Further results on the existence of generalized Steiner triple systems with group size $g \equiv 1, 5 \pmod 6$

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Abstract

Generalized Steiner triple systems, $\mathrm{GS}(2, 3, n, g)$ are equivalent to maximum constant weight codes over an alphabet of size $g + 1$ with distance 3 and weight 3 in which each codeword has length n. The necessary conditions for the existence of a $\mathrm{GS}(2, 3, n, g)$ are $(n - 1)g \equiv 0 \pmod 2$, $n(n - 1)g^2 \equiv 0 \pmod 6$, and $n \geq g + 2$. Recently, we proved that for any given g, $g \equiv 1, 5 \pmod 6$ and $g \geq 11$, if there exists a $\mathrm{GS}(2, 3, n, g)$ for all n, $n \equiv 1, 3 \pmod 6$ and $g + 2 \leq n \leq 9g + 4$, then the necessary conditions are also sufficient. In this paper, the above result is improved and two new results are obtained. First, we show that for any given g, $g \equiv 1, 5 \pmod 6$ and $g \geq 17$, if there exists a $\mathrm{GS}(2, 3, n, g)$ for all n, $n \equiv 1, 3 \pmod 6$ and $g + 2 \leq n \leq 7g + 6$, then the necessary conditions are also sufficient. Second, we prove that the necessary conditions for the existence of a $\mathrm{GS}(2, 3, n, g)$ are also sufficient for $g = 13$.

1 Introduction

A $(g + 1)$-ary *constant weight code* (n, w, d) is a code $C \subseteq (Z_{g+1})^n$ of length n and minimum distance d, such that every $c \in C$ has Hamming weight w. To construct a constant weight code (n, w, d) with $w = 3$, a *group divisible design* (GDD) will be used. A K-GDD is an ordered triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ where $\mathcal{V}$ is a set of n elements, $\mathcal{G}$ is a collection of subsets of $\mathcal{V}$ called *groups* which partition $\mathcal{V}$, and $\mathcal{B}$ is a set of some subsets of $\mathcal{V}$ called *blocks*, such that each block intersects each group in

* Research supported in part by YNSFC Grant 10001026.

at most one element and that each pair of elements from distinct groups occurs together in exactly one block in $\mathcal{B}$, where $|B| \in K$ for any $B \in \mathcal{B}$. The group type is the multiset $\{|G| : G \in \mathcal{G}\}$. A k-GDD(g^n) denotes a K-GDD with n groups of size g and $K = \{k\}$. If all blocks of a GDD can be partitioned into parallel classes, then the GDD is called a *resolvable* GDD and denoted by RGDD, where a parallel class is a set of blocks partitioning the element set $\mathcal{V}$. In a 3-GDD(g^n), let $\mathcal{V} = (Z_{g+1} \setminus \{0\}) \times (Z_{n+1} \setminus \{0\})$ with n groups $G_i \in \mathcal{G}$, $G_i = (Z_{g+1} \setminus \{0\}) \times \{i\}$, $1 \leq i \leq n$ and blocks $\{(a, i), (b, j), (c, k)\} \in \mathcal{B}$. One can construct a constant weight code $(n, 3, d)$ as stated in [5], [8]. From each block we form a codeword of length n by putting an a, b and c in positions i, j and k respectively and zeros elsewhere. This gives a constant weight code over Z_{g+1} with minimum distance 2 or 3. If the minimum distance is 3, then the code is a $(g+1)$-ary *maximum constant weight code* (MCWC) $(n, 3, 3)$ and the 3-GDD(g^n) is called a *generalized Steiner triple system*, denoted by GS$(2, 3, n, g)$. It is easy to see that a 3-GDD(g^n) is a GS$(2, 3, n, g)$ if and only if any two intersecting blocks meet at most two common groups of the GDD. The following result is known.

Lemma 1.1 ([5], [8]) *The following are the necessary conditions for the existence of a GS$(2, 3, n, g)$:*
(1) $(n-1)g \equiv 0 \pmod 2$*;*
(2) $n(n-1)g^2 \equiv 0 \pmod 6$*;*
(3) $n \geq g+2$.

