Linear blocking sets in $PG(2, q^4)$

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Abstract

In this paper, by using the geometric construction of linear blocking sets as projections of canonical subgeometries, we determine all the $GF(q)$-linear blocking sets of the plane $PG(2, q^4)$.

1 Introduction

A *blocking set* B in the projective plane $PG(2, q)$ is a set of points meeting every line. B is called *trivial* if it contains a line, and it is called *minimal* if no proper subset of it is a blocking set. We say B is *small* when its size is less than $3(q+1)/2$ and we call B of *Rédei type* if there exists a line l such that $|B \setminus l| = q$ (the line l is called a Rédei line of B).

A family of small minimal blocking sets in $PG(2, q^t)$, called $GF(q)$-*linear* blocking sets, was introduced by G. Lunardon in [3] (for a survey on linear blocking sets see [8]). Every small minimal blocking set of Rédei type in $PG(2, q)$, $q = p^n$ and $p \neq 2, 3$, is a linear blocking set over some non-trivial subfield of $GF(q)$ (see [1] and [3]), and the known examples of small minimal blocking sets not of Rédei type are linear ([9]). Hence, all the presently known small minimal blocking sets are linear.

In the planes $PG(2, q^2)$ and $PG(2, q^3)$, the $GF(q)$-linear blocking sets are completely classified: in $PG(2, q^2)$ they are Baer subplanes and in $PG(2, q^3)$ they are isomorphic either to the blocking set obtained from the trace function of $GF(q^3)$ over $GF(q)$ or to the blocking set obtained from the function $x \mapsto x^q$ ([10]).

In this paper, we study the $GF(q)$-linear blocking sets in $PG(2, q^4)$. Our main result is the following theorem:

Theorem 1.1 *Let B be a $GF(q)$-linear blocking set in $PG(2, q^4)$. If B is of Rédei type with at least two Rédei lines, then either B is a Baer subplane or B has q^4+q^3+1 points, $q+1$ Rédei lines and it is equivalent to the blocking set obtained from the graph of the trace function of $GF(q^4)$ over $GF(q)$. If B is of Rédei type with a unique Rédei line, then the possible sizes of B are q^4+q^3+1 and $q^4+q^3+q^2+cq+1$ with $c \in \{-1, 0, 1\}$. Finally, if B is not of Rédei type, then B has size $q^4+q^3+q^2+dq+1$ with $d \in \{0, 1\}$.*

Australasian Journal of Combinatorics 26(2002), pp.41–48

Finally, in the last section we prove that there exists at least one example of $GF(q)$-linear blocking set in $PG(2, q^4)$ for each possible cardinality.

2 Linear blocking sets in $PG(2, q^4)$

It is possible to define a linear blocking set in three different and equivalent ways. In this paper we will use the geometric construction of such blocking sets following [4].

Let $\Sigma = PG(t, q)$, $t \geq 3$, be a canonical subgeometry of $\Sigma^* = PG(t, q^t)$. Let Λ be a $(t-3)-$dimensional subspace of Σ^* disjoint from Σ, and let π be a plane of Σ^* disjoint from Λ. Recall that the *projection* of Σ from the *axis* Λ to the plane π is the map $p_{\Lambda,\pi,\Sigma}$ from Σ to π defined by

$$p_{\Lambda,\pi,\Sigma}(P) =< P, \Lambda > \cap \pi$$

for each point P of Σ. The set $p_{\Lambda,\pi,\Sigma}(\Sigma)$ is a $GF(q)-$linear blocking set of $\pi = PG(2, q^t)$. Also, any $GF(q)-$linear blocking set of $PG(2, q^t)$ can be constructed as a projection of a suitable canonical subgeometry of $PG(t, q^t)$ (see [5]). Note that, since Σ is a canonical subgeometry, there is no hyperplane of Σ^* containing Σ and hence the $GF(q)$-linear blocking sets obtained projecting Σ are non-trivial.

