On vertex-magic and edge-magic total injections of graphs

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Abstract

The study of graph labellings has focused on finding classes of graphs which admit a particular type of labelling. Here we consider variations of the well-known edge-magic and vertex-magic total labellings for which all graphs admit such a labelling. In particular, we consider two types of injections of the vertices and edges of a graph with positive integers: (1) for every edge the sum of its label and those of its end-vertices is some *magic* constant (*edge-magic*); and (2) for every vertex the sum of its label and those of the edges incident to it is some *magic* constant (*vertex-magic*). Our aim is to minimise the maximum label or the magic constant associated with the injection. We present upper bounds on these parameters for complete graphs, forests and arbitrary graphs, which in a number of cases are within a constant factor of being optimal. Our results are based on greedy algorithms for computing an antimagic injection, which is then extended to a magic total injection. Of independent interest is our result that every forest has an edge-antimagic vertex labelling.

1 Introduction

The study of graph labellings has focused on finding classes of graphs which admit a particular type of labelling, the more well-known including graceful, harmonious and various types of magic labellings (see the survey by Gallian [13]). In this paper we relax the definition of the well-known edge-magic and vertex-magic labelling schemes by allowing arbitrary positive integer labels. Since every graph admits such a labelling, our aim is to minimise the maximum label or the associated magic constant.

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All graphs $G = (V, E)$ are finite and simple with $n = |V|$ vertices, $m = |E|$ edges and maximum degree Δ. It will be convenient to assume G has at least one edge. The set of edges incident to a vertex v is denoted by $E(v)$. We consider labellings of the vertices and/or edges of a graph with positive integers such that distinct graph elements receive distinct labels. More formally, a *vertex injection* of a graph $G = (V, E)$ is an injective function $\lambda : V \to \mathbb{N}$; an *edge injection* of G is an injective function $\lambda : E \to \mathbb{N}$; and a *total injection* of G is an injective function $\lambda : V \cup E \to \mathbb{N}$. If the image of an injection λ on a set S $(= V, E$ or $V \cup E)$ is $\{1, 2, \ldots, |S|\}$ then λ is a *labelling.*

For some injection of a graph, the *edge-sum* of an edge vw, denoted by $\Lambda(vw)$, is the sum of the labels of the graph elements associated with vw. That is, if λ is an edge injection then $\Lambda(vw) = \lambda(vw)$; if λ is a vertex injection then the $\Lambda(vw) = \lambda(v)+\lambda(w)$; and if λ is a total injection then the $\Lambda(vw) = \lambda(vw) + \lambda(v) + \lambda(w)$. Similarly, the *vertex-sum* of a vertex v, denoted by $\Lambda(v)$, is the sum of the labels of the graph elements associated with v. That is, if λ is a vertex injection then the $\Lambda(v) = \lambda(v)$; if λ is an edge injection then

$$\Lambda(v) \;=\; \sum_{vx\in E(v)} \lambda(vw) \;\; ;$$

and if λ is a total injection then

$$\Lambda(v) \;=\; \lambda(v) + \sum_{vx\in E(v)} \lambda(vw) \;\; .$$

The maximum edge-sum and maximum vertex-sum are denoted by Λ_E and Λ_V, respectively.

An injection of a graph is *edge-magic* if all edge-sums are some constant and *edge-antimagic* if all edge-sums are distinct. Similarly an injection of a graph is *vertex-magic* if all vertex-sums are some constant and *vertex-antimagic* if all vertex-sums are distinct. The constant associated with an edge-magic or vertex-magic injection is called the *magic constant.* For results in magic and antimagic vertex and edge labellings see the survey by Gallian [13]. Edge-magic [5, 7, 14, 21, 23, 24, 29, 31, 32, 34, 38, 39] and vertex-magic [1, 2, 15, 16, 25, 26, 28, 35] total labellings have been studied extensively; see the recent monograph by Wallis [37]. A total injection which is both vertex-magic and edge-magic (possibly with different magic constants) is *totally magic.* Totally magic injections have been studied by McSorley [27] and totally magic labellings have been studied by Exoo *et al.* [10]. An edge-magic or vertex-magic total labelling with the extra requirement that the vertices are labelled $\{1, 2, \ldots, n\}$ (and hence the edges are labelled $\{n + 1, n + 2, \ldots, n + m\}$) is called *strong.* Strong edge-magic total labellings have been studied in [7, 8, 11, 12]

Most results in the papers cited above identify classes of graphs which do or do not admit types of magic labellings. On the other hand, in this paper we show that every graph has an edge-magic total injection, and every graph has a vertex-magic total injection (except for the obvious exception of a graph containing an isolated edge or two isolated vertices). We consider the problems of minimising the maximum label or magic constant in edge-magic or vertex-magic total injections. We conjecture that each of these problems are NP-hard. Other papers to consider various types of magic injections include [6, 19, 20].

This paper is organised as follows. Section 2 generalises some known results on edge-magic and vertex-magic total labellings to the setting of edge-magic and vertex-magic total injections. In particular, we show that antimagic injections can be used as a first step to producing magic injections, thus establishing the basis for the methods to follow. Section 3 describes lower bounds for the magic constant in edge-magic and vertex-magic total injections. We use these lower bounds to prove that the constructions in the latter sections are often within a constant factor of optimal. In Section 4 we establish a parallel between edge-antimagic vertex injections of complete graphs and the so-called Sidon sequences; we derive edge-magic total injections of complete graphs whose magic constant is within a constant factor of being optimal. In Section 5 we present the first known algorithm for constructing an edge-antimagic vertex labelling of a forest, and we give an algorithm for determining a vertex-antimagic edge injection of a forest. Section 6 describes greedy algorithms for determining edge-antimagic vertex injections and vertex-antimagic edge injections of arbitrary graphs and with polynomial-sized labels. The previously best known result, due to Bodendiek and Walther [2], produced vertex-antimagic edge injections with exponential-sized labels. We obtain edge-magic and vertex-magic total injections of an arbitrary graph also with polynomial-sized labels. Our upper bound on the magic constant in the case of vertex-magic total injections is within a constant factor of being optimal.