The necessary conditions are shown to be sufficient by several authors with one exception for $2 \leq g \leq 11$ ([5], [8], [9], [3], [4], [11], [6]). Hence, we have the following lemma.

Lemma 1.2 *The necessary conditions for the existence of a GS(2, 3, n, g) are also sufficient for* $2 \leq g \leq 11$ *with one exception of* $(g, n) = (2, 6)$.

Blake-Wilson and Phelps [2] proved that the necessary conditions for the existence of a GS$(2, 3, n, g)$ are also asymptotically sufficient for any g. Recently, in [6] we proved that for any given g, $g \equiv 1, 5 \pmod 6$ and $g \geq 11$, if there exists a GS$(2, 3, n, g)$ for all n, $n \equiv 1, 3 \pmod 6$ and $g + 2 \leq n \leq 9g + 4$, then the necessary conditions are also sufficient.

Since the existence of a GS$(2, 3, n, g)$ has been solved for $g \leq 11$, we need only consider $g \geq 13$ for the case $g \equiv 1, 5 \pmod 6$. Let $T_g = \{n$: there exists a GS$(2, 3, n, g)\}$, $B_g = \{n$: n satisfying the necessary conditions listed in Lemma 1.1$\}$, $M_g = \{n$: $n \in B_g, n \leq 7g + 6\}$. In this paper, the results of [6] will be improved for $g \geq 17$ and the following results are obtained.

Theorem 1.3 *For any* $g \equiv 1, 5 \pmod 6$ *and* $g \geq 17$*, if* $M_g \subset T_g$*, then* $B_g = T_g$*. That is the necessary conditions for the existence of a GS$(2, 3, n, g)$ are also sufficient.*

Theorem 1.4 $B_{13} = T_{13}$*, that is the necessary conditions for the existence of a GS$(2, 3, n, g)$ are also sufficient for* $g = 13$.

2 Constructions

In this paper, we will use two types of constructions. One is product constructions, the other is direct construction. To show product constructions, we need the concept of both *holey generalized Steiner triple systems* and *disjoint incomplete Latin squares*.

A *holey group divisible design*, K–HGDD, is a four-tuple $(\mathcal{V}, \mathcal{G}, \mathcal{H}, \mathcal{B})$, where $\mathcal{V}$ is a set of points, $\mathcal{G}$ is a partition of $\mathcal{V}$ into subsets called *groups*, $\mathcal{H} \subset \mathcal{G}$, $\mathcal{B}$ is a set of *blocks* such that a group and a block contain at most one common point and every pair of points from distinct groups, not both in $\mathcal{H}$, occurs in a unique block in $\mathcal{B}$, where $|B| \in K$ for any $B \in \mathcal{B}$. A k-HGDD$(g^{(n,u)})$ denotes a K–HGDD with n groups of size g in $\mathcal{G}$, u groups in $\mathcal{H}$ and $K = \{k\}$. A *holey generalized Steiner triple system*, HGS$(2,3,(n,u),g)$, is a 3-HGDD$(g^{(n,u)})$ with the property that any two intersecting blocks meet at most two common groups.

It is easy to see that if $u = 0$ or $u = 1$, then a HGS$(2,3,(n+u,u),g)$ is just a GS$(2,3,n,g)$ or a GS$(2,3,n+1,g)$ respectively.

A *Latin square* of side n, LS(n), is an $n \times n$ array based on some set S of n symbols with the property that every row and every column contains every symbol exactly once. An *incomplete Latin square*, ILS$(n+a,a)$, denotes a LS$(n+a)$ "missing" a sub LS(a). Without loss of generality, we may assume that the missing subsquare, or *hole*, is at the lower right corner. We say $(i,j,s) \in$ ILS$(n+a,a)$ if the entry in the cell (i,j) is s. Let A_1, A_2 be two ILS$(n+a,a)$ on the same symbol set. If $(i,j,s_1) \neq (i,j,s_2)$ for any $(i,j,s_1) \in A_1$, $(i,j,s_2) \in A_2$, then we say that A_1 and A_2 are *disjoint*. We use r DILS$(n+a,a)$ to denote r pairwise disjoint ILS$(n+a,a)$.