Now, suppose that $t = 4$ and let $\Sigma = PG(4, q)$ be a canonical subgeometry of $\Sigma^* = PG(4, q^4)$. Let σ be the unique semilinear collineation of Σ^* which fixes Σ pointwise. Then $\sigma^4 = 1$ and the set of fixed points of σ^2 is a canonical subgeometry $\Sigma' = PG(4, q^2)$ of Σ^* containing Σ. Also, if S_i is an i-dimensional subspace of Σ^*, then $S_i \cap \Sigma$ (resp. $S_i \cap \Sigma'$) is an i-dimensional subspace of Σ (resp. Σ') if and only if $S_i^\sigma = S_i$ (resp. $S_i^{\sigma^2} = S_i$) (see e.g. [4]).

Let π be a plane of Σ^* and let l be a line disjoint from both π and Σ. Denote by p the projection $p_{l,\pi,\Sigma}$ of Σ from l to the plane π and by B_l the $GF(q)$-linear blocking set $p(\Sigma)$ of the plane π. Notice that if R is a point of B_l, then $p^{-1}(R)$ is an i-dimensional subspace of Σ with $i \in \{0, 1, 2\}$, and $< p^{-1}(R) >$ is an i-dimensional subspace of Σ^* containing l. Similarly, if r is a line of π, then $p^{-1}(r \cap B_l)$ is an i-dimensional subspace of Σ with $i \in \{0, 1, 2, 3\}$, and $< p^{-1}(r \cap B_l) >$ is an i-dimensional subspace of Σ^* containing l.

Proposition 2.1 *B_l is of Rédei type if and only if l is contained in a 3-dimensional subspace of Σ^* fixed by σ.*

Proof. Suppose that there exists a 3-dimensional subspace S_3 of Σ^* containing l and fixed by σ. Then $S_3 \cap \Sigma$ is a 3-dimensional subspace of Σ projected from l to the line $r = \pi \cap S_3$, i.e. $p^{-1}(r \cap B_l) = S_3 \cap \Sigma$. Let R be a point of $B_l \setminus r$. Since $p^{-1}(R)$ is a subspace of Σ disjoint from $p^{-1}(r \cap B_l)$, $p^{-1}(R)$ is a point of Σ. This implies that $|B_l \setminus r| = |\Sigma \setminus S_3| = q^4$, i.e. r is a Rédei line of B_l.

Now, suppose that B_l is of Rédei type and let r be a Rédei line of B_l. Since B_l is non-trivial, it contains at least $q^4 + q^2 + 1$ points (see [2]) and $|B_l \cap r| \geq q^2 + 1$. This implies that $p^{-1}(r \cap B_l)$ is either a 2 or a 3-dimensional subspace of Σ. In the latter case, $< p^{-1}(r \cap B_l) >$ is a 3-dimensional subspace of Σ^* containing l and fixed by σ.

In the former case, the q^4+q^3 points of $\Sigma \setminus p^{-1}(r \cap B_l)$ are projected to the q^4 points of $B_l \setminus r$. Hence, there exist at least two points R and T of $B_l \setminus r$ such that $p^{-1}(R)$ and $p^{-1}(T)$ contain some line of Σ. Let n and t be lines of Σ^* such that $\Sigma \cap n$ is a line of $p^{-1}(R)$ and $\Sigma \cap t$ is a line of $p^{-1}(T)$. Then $n \cap t = \emptyset$, $n \cap l \neq \emptyset$ and $t \cap l \neq \emptyset$. Hence $S_3 =< n, t >$ is a 3-dimensional subspace of Σ^* containing l fixed by σ. □

Corollary 2.2 *B_l is of Rédei type if and only if* $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >\leq 3$. *Also, if B_l is not a Baer subplane, then it has a unique Rédei line if and only if* $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >= 3$.