2 Basic Results

We now review some concepts from the literature on edge-magic and vertex-magic total labellings which can easily be generalised to edge-magic and vertex-magic total injections. First we consider the duality structure for edge-magic [38] and vertex-magic [25] total labellings. If an edge-magic total injection does not have a vertex or edge labelled 1 then reducing each label by one less than the minimum label produces an edge-magic total injection. An edge-magic total injection with a vertex or edge labelled 1 is called *minimal*. Given a total injection λ of a graph $G = (V, E)$, let $\hat{\lambda}$ be the maximum label assigned to a vertex or edge of G. The *dual* labelling λ' is given by $\lambda'(x) = 1 + \hat{\lambda} - \lambda(x)$ for all vertices and edges $x \in V \cup E$. The next result easily follows; see [38].

Observation 1. *Let λ be a minimal edge-magic total injection of a graph G with magic constant κ. Then λ' is also a minimal edge-magic total injection of G, and $\hat{\lambda} = \hat{\lambda}'$. If κ' is the magic constant of λ' then* $\min\{\kappa, \kappa'\} \leq \frac{3}{2}(\hat{\lambda} + 1)$.

There is a similar notion of minimality and duality for vertex-magic total injections of regular graphs. If a vertex-magic total injection of a regular graph does not have an edge or vertex labelled 1 then reducing each label by one less than the minimum label produces a vertex-magic total injection. A vertex-magic total injection of a regular graph with an edge or vertex labelled 1 is called *minimal*. The next result easily follows; see [25].

Observation 2. *Let λ be a minimal vertex-magic total injection of a Δ-regular graph G with magic constant κ. Then λ' is also a minimal vertex-magic total injection of G, and $\hat{\lambda} = \hat{\lambda}'$. If κ' is the magic constant of λ' then* $\min\{\kappa, \kappa'\} \leq \frac{1}{2}(\Delta+1)(\hat{\lambda}+1)$.

There are important relationships between magic total labellings and antimagic labellings (see for example [11, 38]) which we now extend to magic total injections. Since the edge-sums in an edge-magic total injection are equal and edge labels are distinct, the edge-sums in the induced vertex injection are distinct; that is, the induced vertex injection is edge-antimagic. Moreover, an edge-antimagic vertex injection can be extended to produce an edge-magic total injection as follows.

Lemma 1. *An edge-antimagic vertex injection λ of a graph $G = (V, E)$ with maximum label $\hat{\lambda}$ and maximum edge-sum Λ_E can be extended to an edge-magic total injection of G with maximum label at most* $\max\{\hat{\lambda}, \Lambda_E + \hat{\lambda} - 2\} \leq \max\{\hat{\lambda}, 3(\hat{\lambda}-1)\}$ *and magic constant at most* $\Lambda_E + \hat{\lambda} + 1 \leq 3\hat{\lambda}$.

Proof. Let κ be the minimum $i \in \mathbb{N}$ such that $i \geq \Lambda_E + 1$ and $i \neq \lambda(x) + \lambda(v) + \lambda(w)$ for all vertices $x \in V$ and edges $vw \in E$. Clearly $\kappa \leq \Lambda_E + \hat{\lambda} + 1$. Since vertex labels are distinct, $\Lambda_E \leq 2\hat{\lambda} - 1$ and thus $\kappa \leq 3\hat{\lambda}$. For each edge $vw \in E$, let $\lambda(vw) = \kappa - (\lambda(v) + \lambda(w))$. Since $\kappa \geq \Lambda_E + 1$ and edge-sums are distinct in an edge-antimagic vertex injection, the produced edge labels are positive and distinct. Now, κ is chosen so that $\kappa - (\lambda(v) + \lambda(w)) \neq \lambda(x)$ for all edges $vw \in E$ and for all vertices $x \in V$. Thus $\lambda(vw) \neq \lambda(x)$, and hence all labels are distinct. For each edge $vw \in E$, $\lambda(v) + \lambda(vw) + \lambda(w) = \kappa$. Hence λ is an edge-magic total injection with magic number κ. The maximum label is at most $\max\{\hat{\lambda}, \kappa - 3\} \leq \max\{\hat{\lambda}, \Lambda_E + \hat{\lambda} - 2\} \leq \max\{\hat{\lambda}, 3(\hat{\lambda}-1)\}$. ☐

Analogous to the edge-antimagic case, in a vertex-magic total injection the induced edge injection is vertex-antimagic, and a vertex-antimagic edge injection can be extended to a vertex-magic total injection.

Lemma 2. *Let $G = (V, E)$ be a graph with no isolated edges, at most one isolated vertex, and maximum degree Δ. A vertex-antimagic edge injection λ of G with maximum edge label $\hat{\lambda}$ can be extended to a vertex-magic total injection of G with maximum label and magic constant at most* $(\Delta+1)\hat{\lambda}$.

