neq dom_2(T)$.

In each case it has been shown that if T contains a royal pair where neither vertex is a king, then $roy(T) \neq dom_2(T)$. ■

A complete characterisation of those tournaments T for which $dom_2(T) = roy(T)$ is unknown, and the converse of Theorem 3 is not true. The condition that every royal pair contains a king is not sufficient. This is illustrated in Figure 1 where the vertices 1, 2, and 6 of the tournament N_6 are all kings but $roy(T)$ and $dom_2(T)$ are not equal. Figure 1(a) shows N_6 drawn in a customary way; only the upward oriented arcs are shown. The tournament N_6 is called the nearly transitive tournament on 6 vertices (see for example [12]). A *nearly transitive* tournament on n vertices N_n ($n \geq 3$) has vertices $\{v_1, \ldots, v_n\}$ and domination so that v_i beats v_j if $i < j$, except for the pair $\{v_1, v_n\}$, for which v_n beats v_1. (If the arc (v_n, v_1) was reversed, the tournament would be transitive). It is straightforward to verify the following lemma concerning the kings of N_n.

Lemma 4 *The vertices v_1, v_2 and v_n are the only kings of N_n.*

Proposition 5 *The tournaments N_n ($n \geq 5$) have the following properties:*

(i) $dom_2(N_n) \subsetneq roy(N_n)$

(ii) Every royal pair contains a king.

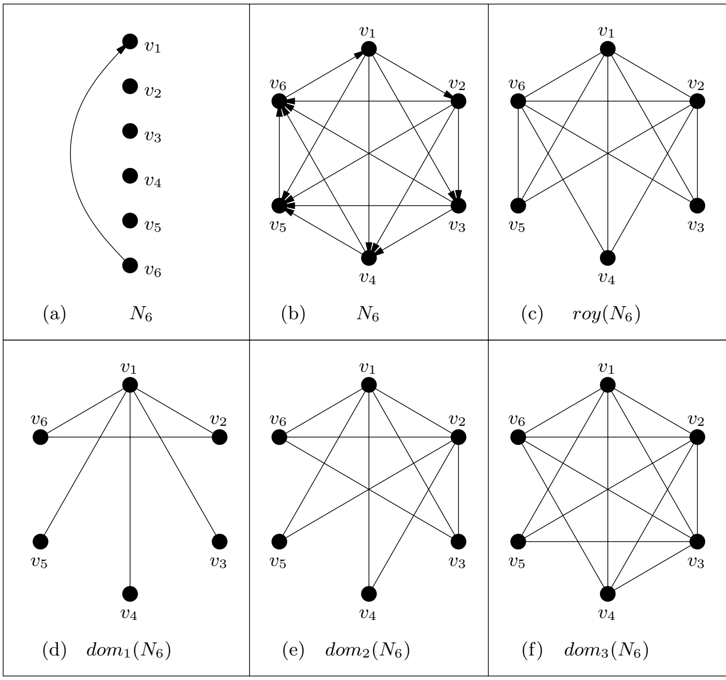

Figure 1: The nearly transitive tournament N_6, its royal graph and first three k-domination graphs.

PROOF.

(i) Since N_n clearly has no emperor, $dom_2(N_n) \subsetneq roy(N_n)$ by Theorem 2. It remains to show that there is an edge in $roy(N_n)$ which is not an edge in $dom_2(N_n)$. Consider the vertices v_{n-1} and v_n. Since v_n is a king, the edge $v_{n-1}v_n$ is in $roy(N_n)$. However, since this pair of vertices does not beat either of the vertices v_2 and v_3, the edge $v_{n-1}v_n$ is not in $dom_2(N_n)$.

(ii) Consider any two vertices v_i and v_j with $3 \leq i < j \leq n-1$. Now, since $in(v_2) = \{v_1\}$, and $in(v_1) = \{v_n\}$, neither of v_i and v_j can beat v_2 in two steps. Hence, v_i, v_j is not an edge in $roy(N_n)$.