Proof. By Proposition 2.1, if B_l is of Rédei type, then l is contained in some 3-dimensional subspace, say S_3, of Σ^* fixed by σ. This implies that l, l^σ, l^{σ^2}, and l^{σ^3} are contained in S_3 and hence $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >\leq 3$. Conversely, suppose that $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >\leq 3$. If $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >= 3$, then $S_3 =< l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >$ is a 3-dimensional subspace of Σ^* containing l and fixed by σ, hence B_l is of Rédei type. If $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >= 2$, for any point $P \in \Sigma \setminus < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >$, it is easy to check that $< P, l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >$ is a 3-dimensional subspace fixed by σ, so B_l is of Rédei type. Finally, suppose that B_l is of Rédei type, but it is not a Baer subplane. In this case $|B_l| \geq q^4 + q^3 + 1$ (see [1]) and hence, if r is a Rédei line of B_l, then $|B_l \cap r| \geq q^3 + 1$. This implies that $p^{-1}(r \cap B_l)$ is a 3-dimensional subspace of Σ, hence $< p^{-1}(r \cap B_l) >$ is a 3-dimensional subspace of Σ^* containing l fixed by σ. Then $< l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >\subset< p^{-1}(r \cap B_l) >$. Thus r is the unique Rédei line of B_l if and only if $< l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >=< p^{-1}(r \cap B_l) >$, i.e. if and only if $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >= 3$. □

Proposition 2.3 *With the previous notation, the following are equivalent:*
1) *B has maximum size $q^4 + q^3 + q^2 + q + 1$.*
2) *There is no line of Σ projected from l to a point of π.*
3) *There is no line of Σ^* fixed by σ incident with l.*

Proof. B_l has maximum size if and only if the map p is a bijection, hence if and only if $p^{-1}(R)$ is a point of Σ for every point $R \in B_l$ (see also [4]). Finally, B_l has not maximum size if and only if there exists a line t of Σ projected from l to a point of B_l. In this case, if t' is the line of Σ^* containing t, then t' is incident with l and $t'^\sigma = t'$. □

3 Proof of Theorem 1.1

The structure of B_l depends on the position of l with respect to the subgeometries Σ and Σ'. In order to determine the different possibilities for B_l, we have to distinguish between the following cases:

(A) $l = l^{\sigma^2} \iff l$ intersects Σ' in a line;

(B) $l \cap l^{\sigma^2}$ is a point $\iff l$ intersects Σ' in a point;

(C) $l \cap l^{\sigma^2} = \emptyset \iff l$ is disjoint from Σ'.

Case A

It is easy to check that in this case $l \cap l^\sigma = \emptyset$, hence $S_3 =< l, l^\sigma >$ is a 3-dimensional subspace of Σ^*. Since S_3 is fixed by σ, by Proposition 2.1, B_l is of Rédei type and $r = S_3 \cap \pi$ is a Rédei line of B_l. Also, if $P \in l \cap \Sigma'$ the line $< P, P^\sigma >$ is fixed by σ and hence $< P, P^\sigma > \cap \Sigma$ is a line of Σ. So $< P, P^\sigma > \cap \Sigma$ is projected from l to a point of $r \cap B_l$, for every point $P \in l \cap \Sigma'$. This implies that the size of $B_l \cap r$ is $q^2 + 1$, and hence $|B_l| = q^4 + q^2 + 1$, i.e. B_l is a Baer subplane of π ([2]).

Case B

Put $l \cap l^{\sigma^2} = \{P\}$. Note that the line $< P, P^\sigma >$ is fixed by σ, hence $< P, P^\sigma >$ intersects Σ in a line.

(B_1) $\quad l \cap l^\sigma \neq \emptyset$

In this case, we have $l^\sigma \cap l^{\sigma^2} \neq \emptyset$, $l^{\sigma^2} \cap l^{\sigma^3} \neq \emptyset$, and $l^{\sigma^3} \cap l \neq \emptyset$, i.e. $\overline{\pi} =< l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >$ is a plane of Σ^* fixed by σ. Hence $\overline{\pi} \cap \Sigma$ is projected from l to a point R of B_l. Also, every 3-dimensional subspace obtained joining $\overline{\pi}$ to a point of $\Sigma \setminus \overline{\pi}$ intersects Σ in a 3-dimensional subspace. Then, through R there pass $q+1$ Rédei lines and $|B_l| = q^4 + q^3 + 1$. This implies that B_l is equivalent to the blocking set obtained from the graph of the trace function of $GF(q^4)$ over $GF(q)$ (see [6]).