Proof. Let κ be the minimum $i \in \mathbb{N}$ such that $i \geq \Lambda_V + 1$ and $i \neq \lambda(vw) + \Lambda(x)$ for all vertices $x \in V$ and edges $vw \in E$. Clearly $\kappa \leq \hat{\lambda} + \Lambda_V + 1$. Let u be the vertex with maximum vertex-sum. Clearly $\deg(u) \geq 1$. If $\deg(u) = 1$ then $\Lambda_V = \hat{\lambda}$ and $\kappa \leq 2\hat{\lambda} + 1 \leq 3\hat{\lambda} \leq (\Delta+1)\hat{\lambda}$ since G has no isolated edges. If $\deg(u) \geq 2$ then

$$\Lambda_V = \Lambda(u) \leq \sum_{i=1}^{\deg(u)} (\hat{\lambda} - i + 1) \leq \Delta\hat{\lambda} - 1 \ .$$

Therefore $\kappa \leq (\Delta+1)\hat{\lambda}$. For each vertex $v \in V$, let $\lambda(v) = \kappa - \Lambda(v)$. (In what follows $\Lambda(v)$ still refers to the vertex-sum of v in the given edge injection.) Since $\kappa \geq \Lambda_V + 1$ and vertex-sums are distinct the produced vertex labels are positive and distinct. Now, κ is chosen so that $\kappa - \Lambda(x) \neq \lambda(vw)$ for all vertices $x \in V$ and for all edges $vw \in E$. Thus vertex labels are distinct from edge labels, and hence all labels are distinct. The new vertex-sum of each vertex v is $\lambda(v) + \Lambda(v) = \kappa$. Therefore λ is a vertex-magic total injection of G. The maximum label is at most $\max\{\hat{\lambda}, \kappa\} \leq (\Delta+1)\hat{\lambda}$. $\square$

Note that it is easily seen that of all the edge-magic total injections which extend a given edge-antimagic vertex injection, the one produced by the algorithm described in the proof of Lemma 1 has the minimum magic constant, and similarly for the vertex-magic total injections of Lemma 2.

3 Lower Bounds for the Magic Constant

The following lower bounds for the magic constant in edge-magic and vertex-magic total injections are proved using a double-counting argument.

Lemma 3. *Let $G = (V, E)$ be an n-vertex m-edge graph with maximum degree Δ and no isolated vertices. Let n_i be the number of vertices with degree i, and let N_i be the number of vertices with degree at least i. The magic constant κ in an edge-magic total injection λ of G is at least*

$$n + \frac{1}{2}(m+1) \ + \ \frac{1}{m}\sum_{i=1}^{\Delta} n_i i \left(N_{i+1} + \frac{1}{2}(n_i+1)\right) \ .$$

Proof. For every edge $vw \in E$, $\lambda(v) + \lambda(vw) + \lambda(w) = \kappa$. Hence,

$$\kappa \cdot m \ = \ \sum_{vw \in E} \lambda(v) + \lambda(vw) + \lambda(w) = \sum_{vw \in E} \lambda(vw) \ + \ \sum_{v \in V} \deg(v) \cdot \lambda(v) \ . \qquad (1)$$

Since $\deg(v) \geq 1$ for all vertices v, (1) is minimised when edges are labelled $\{n+1, n+2, \ldots, n+m\}$ and vertices are labelled $\{1, 2, \ldots, n\}$, with small labels applied to vertices of high degree and high labels applied to vertices of low degree. That is, the vertices of degree i, $1 \leq i \leq \Delta$, are labelled $\{N_{i+1}+1, N_{i+1}+2, \ldots, N_{i+1}+n_i(=N_i)\}$. Thus,

$$\begin{aligned}
\kappa \cdot m \ &\geq \ \sum_{j=1}^{m}(n+j) \ + \ \sum_{i=1}^{\Delta} i \sum_{k=1}^{n_i}(N_{i+1}+k) \\
\kappa \cdot m \ &\geq \ nm + \frac{1}{2}m(m+1) \ + \ \sum_{i=1}^{\Delta} i \left(N_{i+1} \cdot n_i + \frac{1}{2}n_i(n_i+1)\right) \\
\kappa \ &\geq \ n + \frac{1}{2}(m+1) \ + \ \frac{1}{m}\sum_{i=1}^{\Delta} n_i i \left(N_{i+1} + \frac{1}{2}(n_i+1)\right) \ . \quad \square
\end{aligned}$$

Corollary 1. *The magic constant κ in an edge-magic total injection of an n-vertex m-edge Δ-regular graph ($\Delta \geq 1$) is at least $\frac{1}{2}(m+3)+2n$.*

Proof. By Lemma 3, $\kappa \geq n+\frac{1}{2}(m+1)+\frac{n\Delta}{m}\left(0+\frac{1}{2}(n+1)\right) = n+\frac{1}{2}(m+1)+(n+1) = \frac{1}{2}(m+3)+2n$. ☐

We now establish a lower bound for the magic constant in vertex-magic total injections.

Lemma 4. *The magic constant κ in a vertex-magic total injection λ of an n-vertex m-edge graph $G=(V,E)$ is at least $\frac{m}{n}(m+1)+m+\frac{1}{2}(n+1)$.*

Proof. For all vertices $v \in V$,

$$\lambda(v) \;+\; \sum_{vx \in E(v)} \lambda(vx) \;=\; \kappa \;.$$

Hence,

$$\kappa \cdot n \;=\; \sum_{v \in V}\left(\lambda(v) \;+\; \sum_{vx \in E(v)} \lambda(vx)\right) \;=\; 2\sum_{vw \in E} \lambda(vw) \;+\; \sum_{v \in V} \lambda(v) \;. \tag{2}$$

(2) is minimised if edges are assigned small labels and vertices are assigned large labels. Since all labels are distinct positive integers,

$$\begin{aligned}\kappa \cdot n \;&\geq\; 2\sum_{i=1}^{m} i \;+\; \sum_{i=1}^{n}(m+i) \;=\; m(m+1)+nm+\frac{1}{2}n(n+1)\\ \kappa &\geq \frac{m(m+1)}{n}+m+\frac{1}{2}(n+1)\;. \quad \square\end{aligned}$$