For the existence of r DILS$(n+a,a)$, we have the following two lemmas.

Lemma 2.1 ([3]) *There exist $\delta(a)$ DILS$(n+a,a)$, where $\delta(0) = n$ and $\delta(a) = a$ for $1 \leq a \leq n$.*

Lemma 2.2 ([7, 10, 11]) *There exist n DILS$(n+a,a)$ for any positive integer n and for any integer a, $0 \leq a \leq n$ except for $(n,a) = (2,1)$, $(6,5)$.*

The following singular indirect product construction for GS$(2,3,n,g)$ is first stated in [3].

Lemma 2.3 *(Singular Indirect Product (SIP)) Let m, n, t, u and a be integers such that $0 \leq a \leq u < n$. Suppose the following designs exist:*
(1) t DILS$(n+a,a)$;
(2) a 3–GDD(g^m) with the property that all blocks of the design can be partitioned into t sets $S_0, S_1, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3;
(3) a HGS$(2,3,(n+u,u),g)$.
Then there exists a HGS$(2,3,(c,d),g)$, where $c = m(n+a)+u-a, d = ma+u-a$.
Further, if there exists
(4) a GS$(2,3,ma+u-a,g)$,
then there exists a GS$(2,3,m(n+a)+u-a,g)$.

Taking $a = 0$ in Lemma 2.3, we get the singular direct product construction, which first appeared in [9].

Lemma 2.4 *(Singular Direct Product (SDP)) Let m, n, t, and u be integers such that $0 \leq u < n$. Suppose $t \leq n$ and the following designs exist:*
(1) a 3–GDD(g^m) with the propery that all blocks of the design can be partitioned into t sets $S_0, S_1, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3;
(2) a HGS$(2, 3, (n+u, u), g)$.
Then there exists a HGS$(2, 3, (mn+u, u), g)$. Further, if there exists a GS$(2, 3, u, g)$, then there exists a GS$(2, 3, mn+u, g)$.

Taking $u = 0$ or 1 in Lemma 2.4, we get the Construction C or D of Etzion in [5] respectively.

Lemma 2.5 *(Direct Product (DP)) Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a 3-GDD(g^m), and suppose there exists a GS$(2, 3, n, g)$. Then there exists a GS$(2, 3, mn, g)$ if $\mathcal{B}$ can be partitioned into t sets $S_0, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3 and $t \leq n$.*

Lemma 2.6 ([5]) *Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a 3-GDD(g^m), and suppose there exists a GS$(2, 3, n, g)$. Then there exists a GS$(2, 3, m(n-1)+1, g)$ if $\mathcal{B}$ can be partitioned into t sets $S_0, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3 and $t \leq n-1$.*

It is easy to notice that the derived generalized Steiner triple system in Lemma 2.5 and Lemma 2.6 has a sub GS$(2, 3, n, g)$. Hence, we have the following.

Lemma 2.7 *Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a 3-GDD(g^m). Suppose there exists a GS$(2, 3, n, g)$. Then there exists a HGS$(2, 3, (mn, n), g)$ or a HGS$(2, 3, (m(n-1)+1, n), g)$ if $\mathcal{B}$ can be partitioned into t sets $S_0, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3 and $t \leq n$ or $t \leq n-1$ respectively.*

If one uses a 3-RGDD(g^m) in the constructions, then each parallel class becomes an S_r and there are $t = \frac{g(m-1)}{2}$ such classes. The following is stated in [3].

Lemma 2.8 *If there exists a GS$(2, 3, n, g)$ and a 3-RGDD(g^m) with $t = \frac{g(m-1)}{2} \leq n$ or $n-1$, then there exists a GS$(2, 3, mn, g)$ or a GS$(2, 3, m(n-1)+1, g)$ respectively.*

For the existence of a 3-RGDD(g^m), we have the following.