(B_2) $\quad l \cap l^\sigma = \emptyset$ and $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >= 3$

Let $S_3 =< l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >$. By Corollary 2.2 and Proposition 2.1, B_l is of Rédei type and $S_3 \cap \pi$ is a Rédei line of B_l. Note that both lines $< P, P^\sigma >$ and $l' =< l, l^{\sigma^2} > \cap < l^\sigma, l^{\sigma^3} >$ are fixed by σ, so they intersect Σ in a line. Also, any line of Σ^* fixed by σ and incident with l is incident with l^σ, l^{σ^2} and l^{σ^3}. This implies that $< P, P^\sigma >$ and l' are the unique lines of Σ^* fixed by σ and incident with l.

(B_{21}) If $< P, P^\sigma >= l'$, then exactly one line of Σ is projected from l to a point of B_l, so B_l has size $q^4 + q^3 + q^2 + 1$.

(B_{22}) If $< P, P^\sigma > \neq l'$, then exactly two lines of Σ are projected from l to a point of B_l, so B_l has size $q^4 + q^3 + q^2 - q + 1$.

Since the blocking sets B_l obtained in Cases B_{21} and B_{22} are not Baer subplanes, $S_3 \cap \pi$ is the unique Rédei line of B_l (Corollary 2.2).

(B_3) $\quad l \cap l^\sigma = \emptyset$ and $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >= 4$

Since the subspace joining l, l^σ, l^{σ^2}, and l^{σ^3} has dimension four, B_l is not of Rédei type (Corollary 2.2). If m is a line fixed by σ and incident with l, then $m =< P, P^\sigma >$. Thus, B_l has size $q^4 + q^3 + q^2 + 1$.

Case C

(C_1) $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >= 3$
Let $S_3 =< l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >$. In this case B_l is of Rédei type and $r = S_3 \cap \pi$ a Rédei line of B_l (Corollary 2.2).

(C_{11}) Suppose that $l \cap l^\sigma \neq \emptyset$ and let $\{P\} = l \cap l^\sigma$. This implies that $l =< P, P^{\sigma^3} >$. Hence the unique lines intersecting l, l^σ, l^{σ^2} and l^{σ^3} are $< P^{\sigma^2}, P >$ and $< P^{\sigma^3}, P^\sigma >$. Since such lines are not fixed by σ, there is no line of Σ^* projected from l to a point of B_l, i.e. B_l has maximum size (Proposition 2.3).

(C_{12}) Suppose that $l \cap l^\sigma = \emptyset$ and that l, l^σ, l^{σ^2} and l^{σ^3} belong to the same regulus $\mathcal{R}$ of S_3. Since $\mathcal{R}$ is fixed by σ, $\mathcal{R} \cap \Sigma$ is a regulus of $S_3 \cap \Sigma$. This implies that each transversal line to $\mathcal{R} \cap \Sigma$ is projected from l to a point of $r \cap B_l$. Since the transversal lines to $\mathcal{R} \cap \Sigma$ number $q+1$, the size of $r \cap B_l$ is $q^3 + 1$, and $|B_l| = q^4 + q^3 + 1$ (see also [6]).

Now, suppose that $l \cap l^\sigma = \emptyset$ and that l, l^σ, l^{σ^2} and l^{σ^3} do not belong to the same regulus of S_3. Let $\mathcal{R}$ be the regulus determined by l, l^σ, and l^{σ^2} and let $\overline{\mathcal{R}}$ be the opposite regulus to $\mathcal{R}$. A line l' fixed by σ and incident with l, is incident with l^σ, l^{σ^2} and l^{σ^3} and hence it is a transversal line to $\mathcal{R}$, $\mathcal{R}^\sigma$, $\mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$, i.e. $l' \in \overline{\mathcal{R}} \cap \overline{\mathcal{R}}^\sigma \cap \overline{\mathcal{R}}^{\sigma^2} \cap \overline{\mathcal{R}}^{\sigma^3}$. Note that two distinct reguli can have at most two transversal lines in common and that the intersection of $\overline{\mathcal{R}}$, $\overline{\mathcal{R}}^\sigma$, $\overline{\mathcal{R}}^{\sigma^2}$ and $\overline{\mathcal{R}}^{\sigma^3}$ is fixed by σ.