So the family of tournaments N_n $(n \geq 5)$ is an infinite family of counterexamples to the converse of Theorem 3. ■

We now generalise the definition of an emperor by defining a *sub*^{k}*-emperor* to be a vertex which beats all other vertices, except k of them, in one step. That is, a vertex $x \in V$ is a sub^k-emperor if and only if $|in(x)| \leq k$. The next lemma follows by generalising the proof of Theorem 2.

Lemma 6 *If T is a tournament with no sub^k-emperor, where $k \geq 1$, then $dom_{k+2}(T) \subseteq roy(T)$.*

We cannot guarantee that for each tournament T, $dom_i(T) \subseteq roy(T) \subseteq dom_{i+1}(T)$ for some i.

The tournament N_6 in Figure 1 is an example of a tournament which has $dom_3(T) \not\subseteq roy(T)$ and $roy(T) \not\subseteq dom_3(T)$. The nearly transitive tournaments N_n ($n \geq 6$) are a family of tournaments which show that this containment does not always hold for arbitrary k.

Theorem 7 *For any nearly transitive tournament N_n ($n \geq 6$) and any k ($3 \leq k \leq n-3$), $dom_k(N_n) \not\subseteq roy(N_n)$ and $roy(N_n) \not\subseteq dom_k(N_n)$.*

Proof. Consider the vertices v_{n-1} and v_n. Since v_n is a king (Lemma 4), $v_{n-1}v_n$ is an edge in $roy(N_n)$. Since the vertices $v_2, \ldots v_{n-2}$ beat both v_{n-1} and v_n, the edge $v_{n-1}v_n$ is not in $dom_k(N_n)$.

Consider the vertices v_3 and v_4. Theorem 5*(ii)* tells us that v_3v_4 is not an edge in $roy(N_n)$. Since v_3 and v_4 together dominate every vertex except v_1 and v_2, the edge v_3v_4 is in $dom_k(N_n)$.

So $dom_k(N_n) \not\subseteq roy(N_n)$ and $roy(N_n) \not\subseteq dom_k(N_n)$. ■

The generalisation of domination graphs to k-domination graphs provides a means of classifying some pairs of vertices as more dominant within the tournament than others.

There are a number of interesting, but as yet unanswered, questions concerning the classification of tournaments which have a particular type of k-domination or royal graph.

- How can Theorem 3 be extended to a classification of tournaments T which have $dom_2(T) = roy(T)$?
- Can a structural classification of 2-domination graphs be found, similar to domination graphs being odd cycles with possible isolated and pendant vertices or forests of caterpillars? Can this be extended to k-domination graphs?
- Which tournaments have no kingless royal pairs?
- Which tournaments have empty k-domination graphs?
- Do k-domination graphs always provide a separation of the pairs of vertices, classifying some pairs as more dominant than others?

Rotational tournaments provide some partial answers to some of these questions and are a good source of examples. A sampling of these are presented in the next two sections.

3 Rotational tournaments

Let S be a subset of $\mathbf{Z}_n \backslash \{0\}$ with the property that $i \in S$ if and only if $n - i \in S$. Define a graph G to have a vertex set $v_1, v_2, \ldots, v_n$, with an edge joining vertices v_i and v_j if and only if $i - j \in S$. A graph G of this type is called a *circulant graph* and the set S is known as its *symbol*. A *rotational tournament* $T(S)$ is defined in a similar fashion with the symbol S again being a subset of $\mathbf{Z}_n \backslash \{0\}$ but having slightly different restrictions, namely that it must contain $\frac{n-1}{2}$ elements and $s_i + s_j \neq 0$ for all $s_i, s_j \in S$. For such a rotational tournament (v_i, v_j) is an arc if and only if $j - i = s \bmod n$ for some $s \in S$. For example, if $\{1, 2, 3\}$ is the symbol for a tournament T on seven vertices, then vertex v_1 beats vertices v_2, v_3 and v_4, and is beaten by vertices v_5, v_6 and v_7 [1].