Corollary 2. *The magic constant κ in a vertex-magic total injection of an n-vertex m-edge Δ-regular graph is at least $(\frac{1}{2}\Delta+1)m+\frac{1}{2}(n+1+\Delta)$.*

Proof. By Lemma 4, $\kappa \geq \frac{m}{n}(m+1)+m+\frac{1}{2}(n+1) = \frac{1}{2}\Delta(m+1)+m+\frac{1}{2}(n+1) = (\frac{1}{2}\Delta+1)m+\frac{1}{2}(n+1+\Delta)$. ☐

4 Edge-Magic Total Injections of Complete Graphs

In this section we show that edge-magic total injections of complete graphs are closely related to the so-called Sidon sequences [33, 36], which have been rediscovered under the guise of *well-spread* sequences [22, 24, 30]. The positive integers $a_1, a_2, \ldots, a_n$ are called a *Sidon* sequence (or a B_2 sequence[1]) if $a_i + a_j \neq a_k + a_l$ for all $i,j,k,l \in \{1,2,\ldots,n\}$; see Halberstam and Roth [17] for an overview. The next result follows immediately from the definitions of a Sidon sequence and an edge-antimagic vertex injection.

[1] $(a_1, a_2, \ldots, a_n)$ is a B_h sequence ($h \geq 2$) if every integer x has at most one representation of the form $x = b_1 + b_2 + \cdots + b_h$ ($b_i \in \{a_1, a_2, \ldots, a_n\}$) with $b_1 \leq b_2 \leq \cdots \leq b_h$.

Observation 3. *There exists an edge-antimagic vertex injection of the complete graph K_n with maximum label M if and only if there exists a Sidon sequence $a_1 < a_2 < \cdots < a_n = M$.*

Theorem 1. *For all n, the complete graph K_n has an edge-magic total injection with maximum label and magic constant at most $3n^2 + o(n^2)$.*

Proof. Define $f(x)$ to be the maximum number of elements from $\{1, 2, \ldots, x\}$ which form a Sidon sequence, and $g(n)$ to be the minimum maximum element taken over all Sidon sequences of length n; that is, g is the inverse function of f, and a lower bound for f or an upper bound for g corresponds to the construction of a Sidon sequence. An elementary result of Sidon [33] is that $f(x) = \Omega(x^{1/4})$. A much stronger result is that $f(x) = \sqrt{x} + o(\sqrt{x})$ (see Halberstam and Roth [17, Theorem 7]; in particular, Erdös and Turán [9] show that $f(x) < \sqrt{x} + O(x^{1/4})$ and Bose and Chowla [3] show that $f(x) > \sqrt{x} - O(x^{5/16})$). Since f and g are inverse functions, $g(n) = n^2 + o(n^2)$; that is, there exists a Sidon sequence of length n with a maximum element of $n^2 + o(n^2)$. By Observation 3 there exists an edge-antimagic vertex injection of K_n with a maximum label of $n^2 + o(n^2)$. Thus, by Lemma 1, K_n has an edge-magic total injection with maximum label and magic constant at most $3n^2 + o(n^2)$. □

By the lower bound in Corollary 1, the above upper bound for the magic constant in an edge-magic total injection of a complete graph is within a constant factor of being optimal. Note that for each n-vertex graph G, by deleting the appropriate edges from K_n, we obtain an edge-magic total injection with magic constant at most $3n^2 + o(n^2)$.

5 Antimagic Injections of Forests

In this section we present algorithms for constructing edge-antimagic vertex labellings and vertex-antimagic edge injections of a forest. Given a tree $T = (V, E)$, let $\operatorname{dist}(v, w)$ be the graph-theoretic distance between vertices $v, w \in V$. For some *root* vertex $r \in V$ and edge $vw \in E$, let $\operatorname{dist}(vw, r) = \min\{\operatorname{dist}(v, r), \operatorname{dist}(w, r)\}$, and for each vertex $v \neq r$, let $\operatorname{parent}(v)$ be the unique vertex adjacent to v with $\operatorname{dist}(\operatorname{parent}(v), r) = \operatorname{dist}(v, r) - 1$.

Theorem 2. *Every n-vertex forest F has an edge-antimagic vertex labelling.*

Proof. Initially assume F is a (connected) tree T. Let r be a vertex of T. For each $i \geq 0$ define $V_i = \{v \in V : \operatorname{dist}(v, r) = i\}$, and let

$$n_i = |V_i| \text{ and } S_i = \sum_{0 \leq j \leq i} n_j \ .$$

The labels $\{1, 2, \ldots, n\}$ are assigned in this order to vertices via a breadth-first search from r. That is, set $\lambda(r) = 1$, then assign the labels $\{2, 3, \ldots, S_1\}$ to the vertices in V_1, then assign the labels $\{1 + S_1, \ldots, S_2\}$ to the vertices in V_2, and so on with the vertices in V_i receiving the labels $\{1 + S_{i-1}, \ldots, S_i\}$. The particular label assigned to a vertex in V_i depends on the label assigned to its parent (in V_{i-1}). In particular, let V_i be ordered $(v_{i,1}, v_{i,2}, \ldots, v_{i,n_i})$ by increasing $\lambda(\text{parent}(v_{i,j}))$, and let $\lambda(v_{i,j}) = S_{i-1} + j$, $1 \leq j \leq n_i$. Clearly the vertices are labelled $\{1, 2, \ldots, n\}$.