Lemma 2.9 ([1]) *A 3-RGDD(g^m) exists if and only if $(m-1)g \equiv 0 \pmod 2, mg \equiv 0 \pmod 3$ and $g^m \neq 2^3, 2^6$ and 6^3.*

Combining Lemmas 2.7-2.9, we have the following.

Lemma 2.10 *For any $g \geq 13$, if there exists a GS$(2, 3, n, g)$, then there exists a GS$(2, 3, 3n, g)$ and a GS$(2, 3, 3(n-1)+1, g)$. Consequently, there exists a HGS$(2, 3, (3n, n), g)$ and a HGS$(2, 3, (3(n-1)+1, n), g)$.*

3 Proof of Theorem 1.3

In this section, we will show the proof of Theorem 1.3. First, we need the following lemmas.

Lemma 3.1 *For $g \equiv 1 \pmod 6$ and $g \geq 19$, suppose $v = 42p + 6j + 12 + k$, where $0 \leq j \leq 6$ and $k = 3$ or 7. If $6p+3 \in T_g$, $6p+6j+k \in T_g$, and $p \geq \lceil \frac{j}{2} \rceil$, then $v \in T_g$.*

Proof. Apply Lemma 2.3 with $m = 3, n = 12p+4, t = g, u = 6p+3$ and $a = 3j+\frac{k-3}{2}$. Since $p \geq \lceil \frac{j}{2} \rceil$, it is easy to check that $0 \leq a \leq u < n$. From Lemma 2.2, there exist n DILS$(n+a, a)$ for $0 \leq a \leq n$. We further have t DILS$(n+a, a)$ since $t \leq u-2 < n$. Thus the condition (1) of Lemma 2.3 is satisfied. For $g \geq 19$, a 3-RGDD(g^3) always exists by Lemma 2.9, which has g parallel classes. So, condition (2) is also satisfied. From $u \in T_g$, we apply Lemma 2.10 to obtain a HGS$(2, 3, (n + u, u), g)$, providing the design in condition (3). Finally, we have $ma + u - a = 6p + 6j + k \in T_g$, the condition (4) is satisfied. Therefore, we have $v \in T_g$. This completes the proof. ◻

Lemma 3.2 *For $g \equiv 5 \pmod 6$ and $g \geq 17$, suppose $v = 42p + 6j + k$, where $0 \leq j \leq 6$ and $k = 1$ or 3. If $6p+1 \in T_g$, $6p+6j+k \in T_g$, and $p \geq \lceil \frac{j}{2} \rceil$, then $v \in T_g$.*

Proof. Apply Lemma 2.3 with $m = 3, n = 12p, t = g, u = 6p + 1$ and $a = 3j + \frac{k-1}{2}$. Then the proof is completed analogously to that of Lemma 3.1. ◻

Now, we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. We need to show that $M_g \subset T_g$ implies that $B_g \subset T_g$. The proof is by induction on n. Suppose $n \in B_g$. If $n \in M_g$, then $n \in T_g$. Otherwise, we have $n \geq 7g + 8$ and distinguish between the following cases:

Case 1: $g \equiv 1 \pmod 6$ and $g \geq 19$. Write $n = 42p + 6j + 12 + k \geq 7g + 8$, where $0 \leq j \leq 6$ and $k = 3$ or 7. We first claim that $p \geq \lceil \frac{j}{2} \rceil$. If not, then $p \leq \lceil \frac{j}{2} \rceil - 1 \leq 2$. Thus $n \leq 84 + 6j + 12 + k$. Since $g \geq 19$, $0 \leq j \leq 6$ and $k = 3$ or 7, we have $n \leq 139 < 141 \leq 7g + 8$, a contradiction.