Rotational tournaments are a subclass of regular tournaments. They are vertex transitive, that is, for every pair of vertices v_i and v_j, there is an automorphism which maps v_i to v_j. Let S be a symbol for a rotational tournament T with n vertices. It is routine to show that if r is relatively prime to n, then rS is also a valid symbol for some tournament. Proposition 8 shows that this tournament will be isomorphic to T.

Proposition 8 *A rotational tournament T on n vertices with symbol S is isomorphic to any tournament T_r with symbol rS where r is relatively prime to n.*

Proof. Suppose $S = \{s_1, \ldots, s_t\}$, and the vertices of T and T_r are labelled $v_1, \ldots, v_n$. The symbol for T_r is $\{rs_1, \ldots, rs_t\}$. Define a mapping ϕ from the vertices of T to the vertices of T_r by $\phi(v_\ell) = v_{r\ell}$. Since r and n are relatively prime, ϕ reorders the vertices. It also gives an isomorphism between T and T_r. Suppose vertex v_i beats vertex v_j for some $1 \leq i, j \leq n$ in T, then $j - i \in S$. Consequently, $r(j - i) = rj - ri \in rS$. Hence $v_{ri} = \phi(v_i)$ beats $v_{rj} = \phi(v_j)$ in T_r. Hence ϕ is an isomorphism. ■

Proposition 8 does not determine all possible symbols for a rotational tournament. For example, the symbols $\{1, 3, 4, 7\}$ and $\{1, 4, 6, 7\}$ form isomorphic tournaments on nine vertices, but one is not a multiple of the other.

Let T be a tournament on n vertices. The *arc reversal* of T is defined to be the tournament with the same vertex set as T but with the direction of each arc reversed. By Proposition 8 it is clear that a rotational tournament is isomorphic to its arc reversal.

Let U_n, $n \geq 3$, be the rotational tournament with symbol $\{1, 3, \ldots, n-2\}$, (that is all odd numbers from 1 to $n - 2$.) Fisher et al. [4] showed that if T is a rotational tournament on n vertices then either $dom_1(T)$ is the cycle graph C_n and T is U_n or $dom_1(T)$ is the empty graph. A related result is now proved; namely that each domination graph $dom_k(T)$, for $k = 1, 2, \ldots, \frac{n-3}{2}$, is either the empty graph or a circulant graph. (Note that for a rotational tournament the out degree of every vertex is $\frac{n-1}{2}$, hence any pair of vertices must beat at least $\frac{n-1}{2}$ other vertices, consequently in the definition for $dom_k(T)$ it is only necessary to consider $k = 1, 2, \ldots, \frac{n-3}{2}$.)

For a rotational tournament T with symbol S on vertices $v_1, \ldots v_n$, we define the following for use in the consequent theorems. Let D be the sequence of all possible differences modulo n between pairs of elements in S. For each element d in D let $N(d)$ be the number of times the value d occurs in D.

Theorem 9 *Let T be a rotational tournament with symbol S and vertices labelled $v_1, v_2, \ldots, v_n$. Then, for $1 \leq k \leq \frac{n-3}{2}$, $dom_k(T)$ is either the empty graph or a circulant graph whose symbol consists of those values d in D for which which $N(d) < k$.*

Before presenting the proof of Theorem 9 a short example is given. Let T be the rotational tournament on nine vertices with symbol $S = \{1, 5, 6, 7\}$. We form the difference table for this symbol by calculating the difference between all pairs of elements in S and counting how many times each difference occurs. The difference table for this example is:

d	1	2	3	4	5	6	7	8
$N(d)$	2	1	1	2	2	1	1	2

Since there are no values of d such that $N(d) = 0$ it follows that $dom_1(T)$ is the empty graph. There are four values of d such that $N(d) \leq 1$, hence $dom_2(T)$ is the circulant graph with symbol $\{2, 3, 6, 7\}$, the set of those differences d with $N(d) \leq 1$. Similarly it follows that $dom_3(T)$ is the circulant graph with symbol $\{1, 2, 3, 4, 5, 6, 7, 8\}$; that is, $dom_3(T)$ is the complete graph K_9.