We now show that the edge-sums are distinct. Consider edges $vw, xy \in E$ with $v = \text{parent}(w)$ and $x = \text{parent}(y)$. Suppose $v \in V_i$ and $x \in V_j$. Thus $w \in V_{i+1}$ and $y \in V_{j+1}$.

Consider the case that $i = j$. If $v = x$ then $\lambda(v)+\lambda(w) \neq \lambda(x)+\lambda(y)$ since w and y receive different labels. Otherwise $v \neq x$. Assume without loss of generality that $\lambda(v) < \lambda(x)$. Then by the choice of labels for vertices in V_{i+1}, we have $\lambda(w) < \lambda(y)$. Hence $\lambda(v) + \lambda(w) < \lambda(x) + \lambda(y)$.

Now suppose that $i \neq j$ and assume without loss of generality that $i < j$. Then $\lambda(v) + \lambda(w) \leq S_i + S_{i+1}$ and $S_{j-1} + S_j + 2 \leq \lambda(x) + \lambda(y)$. Clearly $S_i \leq S_{j-1}$ and $S_{i+1} \leq S_j$. Hence $\lambda(v) + \lambda(w) + 2 \leq \lambda(x) + \lambda(y)$. Therefore the edge-sums are distinct, and thus λ is an edge-antimagic vertex labelling of T.

Now suppose F consists of trees $T_1, T_2, \ldots, T_k$. By running the above algorithm for each T_i and adding $\sum_{j=1}^{i-1} |T_j|$ to each label, where $|T_j|$ is the number of vertices in T_j, we obtain an edge-antimagic vertex labelling of F. □

By Lemma 1 and Theorem 2 we have the following.

Corollary 3. *Every n-vertex forest has an edge-magic total injection with maximum label at most $3(n-1)$ and magic constant at most $3n$.* □

Note that for trees, in an edge-magic total injection determined by Corollary 3 the vertices are labelled $\{1, 2, \ldots, n\}$ and the edges are labelled from $\{n+1, n+2, ..., n+2m\}$, thus providing a result closely related to the conjecture of Enomoto *et al.* [7] that every tree has a strong edge-magic total labelling (with vertex labels $\{1, 2, \ldots, n\}$ and edge labels $\{n+1, n+2, ..., n+m\}$). Our upper bounds in Corollary 3 can be improved in the case of complete d-ary trees to a little more than $n + m$.

Lemma 5. *Every n-vertex m-edge complete d-ary tree ($d \geq 2$) has an edge-magic total injection with maximum label and magic constant at most $n + \frac{d+1}{d}m + O(1)$.*

Proof. Let T be the complete d-ary tree of height h for some $d \geq 2$ and $h \geq 1$. It is well-known that T has $n = \frac{1}{d-1}(d^{h+1} - 1)$ vertices. Let r be the root-vertex of T, and apply the algorithm described in Theorem 2 to determine an edge-antimagic vertex labelling of T. Let $v_{i,j}$ be the jth vertex at depth i ($0 \leq i \leq h$, $1 \leq j \leq d^i$), where vertices within depth i are ordered by increasing label. The label of $v_{i,j}$ is $\lambda(v_{i,j}) = \frac{1}{d-1}(d^i - 1) + j$. Clearly the edge from v_{h,d^h} to $v_{h-1,d^{h-1}}$ has the maximum edge-sum. Hence the maximum edge-sum

$$\begin{aligned}
\Lambda_E &= \max\{\lambda(x)+\lambda(y) : xy \in E\} \;=\; \lambda(v_{h,d^h}) + \lambda(v_{h-1,d^{h-1}}) \\
&= \frac{d^h-1}{d-1} + d^h + \frac{d^{h-1}-1}{d-1} + d^{h-1} \\
&= \frac{d^{h+1}-1+d^h-1}{d-1} \\
&= \frac{d^{h+2}-d+d^{h+1}-d}{d(d-1)} \\
&= \frac{(d+1)(d^{h+1}-1)-(d-1)}{d(d-1)} \\
&= \frac{(d+1)n-1}{d} \quad .
\end{aligned}$$

Applying Lemma 1 we obtain an edge-magic total injection with magic constant and maximum label at most $n+\Lambda_E+O(1) \;=\; n+\frac{d+1}{d}n+O(1) \;=\; n+\frac{d+1}{d}m+O(1)$. $\square$

We now evaluate the lower bound in Lemma 3 for the magic constant in an edge-magic total injection of a complete d-ary tree.

Lemma 6. *In an edge-magic total injection of a complete d-ary tree ($d \geq 2$) the magic constant $\kappa \geq \frac{4d+1}{2d}n - O(\frac{n}{d^2})$.*

Proof. Let T be the complete d-ary tree of height h. T has $n = \frac{1}{d-1}(d^{h+1}-1)$ vertices, of which one has degree d, $d^h = \frac{d^{h+1}-1+1}{d} = \frac{(d-1)n+1}{d}$ have degree one, and the remainder have degree $d+1$; that is $n-d^h-1 = n-\frac{1}{d}((d-1)n+1)-1 = \frac{1}{d}(nd-(d-1)n-1-d) = \frac{1}{d}(n-1-d)$ vertices. By Lemma 3 (with $i=1$ and $i=\Delta=d+1$ only),