Next, it is noticed that $7g + 8 \equiv 42p + 15 \pmod{42}$ and it is easy to see that $n \geq 7g + 8$ implies $n = 42p + 15 + 6j + k - 3 \geq 7g + 8 + 6j + k - 3$. Then, it is easily checked that $\alpha = 6p + 3 \geq g + 2$ and $\beta = 6p + 6j + k \geq g + 2$. Since $\beta \equiv 1$ or $3 \pmod 6$, we see that $\alpha \in B_g$ and $\beta \in B_g$. If we have both $\alpha \in M_g$ and $\beta \in M_g$, then Lemma 3.1 guarantees that $n \in T_g$ and the proof is completed. If at least one of α and β is not in M_g, then we can repeat the induction process taking the number α, β not in M_g as n'.

Case 2: $g \equiv 5 \pmod 6$ and $g \geq 17$. Write $n = 42p + 6j + k \geq 7g + 8$, where $0 \leq j \leq 6$ and $k = 1$ or 3. Apply Lemma 3.2; the proof of this case is similar to that of Case 1. We need only to check that $p \geq \lceil \frac{j}{2} \rceil$ and $6p + 1 \geq g + 2$. We first claim that $p \geq \lceil \frac{j}{2} \rceil$. If not, then $p \leq \lceil \frac{j}{2} \rceil - 1 \leq 2$. Thus $n < 84 + 6j + k$. Since $g \geq 17$, $0 \leq j \leq 6$ and $k = 1$ or 3, we have $n \leq 123 < 127 \leq 7g + 8$, a contradiction.

Next, it is noticed that $7g + 8 \equiv 42p + 1 \pmod{42}$ and it is easy to see that $n \geq 7g+8$ implies $n = 42p+6j+k \geq 7g+7+6j+k$. Hence, we have $6p+1 \geq g+2$.

After certain steps of induction on n, n' will be small enough so that n' is in M_g, consequently, $n \in T_g$. This completes the proof. ◻

4 Proof of Theorem 1.4

In this section, we will show that the necessary conditions for the existence of a GS$(2,3,n,13)$ are also sufficient. From Theorem 1.3 of [6], we need only to consider the case $n \in E = \{n : n \equiv 1, 3 \pmod 6 \text{ and } 15 \le n \le 121\}$.

For $n \equiv 3 \pmod 6$, to construct a GS$(2,3,n,13)$ in Z_{13n}, it suffices to find a set of generalized base blocks, $\mathcal{A} = \{B_1, \cdots, B_s\}$, $s = \frac{13(n-1)}{2}$, such that $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ forms a GS$(2,3,n,13)$, where $\mathcal{V} = Z_{13n}$, $G = \{G_1, G_2, \cdots, G_n\}, G_i = \{i + nj : 0 \le j \le 12\}, 1 \le i \le n$, and $\mathcal{B} = \{B + 3j : B \in \mathcal{A}, 0 \le j \le \frac{13n}{3} - 1\}$. For convenience, we write $\mathcal{A} = \bigcup_{i=1}^{3} \{\{i, x, y\} : \{x, y\} \in S_i\}$. So, for each $\mathcal{A}$ we need only display the corresponding $S_i, 1 \le i \le 3$.

Lemma 4.1 *There exists a GS*$(2,3,n,13)$ *for all* $n \in F_1$*, where* $F_1 = \{15, 21, 27, 33, 39, 51, 69, 87\}$.

Proof. For the values $n \in F_1$, with the aid of a computer, we have found a set of generalized base blocks of a GS$(2,3,n,13)$. Here, we only list the $S_i, 1 \le i \le 3$ for $n = 15$. For the remaining values n, the corresponding $S_i, 1 \le i \le 3$ are listed in the Appendix. (In order to save space, we omit the Appendix; the interested reader may contact the author for a copy.)