Proof. Since T is vertex transitive it suffices to consider any pair of vertices v_i and v_j in T. By counting the number of vertices $z = v_\ell$ which beat both v_i and v_j the equality $N(i-j) = |in(v_i) \cap in(v_j)|$ is obtained. Indeed, if v_ℓ beats v_i and v_j then $i - \ell = s_i \bmod n$ and $j - \ell = s_j \bmod n$ for some $s_i, s_j \in S$, and consequently $i - j = s_i - s_j$.

Thus the symbol S_k for $dom_k(T)$ consists of the values of those differences d, from pairs of elements in $\{1, 2, \ldots, n\}$, for which $N(d) < k$. It is clear from the construction that $\ell \in S_k$ if and only if $n - \ell \in S_k$; consequently this symbol fulfils the appropriate condition for a circulant graph. ■

Let U'_n, $n \geq 7$, represent the rotational tournament with symbol $\{1, 2, 4, \ldots, n-3\}$, namely one and all even numbers except $n-1$. It follows from Theorem 9 that $dom_1(U'_n)$ is the empty graph and $dom_{\frac{n-3}{2}}(U'_n)$ is the complete graph K_n. Fisher et al. [5] showed that for a tournament with n vertices the maximum possible number of edges in $dom_1(T)$ is n. It is not surprising that more than n edges may occur in $dom_k(T)$, and in the fact the maximum possible number of edges, namely $\frac{n(n-1)}{2}$, is obtained in $dom_{\frac{n-3}{2}}(U'_n)$.

Since every vertex in U'_n is a king it is clear that $roy(U'_n)$ is complete. However $dom_1(U'_n)$, $n \geq 7$ is empty. This gives a situation where the graphs of $dom_1(T)$ and $roy(T)$ are as different as possible. The next section expands the construction of the symbol of U'_n to give a family of rotational tournaments where, for each member, the k-domination graph is empty and the $(k+1)$-domination graph is not.

4 Construction of empty dom_k

In this section an infinite family of rotational tournaments on n vertices is described, in which for sufficiently large n, the k-domination graph is empty and the $(k+1)$-domination graph is not empty. Additionally, we show that if T is a quadratic residue tournament [8] on $4k+3$ vertices, then the $dom_k(T)$ is empty and $dom_{k+1}(T)$ is complete.

Let $E_{n,k}$, $k \geq 1$, be a rotational tournament on n vertices with symbol

$$S_{n,k} = \{1, 2, \ldots, 2k-1, 2k\} \cup \{2k+2, 2k+4, \ldots, n-2k-3, n-2k-1\}.$$

For example, the symbol for $E_{17,2}$ is $S_{17,2} = \{1, 2, 3, 4, 6, 8, 10, 12\}$, and the symbol for $E_{17,3}$ is $S_{17,3} = \{1, 2, 3, 4, 5, 6, 8, 10\}$. Notice that the symbols for $E_{n,1}$ and U'_n are equal.

Theorem 10 *The k-domination graph $dom_k(E_{n,k})$ is empty when $n > 8k$.*

Proof. Let D be the sequence of all possible differences modulo n between pairs of elements in $S_{n,k}$. By Theorem 9 it suffices to show that $N(d) \geq k$ for all $d \in D$. Note that it is only necessary to consider $1 \leq d \leq \frac{n-1}{2}$, since the difference table is symmetrical; $N(d) = N(n-d)$. Odd and even values of d are considered separately.

If d is even then each of the k differences in the sequence

$$(d+2)-2,\ (d+4)-4,\ \ldots,\ (d+2k)-2k$$

has value d and occurs in D provided $d + 2k < n - 2k + 1$. The condition $n > 8k$ ensures this inequality holds.