$$\begin{aligned}
\kappa \geq{}& n+\frac{1}{2}(m+1)+\frac{1}{m}\cdot\frac{(d-1)n+1}{d}\left(\left(\frac{n-d-1}{d}+1\right)+\frac{1}{2}\left(\frac{(d-1)n+1}{d}+1\right)\right) \\
&+\frac{1}{m}\cdot\frac{n-d-1}{d}\cdot(d+1)\left(0+\frac{1}{2}\left(\frac{n-d-1}{d}+1\right)\right) \\
\geq{}& \frac{3n}{2}+\frac{1}{n-1}\cdot\frac{(d-1)n}{d}\left(\frac{n-1}{d}+\frac{dn-n+d+1}{2d}\right)+\frac{n-(d+1)}{n-1}\cdot\frac{n-1}{2d} \\
\geq{}& \frac{3n}{2}+\frac{1}{n-1}\cdot\frac{(d-1)n}{d}\cdot\frac{(d+1)n}{2d}+\frac{n-(d+1)}{2d} \\
\geq{}& \frac{3n}{2}+\frac{(d^2-1)(n+1)}{2d^2}+\frac{n-(d+1)}{2d} \\
\geq{}& \frac{3d^2n+d^2n+d^2-n-1+dn-d^2-d}{2d^2} \\
\geq{}& \frac{(4d+1)n}{2d}-O\left(\frac{n}{d^2}\right) \quad .
\end{aligned}$$

$\square$

It follows from Lemma 5 and Lemma 6 that the ratio between our lower and upper bounds for the magic constant in an edge-magic total injection of a complete d-ary tree is $1 + O(\frac{1}{d})$, and similarly for the maximum label.

We now consider the problem of determining a vertex-antimagic edge injection of a forest. The following result will be used in Theorem 5 to determine a vertex-antimagic edge injection of an arbitrary graph.

Theorem 3. *Every n-vertex m-edge forest F with no isolated edges and at most one isolated vertex has a vertex-antimagic edge injection with maximum label at most $m + 2(n-2)$.*

Proof. Let $T = (V, E)$ be a connected component (tree) of F with $E \neq \emptyset$. Let r be a vertex of T with $\deg(r) \geq 2$. Since $T \neq K_2$ such a vertex exists. The following algorithm, which proceeds via a breadth-first search from r, sequentially chooses the minimum possible label for each edge of T. More formally, label the edges of T in non-decreasing order of $\text{dist}(vw, r)$, assigning to each edge vw with $v = \text{parent}(w)$ the minimum positive integer $\lambda(vw)$ such that (1) $\lambda(vw) \neq \lambda(xy)$ for all labelled edges $xy \in E$, and (2) $\lambda(vw) \neq \Lambda(x) - \Lambda(v)$ and $\lambda(vw) \neq \Lambda(x)$ for all vertices $x \in V \setminus \{v, w\}$ (only counting labelled edges in each vertex-sum).

Suppose rx and ry are the first and second edges to be labelled by the above algorithm. Then $\Lambda(r) = \lambda(rx) + \lambda(ry)$, $\Lambda(x) = \lambda(rx)$ and $\Lambda(y) = \lambda(ry)$. Thus $\Lambda(r)$, $\Lambda(x)$ and $\Lambda(y)$ are pairwise distinct. All other vertices v have $\Lambda(v) = 0$. We shall prove that the the following property is maintained, which as we have just shown holds after two edges are labelled:

$$\text{For all distinct vertices } v, w \in V, \text{ if } \Lambda(v) = \Lambda(w) \text{ then } \Lambda(v) = 0 = \Lambda(w). \tag{3}$$

Consider when the edge vw is labelled. Since $v = \text{parent}(w)$, all other edges incident to w are unlabelled, and hence $\Lambda(w) = 0$. By (3), $\Lambda(v) > 0$. The new vertex-sum $\Lambda'(v)$ of v is $\Lambda(v) + \lambda(vw)$, the new vertex-sum $\Lambda'(w)$ of w is $\lambda(vw)$, and $\Lambda(x)$ does not change for any other vertex x. Since $\Lambda(v) > 0$, we have $\Lambda'(v) = \Lambda(v) + \lambda(vw) \neq \lambda(vw) = \Lambda'(w)$. For all vertices $x \in V \setminus \{v, w\}$, by the choice of $\lambda(vw)$, we have $\Lambda'(v) = \Lambda(v) + \lambda(vw) \neq \Lambda(x)$ and $\Lambda'(w) = \lambda(vw) \neq \Lambda(x)$. Hence (3) holds after labelling vw. Therefore (3) holds after all edges are labelled, in which case $\Lambda(v) > 0$ for all vertices v; that is, for all distinct vertices $v, w \in V$, $\Lambda(v) \neq \Lambda(w)$. Hence λ is a vertex-antimagic edge injection of T.

By running the above algorithm for each connected component T of F we obtain a vertex-antimagic edge injection of F. For each edge vw, there are at most $(m - 1) + 2(n-2)$ values which the label $\lambda(vw)$ cannot take on. Hence the maximum label is at most $m + 2(n-2)$. □

It is expected that better upper bounds are achievable for the maximum label in a vertex-antimagic edge injection. In fact, Ringel and Llado [31] (see also Hartsfield and Ringel [18]) conjecture that every *graph* has a vertex-antimagic edge labelling (with labels $\{1, 2, \ldots, m\}$).

6 Greedy Algorithms for Arbitrary Graphs

We now present methods for producing edge-antimagic and vertex-antimagic injections of arbitrary graphs, which we then extend to edge-magic and vertex-magic injections. These greedy algorithms sequentially choose the minimum possible label for the vertices or edges of the given graph.