(C_{13}) $\mathcal{R}$, $\mathcal{R}^\sigma$, $\mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$ have two transversal lines in common, both fixed by σ. Then B_l has size $q^4 + q^3 + q^2 - q + 1$.

(C_{14}) $\mathcal{R}$, $\mathcal{R}^\sigma$, $\mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$ have two transversal lines in common, each one not fixed by σ. Then B_l has maximum size.

(C_{15}) $\mathcal{R}$, $\mathcal{R}^\sigma$, $\mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$ have a unique transversal line in common. Such transversal is fixed by σ, so B_l has size $q^4 + q^3 + q^2 + 1$.

(C_{16}) $\mathcal{R}$, $\mathcal{R}^\sigma$, $\mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$ have no transversal line in common. Then B_l has maximum size.

Since the blocking sets B_l obtained in Cases (C_{1i}), for $i = 1, \ldots, 6$, are not Baer subplanes, $r = S_3 \cap \pi$ is the unique Rédei line of B_l (Corollary 2.2).

(C_2) $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >= 4$
By Corollary 2.2, B_l is not of Rédei type. Also, l, l^σ, l^{σ^2} and l^{σ^3} are pairwise disjoint. Let $S_3 =< l, l^\sigma >$ and let $L = S_3 \cap S_3^\sigma \cap S_3^{\sigma^2} \cap S_3^{\sigma^3}$. Note that L is fixed by σ and, since $S_3^\sigma \neq S_3$, $\dim L \in \{0, 1, 2\}$. If $\dim L = 2$, then $S_3 \cap S_3^\sigma = S_3^{\sigma^2} \cap S_3^{\sigma^3}$. Hence l^σ and l^{σ^3} are contained in the plane $S_3 \cap S_3^\sigma$, a contradiction. So, $0 \leq \dim L \leq 1$. If l' is a line fixed by σ and incident with l, then l' is contained in L.

(C_{21}) Suppose that $\dim L = 1$. Then L is the unique line of Σ projected from l to a point of B_l. So $|B_l| = q^4 + q^3 + q^2 + 1$.

(C_{22}) Suppose that $\dim L = 0$. In this case there is no line of Σ projected from l to a point of B_l. Hence B_l has maximum size.

This completes the proof of Theorem 1.1. $\square$

4 Examples

In this section we show that all the cases discussed in the proof of Theorem 1.1, but Case (C_{16}), effectively occur.

Let σ be the semilinear collineation of $\Sigma^* = PG(4, q^4)$ defined by

$$\sigma\colon (x_0, x_1, x_2, x_3, x_4) \longmapsto (x_0^q, x_4^q, x_1^q, x_2^q, x_3^q).$$

Then the set $\Sigma = \{(\alpha, x, x^q, x^{q^2}, x^{q^3}) : \alpha \in GF(q), x \in GF(q^4)\}$ of fixed points of σ is a 4-dimensional canonical subgeometry of Σ^*. Let l be the line with equations

$$x_0 = 0, \quad x_1 = \beta x_3, \quad x_2 = a x_3 + b x_4,$$

where $\beta, a, b \in GF(q^4)$. The lines l, l^σ, l^{σ^2} and l^{σ^3} are contained in the 3-dimensional subspace with equation $x_0 = 0$. Hence, if $l \cap \Sigma = \emptyset$, projecting Σ from l to a plane π disjoint from l, we obtain a $GF(q)$-linear blocking set of Rédei type of π. By different choices of the coefficients β, a and b we get all the $GF(q)$-linear blocking sets of Rédei type listed in Theorem 1.1, but Case (C_{16}):