The case for d odd is similar. The required sequence of k differences, each with value d, is

$$(d+1)-1,\ (d+3)-3,\ \ldots,\ (d+2k-1)-(2k-1)$$

and it is clear that $d + 2k - 1 < n - 2k + 1$.

Hence $N(d) \geq k$ irrespective of whether d is even or odd, so $dom_k(E_{n,k})$ is the empty graph. ∎

Theorem 11 *The $(k+1)$-domination graph $dom_{k+1}(E_{n,k})$ is not empty.*

Proof. It suffices to show that there is a value of d such that $N(d) < k+1$, for then there will be edges in $dom_{k+1}(E_{n,k})$. Let $d = 2k+1$. Since d is odd the only pairs of elements in $S_{n,k}$ with difference equal to $2k+1$ are given by the k differences

$$(1+2k+1)-1,\ (3+2k+1)-3,\ \ldots,\ (2k-1+2k+1)-(2k-1)$$

(None of the even symbol elements between 1 and $2k$ can be used to form a difference with value $2k+1$ in this fashion, since adding $2k+1$ to any of them gives a larger odd value which isn't in the symbol.) Thus $N(2k+1) = k < k+1$ giving the required result. ∎

Theorems 10 and 11 give an example of a family of rotational tournaments, where for any member T, $dom_k(T)$ is empty and the next graph in the chain of domination graphs, namely $dom_{k+1}(T)$ is not empty. Moreover $roy(T)$ is complete.

Conjecture 12 *Theorems 10 and 11 hold for the improved condition of* $n > 6k$.

Note that this construction works for $E_{17,2}$ where $n > 8k$ and fails for $E_{17,3}$ where $n < 6k$. Observations seem to suggest this construction will in fact work when $n > 6k$, and fail when $n < 6k$. A computer check shows that the condition $n > 6k$ is satisfactory for all constructions of $E_{n,k}$ with $k < 35$. ■

Finally, we present a family of tournaments for which the k-domination graph is empty, and the $(k+1)$-domination graph is complete. They are rotational tournaments with quadratic residues as the elements of the symbol.

A *quadratic residue tournament* is a rotational tournament on $n = 4k+3$ vertices, where n is prime [8]. The symbol for the quadratic residue tournament on $n = 4k+3$ vertices is of the form $\{a^2 \bmod n | 1 \leq a \leq n\}$, the quadratic residues mod n.

Theorem 13 *If* T *is a quadratic residue tournament with* $4k+3$ *vertices, then* $N(d) = k$ *for all* d.

Proof. Recall from Proposition 8 that every (relatively prime) multiple of a symbol for a tournament yields an isomorphic tournament. When considering quadratic residues, multiplying the symbol S by a quadratic residue yields exactly the same symbol S.

It follows that entries for all quadratic residues in the difference table are equal. A similar argument shows that all non-square entries in the difference table are equal. It now remains to show that the entries for square and non-square differences are the same.

Applying Theorem 9, we see that the quadratic residue tournament can also be generated from the complement of the quadratic residues. Applying this isomorphism shows the number of entries for square and non-square differences are the same.

Now, since there are $4k+2$ differences in the difference sequence, and $2k(2k+1)$ ordered pairs to form differences from in the symbol, each difference occurs exactly k times. ■

It follows from this theorem and the use of difference tables as outlined earlier that the k-domination graph of a quadratic residue tournament with $4k+3$ vertices is empty. Moreover, its $(k+1)$-domination graph is complete.

This family of quadratic residue tournaments demonstrates that there are an infinite number of tournaments for which generating a chain of domination graphs to determine which pairs are 'more dominant' cannot always work. For these special tournaments, the number of edges in the k-domination graphs is either zero or $\frac{n(n-1)}{2}$.

5 Conclusion

The formation of domination graphs of tournaments extends to the family of k-domination graphs. These generalisations of domination graphs give a means of indicating which pairs of vertices are more dominant in a tournament.

The royal graph generalisation of domination graphs give a different means of measuring which pairs of vertices are more dominant based on the concept of kings. Royal graphs and 2-domination graphs are closely related.