Theorem 4. *Every n-vertex m-edge graph G with maximum degree Δ has an edge-antimagic vertex labelling with maximum label at most $\Delta(m-\Delta)+n$.*

Proof. Consider the following algorithm. Order the vertices $(v_1, v_2, \dots, v_n)$ by non-decreasing degree. For $i = 1, 2, \dots, n$, set $\lambda(v_i)$ to be the minimum $l \in \mathbb{N}$ such that (1) $l \neq \lambda(v_j)$ for all $j < i$, and (2) $l \neq \lambda(v_p)+\lambda(v_q)-\lambda(v_j)$ for all edges $v_iv_j, v_pv_q \in E$ with $j, p, q < i$ and $p \neq j \neq q$.

At each step of the algorithm, by the choice of $\lambda(v_i)$, for all edges xy with both end-vertices labelled, the sum $\lambda(x)+\lambda(y)$ is unique. Thus at the end of the algorithm, λ is an edge-antimagic vertex injection of G. When choosing $\lambda(v_i)$, there are at most $(i-1)+\text{pred}(v_i)\cdot m_i$ positive integers which $\lambda(v_i)$ cannot become, where $\text{pred}(v_i)$ is the number of edges v_iv_j with $j < i$, and m_i is the number of edges $v_pv_q \in E$ with $p, q < i$. Hence $\lambda(v_i) \leq i + \text{pred}(v_i) \cdot m_i$. Now $\text{pred}(v_i) \leq \Delta$ and $m_i \leq m - \deg(v_n) = m - \Delta$ since vertices are ordered by non-decreasing degree. Hence the maximum label is at most $n + \Delta(m-\Delta)$. □

By Lemma 1 and Theorem 4 we have the following result.

Corollary 4. *Every n-vertex m-edge graph with maximum degree Δ has an edge-magic total injection with maximum label at most $3(\Delta(m-\Delta)+n)-2$ and magic constant at most $3(\Delta(m-\Delta)+n)$.* □

Theorem 5. *Every n-vertex m-edge graph $G=(V,E)$ with no isolated edge and at most one isolated vertex has a vertex-antimagic edge injection with maximum label at most $m+2(n-2)$.*

Proof. Consider the following two-step algorithm. First, let $T=(V,E')$ be a spanning forest of G, and apply Theorem 3 to T to obtain a vertex-antimagic edge injection λ. In the second step, for each edge $vw \in E \setminus E'$ in turn, let $\lambda(vw)$ to be the minimum $l \in \mathbb{N}$ such that (1) $l \neq \lambda(xy)$ for all labelled edges $xy \in E$, and (2) $l \neq \Lambda(x)-\Lambda(v)$ and $l \neq \Lambda(x)-\Lambda(w)$ for all vertices $x \in V \setminus \{v,w\}$ (only counting labelled edges in each vertex-sum).

After the first step (since T is a spanning forest and λ is vertex-antimagic) for all distinct vertices $v, w \in V$, $\Lambda(v) \neq \Lambda(w)$. Suppose, at some point during the second step, λ is still vertex-antimagic on the set of labelled edges, and the label l is chosen for some edge vw. The new vertex-sum $\Lambda'(v)$ of v is $\Lambda(v)+l$, the new vertex-sum $\Lambda'(w)$ of w is $\Lambda(w)+l$, and $\Lambda(x)$ does not change for any other vertex x. Since λ is vertex-antimagic, $\Lambda(v) \neq \Lambda(w)$, and hence $\Lambda'(v)=\Lambda(v)+l \neq \Lambda(w)+l=\Lambda'(w)$. By the choice of l, for all other vertices $x \in V$, $\Lambda'(v)=\Lambda(v)+l \neq \Lambda(x)$ and $\Lambda'(w)=\Lambda(w)+l \neq \Lambda(x)$. Hence λ remains vertex-antimagic throughout the second step.

By Theorem 3 the maximum label in T is $|E'|+2(n-2)$. For each edge $vw \in E\backslash E'$, there are at most $(m-1)+2(n-2)$ values which the label $\lambda(vw)$ cannot take on. Hence the maximum label is at most $\max\{|E'|+2(n-2), m+2(n-2)\} = m+2(n-2)$. □

By Lemma 2 and Theorem 5 we have the following result.

Corollary 5. *Every n-vertex m-edge graph with maximum degree Δ has a vertex-magic total injection with maximum label and magic constant at most* $(\Delta+1)(m+2(n-2))$. □

By Corollary 2 the above upper bound on the magic constant is within a constant factor of being optimal for regular graphs.

7 Conclusion

In this paper we have studied edge-magic and vertex-magic total injections of graphs. These variations of the well-known edge-magic and vertex-magic total labellings allow labels to be arbitrary positive integers. All graphs admit such magic injections (except for some trivial exceptions). We have presented lower and upper bounds on the magic constant for complete graphs, trees and arbitrary graphs, which in a number of cases are within a constant factor. We expect that better upper bounds are possible for the magic constant in edge-magic total injections. In particular, we conjecture that every n-vertex m-edge graph has an edge-magic total injection with magic constant $O(m+n)$. This conjecture has been verified in the case of complete graphs, complete bipartite graphs, trees and cycles.

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References

[1] M. Bača, M. Miller and Slamin, Every generalized Petersen graph has a vertex-magic total labeling. In [4], pp. 1–6.

[2] R. Bodendiek and G. Walther, On the relations between certain graph labelings. *Discrete Math.*, **134(1-3)**:9–16, 1994.

[3] R.C. Bose and S. Chowla, Theorems in the additive theory of numbers. *Comment. Math. Helv.*, **37**:141–147, 1962/1963.

[4] L. Brankovic and J. Ryan, eds., *Proc. 11th Australasian Workshop on Combinatorial Algorithms (AWOCA '00)*, The University of Newcastle, Australia, 2000.

[5] D. Craft and E.H. Tesar, On a question by Erdős about edge-magic graphs. *Discrete Math.*, **207(1-3)**:271–276, 1999.