- If $\beta = 1$, $a = 0$, $b^{q^2+1} = 1$ and $b \neq 1$, then $l \cap \Sigma = \emptyset$, $l = l^{\sigma^2}$ and hence Case (A) occurs.
- If $\beta = 0$ and $b^{q^2+1} = 1$, then $l \cap \Sigma = \emptyset$ and $l \cap l^{\sigma^2} = \{P\}$ with $P = (0, 0, b, 0, 1)$. In this case, if $a = b = -1$, since $l \cap l^\sigma \neq \emptyset$, we get Case ($B_1$). If $a = 1$ and $b \neq -1$, since $l \cap l^\sigma = \emptyset$ and $P^\sigma \in < l, l^{\sigma^2} >$, we get Case ($B_{21}$). Finally, if $b = 1$ and $a \notin GF(q^2)$, since $l \cap l^\sigma = \emptyset$ and $P^\sigma \notin < l, l^{\sigma^2} >$, we get Case ($B_{22}$).
- If $\beta = a = b = 0$, then $l \cap \Sigma = \emptyset$, $l \cap l^{\sigma^2} = \emptyset$ and $l \cap l^\sigma \neq \emptyset$. So Case ($C_{11}$) occurs.
- If $\beta = a = 0$ and $b^{q^2+1} \neq 1$ with $b \neq 0$, then $l \cap \Sigma = \emptyset$, l, l^σ and l^{σ^2} are mutually disjoint and determine a regulus $\mathcal{R}$ of the quadric with equations

  $$x_0 = 0, \quad b^{q^2+q+1} x_1 x_2 - b^{q+1} x_1 x_4 - x_2 x_3 + b x_3 x_4 = 0.$$

  If $b^{q^3+q^2+q+1} = 1$, then $l^{\sigma^3} \in \mathcal{R}$ (Case (C_{12})). If $b^{q^3+q^2+q+1} \neq 1$ then $\mathcal{R}$, $\mathcal{R}^\sigma$, $\mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$ have in common the transversal lines with equations $x_0 = x_4 = x_2 = 0$ and $x_0 = x_3 = x_1 = 0$, each one not fixed by σ (Case (C_{14})).

- If $\beta = b = 0$ and $a \neq 0$, then $l \cap \Sigma = \emptyset$, l, l^σ and l^{σ^2} are mutually disjoint and determine a regulus $\mathcal{R}$ of the quadric with equations

$$x_0 = 0, \quad ax_3^2 - a^{q^2+q}x_1x_2 + a^q x_2x_4 - x_2x_3 - a^{q+1}x_3x_4 = 0.$$

  By direct calculations it is possible to prove that if either q is even or q is odd and $1 + 4a^{q^3+q^2+q+1}$ is a non square in $G(q)$, then the reguli $\mathcal{R}$, $\mathcal{R}^\sigma$, $\mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$ have two transversal lines in common, both fixed by σ (Case (C_{13})). Also, if q is odd and $a^{q^3+q^2+q+1} = -1/4$, then the reguli $\mathcal{R}$, $\mathcal{R}^\sigma$, $\mathcal{R}^{\sigma^2}$ and $\mathcal{R}^{\sigma^3}$ have a unique transversal line in common (Case (C_{15})).

In order to obtain the $GF(q)$-linear blocking sets not of Rédei type consider the following lines:

- Let l be the line of Σ^* with equations $x_0 = x_4,\ x_1 = x_3,\ x_2 = 0$. Since $l \cap \Sigma = \emptyset$, $l \cap l^{\sigma^2} = \{P\}$ with $P = (0,1,0,1,0)$, $l \cap l^\sigma = \emptyset$ and $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >= 4$, projecting Σ from l to a plane π disjoint from l, we obtain a $GF(q)$-linear blocking set of π as discussed in Case (B_3).
- Let l be the line of Σ^* with equations $x_0 = x_4,\ x_1 = 0,\ x_2 = 0$ and let $S_3 =< l, l^\sigma >$. Since $l \cap \Sigma = \emptyset$, $l \cap l^{\sigma^2} = \emptyset$, $\dim < l, l^\sigma, l^{\sigma^2}, l^{\sigma^3} >= 4$ and $\dim(S_3 \cap S_3^\sigma \cap S_3^{\sigma^2} \cap S_3^{\sigma^3}) = 0$, projecting Σ from l to a plane disjoint from l, we get Case (C_{22}).
- Finally, let l be the line with equations $x_1 = ax_0$, $x_1 = x_2$ and $x_2 = x_3$ with $a \notin GF(q^2)$ and let $S_3 =< l, l^\sigma >$. Since $l \cap l^{\sigma^2} = \emptyset$, $l \cap l^\sigma = \emptyset$ and $L = S_3 \cap S_3^\sigma \cap S_3^{\sigma^2} \cap S_3^{\sigma^3}$ is the line with equations $x_1 = x_2$, $x_2 = x_3$ and $x_3 = x_4$, Case (C_{21}) occurs.

We close this section by noting that different positions of the axis of the projection with respect to Σ and Σ' can produce non-equivalent linear blocking sets of the same type and of the same size. Indeed, projecting Σ to the plane π with equations $x_3 = x_4 = 0$ from the line $l_{a,b}$ with equations $x_0 = 0,\ x_1 = 0,\ x_2 = ax_3 + bx_4$, we get the following $GF(q)$-linear blocking set of π

$$B_{l_{a,b}} = \{(\alpha, x, x^q - ax^{q^2} - bx^{q^3}) \,:\, \alpha \in GF(q), x \in GF(q^4)\}.$$

As previously noted, if $a = b = 0$, then $B_{l_{0,0}}$ is a $GF(q)$-linear blocking set of Rédei type of maximum size of Case (C_{11}) and, if $a = 0$ and $b^{q^3+q^2+q+1} \neq 1$, $B_{l_{0,b}}$ is a $GF(q)$-linear blocking set of Rédei type of maximum size of Case (C_{14}). It is possible to prove (see [7]) that $B_{l_{0,0}}$ and $B_{l_{0,b}}$ with $b^{q^3+q^2+q+1} \neq 1$ are not isomorphic if $q > 3$.

References

[1] S. Ball, A. Blokhuis, A.E. Brouwer, L. Storme and T. Szőnyi, On the number of slopes of the graph of a function defined on a finite field. *J. Combin. Theory (A)*, **86** (1998), 187–196.

[2] A.A, Bruen, Baer subplanes and blocking sets. *Bull. Amer. Math. Soc.*, **76** (1970), 342–344.

[3] G. Lunardon, Normal spreads. *Geom. Dedicata*, **75** (1999), 245–261.

[4] G. Lunardon, Linear k-blocking sets. *Combinatorica*, to appear.

[5] G. Lunardon, P. Polito and O. Polverino, A geometric characterisation of k-linear blocking sets. *J. Geometry*, to appear.

[6] G. Lunardon and O. Polverino, Blocking sets of size $q^t + q^{t-1} + 1$. *J. Combin. Theory (A)*, **90** (2000), 148–158.

[7] G. Lunardon and O. Polverino, Blocking sets and derivable partial spreads. *J. Alg. Combin.*, to appear.

[8] G. Lunardon and O. Polverino, Linear Blocking Sets: a survey. *Finite Fields Appl.*, to appear.

[9] P. Polito and O. Polverino, On small blocking sets. *Combinatorica*, **18** (1998), 133–137.

[10] O. Polverino, Blocking set nei piani proiettivi, Ph.D. Thesis, University of Naples "Federico II", 1998.

[11] O. Polverino and L. Storme, Small minimal blocking sets in $PG(2, q^3)$, *European J. Combin.*, to appear.

[12] T. Szőnyi, Blocking sets in desarguesian affine and projective planes. *Finite Fields Appl.*, **3** (1997), 187–202

(Received 14/3/2001)