$n = 15$, $\mathcal{A} = \bigcup_{i=1}^{3} \{\{i, x, y\} : \{x, y\} \in S_i\}$,

$S_1 = \{\{147, 191\}, \{41, 83\}, \{23, 87\}, \{74, 129\}, \{34, 35\}, \{75, 188\}, \{148, 187\}, \{43, 70\}, \{72, 95\}, \{36, 64\}, \{8, 125\}, \{146, 178\}, \{90, 101\}, \{173, 182\}, \{120, 177\}, \{39, 116\}, \{17, 57\}, \{54, 96\}, \{132, 193\}, \{18, 69\}, \{11, 119\}, \{26, 186\}, \{30, 142\}, \{20, 22\}, \{14, 103\}, \{104, 143\}, \{100, 128\}, \{109, 176\}, \{53, 184\}, \{162, 189\}, \{153, 155\}, \{56, 80\}, \{71, 138\}, \{12, 67\}\}$;

$S_2 = \{\{50, 76\}, \{48, 185\}, \{3, 19\}, \{105, 159\}, \{33, 126\}, \{144, 166\}, \{5, 141\}, \{6, 154\}, \{81, 106\}, \{111, 131\}, \{132, 150\}, \{31, 179\}, \{29, 101\}, \{58, 146\}, \{61, 63\}, \{147, 193\}, \{100, 180\}, \{40, 46\}, \{112, 148\}, \{37, 121\}, \{99, 113\}, \{64, 114\}, \{67, 128\}, \{143, 176\}, \{21, 27\}, \{88, 96\}, \{15, 151\}\}$;

$S_3 = \{\{29, 110\}, \{97, 162\}, \{107, 113\}, \{102, 182\}, \{66, 136\}, \{12, 193\}, \{85, 157\}, \{120, 188\}, \{8, 65\}, \{7, 129\}, \{82, 106\}, \{165, 166\}, \{100, 178\}, \{44, 176\}, \{77, 88\}, \{27, 146\}, \{91, 128\}, \{40, 121\}, \{51, 132\}, \{11, 170\}, \{15, 161\}, \{24, 55\}, \{16, 154\}, \{98, 191\}, \{42, 164\}, \{35, 172\}, \{92, 175\}, \{32, 75\}, \{87, 90\}, \{43, 94\}\}$. □

For $n \equiv 1 \pmod 6$, to construct a GS$(2,3,n,13)$ in Z_{13n}, it suffices to find a set of base blocks, $\mathcal{A} = \{B_1, \cdots, B_s\}$, $s = 13(n-1)/6$, such that $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ forms a GS$(2,3,n,13)$, where $\mathcal{V} = Z_{13n}$, $G = \{G_0, G_1, \cdots, G_{n-1}\}, G_i = \{i + nj : 0 \le j \le 12\}, 0 \le i \le n-1$, and $\mathcal{B} = \{B + j : B \in \mathcal{A}, 0 \le j \le 13n - 1\}$. For convenience, we write $\mathcal{A} = \bigcup \{\{0, x, y\} : \{x, y\} \in S\}$. So, for each $\mathcal{A}$ we need only to display the corresponding S.

If the set of base blocks has some structure, say some base block is a multiple of the other, then we may present the set in a shortened way. Suppose we have two sets of blocks P_0 and R, a suitable multiplier $m \in Z_{13n} \setminus \{0, n, \cdots, 12n\}$ and a suitable integer t, such that $\mathcal{A} = P \cup R$ forms a set of base blocks of a GS$(2, 3, n, 13)$, where $P = \bigcup_{i=0}^{t-1} P_i$ and $P_i = m^i P_0$, $1 \le i \le t-1$. We call such a set P_0 *partial set* and R *remainder*.

Lemma 4.2 *There exists a GS*$(2, 3, n, 13)$ *for all* $n \in F_2 = \{19, 25, 31, 37, 49, 85\}$.

Proof. With the aid of a computer, we have found a set of base blocks $\mathcal{A}$ of a GS$(2, 3, n, 13)$ for all $n \in F_2$. As mentioned above, for a GS$(2, 3, n, 13)$, we need only to list the corresponding multiplier m, integer t, partial set P_0 and remainder R as follows.