[6] M. Doob, Characterizations of regular magic graphs. *J. Combin. Theory Ser. B*, **25(1)**:94–104, 1978.

[7] H. Enomoto, A.S. Llado, T. Nakamigawa and G. Ringel, Super edge-magic graphs. *SUT J. Math.*, **34(2)**:105–109, 1998.

[8] H. Enomoto, K. Masuda and T. Nakamigawa, Induced graph theorem on magic valuations. *Ars Combin.*, **56**:25–32, 2000.

[9] P. Erdös and P. Turán, On a problem of Sidon in additive number theory, and on some related problems. *J. London Math. Soc.*, **16**:212–215, 1941.

[10] G. Exoo, A.C.H. Ling, J.P. McSorley, N.C.K. Phillips and W.D. Wallis, Totally magic graphs. *Discrete Math.*, **254**:103–113, 2002.

[11] R.M. Figueroa-Centeno, R.Ichishima and F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings. *Discrete Math.*, **231(1-3)**:153–168, 2001.

[12] Y.Fukuchi, A recursive theorem for super-edge-magic labelings of trees. *SUT J. Math.*, **36(2)**:279–285, 2000.

[13] J.A. Gallian, A dynamic survey of graph labeling. *Electron. J. Combin.*, **#DS6**, 2000.

[14] R.D. Godbold and P.J. Slater, All cycles are edge-magic. *Bull. Inst. Combin. Appl.*, **22**:93–97, 1998.

[15] I.D. Gray, J.A. MacDougall, R.J. Simpson and W.D. Wallis, Vertex-magic total labelings of complete bipartite graphs. *Ars Combinatoria*, to appear.

[16] I.D. Gray, J.A. MacDougall and W.D. Wallis, A unified construction of vertex-magic total labelings for the complete graphs. Preprint.

[17] H. Halberstam and K.F. Roth, *Sequences. Vol. I.* Clarendon Press, Oxford, 1966.

[18] N. Hartsfield and G. Ringel, *Pearls in graph theory: a comprehensive introduction.* Academic Press, 1994.

[19] R.H. Jeurissen, Pseudomagic graphs. *Discrete Math.*, **43(2-3)**:207–214, 1983.

[20] R.H. Jeurissen, Magic graphs, a characterization. *European J. Combin.*, **9(4)**:363–368, 1988.

[21] A. Kotzig, On magic valuations of trichromatic graphs. *Publications du Centre de Recherches Mathématiques Université de Montréal*, **148**, 1971.

[22] A. KOTZIG, On well spread sets of integers. *Publications du Centre de Recherches Mathématiques Université de Montréal*, **161**, 1972.

[23] A. KOTZIG AND A. ROSA, Magic valuations of finite graphs. *Canad. Math. Bull.*, **13**:451–461, 1970.

[24] A. KOTZIG AND A. ROSA, Magic valuations of complete graphs. *Publications du Centre de Recherches Mathématiques Université de Montréal*, **175**, 1972.

[25] J. MACDOUGALL, M. MILLER, SLAMIN AND W.D. WALLIS, Vertex-magic total labelings of graphs. *Utilitas Math.*, **61**:3–21, 2002.

[26] J. MACDOUGALL, M. MILLER AND W.D. WALLIS, Vertex-magic total labelings of wheels and related graphs. *Utilitas Math.*, to appear.

[27] J.P. MCSORLEY, Totally magic injections of graphs. *J. Combin. Math. Combin. Comput.*, to appear.

[28] M. MILLER, J.A. MACDOUGALL, SLAMIN AND W.D. WALLIS, Problems in total graph labelings. In R. RAMAN AND J. SIMPSON, eds., *Proc. 8th Australasian Workshop on Combinatorial Algorithms (AWOCA '97)*, pp. 19–25, Curtin University of Technology, Australia, 1999.

[29] N.C.K. PHILLIPS, R.S. REES AND W.D. WALLIS, Edge-magic total labelings of wheels. *Bull. Inst. Combin. Appl.*, **31**:21–30, 2001.

[30] N.C.K. PHILLIPS AND W.D. WALLIS, Well-spread sequences. *J. Combin. Math. Combin. Comput.*, **31**:91–96, 1999.

[31] G. RINGEL AND A.S. LLADO, Another tree conjecture. *Bull. Inst. Combin. Appl.*, **18**:83–85, 1996.

[32] Y. RODITTY AND T. BACHAR, A note on edge-magic cycles. *Bull. Inst. Combin. Appl.*, **29**:94–96, 2000.

[33] S. SIDON, Ein Satz űber trigonometrische Polynome und seine Anwendung in der Theorie der Fourier-Reihen. *Math. Ann.*, **106**:536–539, 1932.

[34] SLAMIN, M. BAČA, Y. LIN, M. MILLER AND R. SIMANJUNTAK, Edge-magic total labelings of wheels, fans and friendship graphs. *Bull. Inst. Combin. Appl.*, **35**:89–98, 2002.

[35] SLAMIN AND M. MILLER, On two conjectures concerning vertex-magic total labelings of generalized Petersen graphs. *Bull. Inst. Combin. Appl.*, **32**:9–16, 2001.

[36] N.J.A. SLOANE, Sequence A011185. *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/`.

[37] W.D. WALLIS, *Magic Graphs.* Birkhäuser, 2001.

[38] W.D. Wallis, E.T. Baskoro, M. Miller and Slamin, Edge-magic total labelings. *Austral. J. Combin.*, **22**:177–190, 2000.

[39] K. Wijaya and E.T. Baskoro, Edge-magic total labelling on disconnected graphs. In [4], pp. 139–144.

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