Examples with rotational tournaments show that k-domination graphs can be empty for arbitrarily large k, and the family of quadratic residue tournaments shows that in some special cases, the chain of k-domination graphs give no information about which pairs of vertices are more dominant.

Further work on these generalisations could include the answering of the open questions presented at the end of Section 2, and a proof of Conjecture 12.

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Dedication

MARGARET MORTON (1944-2000)

With much sadness we report the death of Margaret Morton, aged 55, in August 2000. Born in Auckland, New Zealand, Margaret graduated MSc with first class honours in mathematics from the University of Auckland in 1966. The following year she enrolled at Pennsylvania State University, and subsequently had three children before completing her PhD in 1975. After a succession of part-time teaching positions in Pennsylvania and Texas she returned to Auckland with her children in 1984. She worked as a computer programmer before gaining a Tutorship in Mathematics at the University of Auckland and began to establish an academic career with collaborative research in combinatorics and mathematics education. Margaret was generous with her time and energy and served on many committees, as Teaching Coordinator for the Mathematics Department for a number of years and as secretary of the NZ Mathematical Society 1993–1995. In 1998 she was promoted to Senior Lecturer and won a Marsden Fund Research Grant with a colleague for their work in graph theory. Her cancer was diagnosed during a year of sabbatical leave and prevented her from completing a visiting professorship at the University of North Texas.

Margaret's coauthors are grateful to have been able to work with her at the University of Auckland. Patricia met and worked with Margaret while there as a Postdoctoral Fellow, and Jamie had Margaret as one of his PhD supervisors.

References

[1] B. Alspach, On point-symmetric tournaments, *Canad. Math. Bull.* **13** (1970), 317–323.

[2] H. Cho, S-R. Kim and J.R. Lundgren, Domination graphs of regular tournaments I, *Discrete Math.*, to appear.

[3] H. Cho, F. Doherty, S-R. Kim and J.R. Lundgren, Domination graphs of regular tournaments II, *Congr. Numer.* **130** (1995), 95–111.

[4] D. Fisher, J.R. Lundgren, S. Merz and K. Reid, The domination and competition graphs of a tournament, *J. Graph Theory* **29** no. 2 (1998), 103–110.

[5] D. Fisher, J.R. Lundgren, S. Merz and K. Reid, Domination graphs of tournaments and digraphs, *Congr. Numer.* **108** (1995), 97–107.

[6] D. Fisher, J.R. Lundgren, S. Merz and K. Reid, Connected domination graphs of tournaments, *J. Combin. Math. Combin. Comput.* **31** (1999), 169–176.

[7] D. Fisher, D. Guichard, J.R. Lundgren, S. Merz and K. Reid, Domination graphs with nontrivial components, *Graphs Combin.* **17** no. 2 (2001), 227–236.

[8] M. Goldberg, The group of the quadratic residue tournament, *Canad. Math. Bull.* **13** (1970), 51–54.

[9] G. Landau, On domination relations and the structure of animal societies: III, the condition for a score structure, *Bull. Math. Biophys.* **15** (1953), 143–148.

[10] S. Maurer, The king chicken theorems, *Math. Mag.* **53(2)** (1980), 67–80.

[11] J. Moon, *Topics on tournaments.* Holt, Rinehart and Winston, 1968.

[12] J. Poet and B. Shader, Short score certificates for upset tournaments, *Electronic J. Combin.* **5** #R24 (1998), 18pp., `http://www.combinatorics.org/Volume_5/Abstracts/v5i1r24.html`

[13] K. Reid and L. Beineke, Tournaments. In L.W. Beineke and R.J. Wilson, *Selected topics in graph theory*, 169–204. Academic Press, New York, 1979.

[14] K. Reid, Tournaments: scores, kings, generalisations and special topics, in G. Chartrand and M. Jacobson (ed.), Surveys in graph theory, *Congr. Numer.* **115** (1996), 171–211.

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