$n = 19$, $m = 2$, $t = 9$, $P_0 = \{\{1, 6\}, \{7, 25\}\}$, $R = \{\{43, 136\}, \{88, 226\}, \{104, 203\}, \{181, 210\}, \{106, 195\}, \{121, 179\}, \{178, 212\}, \{33, 123\}, \{79, 196\}, \{74,149\}, \{17,186\}, \{15, 176\}, \{77, 116\}, \{31, 194\}, \{122, 187\}, \{62, 177\}, \{49, 205\}, \{13, 120\}, \{150, 217\}, \{26, 134\}, \{3, 105\}\}$.

$n = 25$, $m = 7$, $t = 4$, $P_0 = \{\{1, 3\}, \{4, 9\}, \{6, 16\}, \{8, 19\}, \{12, 27\}, \{22, 51\}, \{24, 58\}, \{30, 148\}\}$, $R = \{\{53, 185\}, \{13, 86\}, \{59, 211\}, \{47, 231\}, \{45, 142\}, \{229, 305\}, \{39, 82\}, \{167, 259\}, \{79, 273\}, \{106, 232\}, \{89, 226\}, \{40, 230\}, \{143, 208\}, \{159, 247\}, \{138, 251\}, \{123, 169\}, \{119, 265\}, \{26, 130\}, \{172, 292\}, \{48, 234\}\}$.

$n = 31$, $m = 5$, $t = 3$, $P_0 = \{\{4, 12\}, \{6, 13\}, \{9, 26\}, \{11, 29\}, \{14, 38\}, \{19, 42\}, \{21, 49\}, \{27, 61\}, \{36, 88\}, \{39, 108\}, \{41, 92\}, \{43, 110\}, \{46,102\}, \{54,196\}, \{74,199\}, \{1,117\}, \{80, 267\}\}$, $R = \{\{32, 146\}, \{157, 232\}, \{194, 321\}, \{10, 86\}, \{153, 254\}, \{151, 330\}, \{2, 166\}, \{168, 265\}, \{89, 249\}, \{50, 291\}, \{111, 220\}, \{163, 355\}, \{83, 202\}, \{84, 251\}\}$.

$n = 37$, $m = 2$, $t = 18$, $P_0 = \{\{1, 6\}, \{7, 18\}, \{13, 34\}\}$, $R = \{\{50, 196\}, \{268, 432\}, \{329, 367\}, \{344, 371\}, \{19, 157\}, \{100, 428\}, \{76, 127\}, \{98, 392\}, \{102, 175\}, \{69,200\}, \{204, 424\}, \{43, 319\}, \{373, 414\}, \{103, 174\}, \{125, 231\}, \{219, 274\}, \{197, 374\}, \{216, 303\}, \{54, 266\}, \{189, 275\}, \{133, 267\}, \{253, 400\}, \{172, 254\}, \{25, 167\}\}$.

$n = 49$, $m = 8$, $t = 7$, $P_0 = \{\{1, 3\}, \{4, 9\}, \{6, 29\}, \{11, 26\}, \{13, 31\}, \{22, 69\}, \{27, 130\}, \{33, 87\}, \{45, 174\}\}$, $R = \{\{258, 309\}, \{199, 427\}, \{486, 528\}, \{272, 399\}, \{276, 346\}, \{80, 252\}, \{139, 398\}, \{179, 623\}, \{306, 469\}, \{58, 476\}, \{154, 571\}, \{146, 167\}, \{408, 556\}, \{182, 425\}, \{162, 477\}, \{84, 546\}, \{35,266\}, \{173, 270\}, \{426,497\}, \{30,284\}, \{126, 413\}, \{352, 414\}, \{329, 532\}, \{102, 627\}, \{340, 423\}, \{372, 531\}, \{119, 354\}, \{400, 558\}, \{290, 609\}, \{471, 547\}, \{348, 484\}, \{116, 249\}, \{357, 630\}, \{341, 574\}, \{44, 100\}, \{401, 587\}, \{428, 617\}, \{369, 560\}, \{110, 496\}, \{215, 336\}, \{180, 397\}\}$.

$n = 85$, $m = 12$, $t = 16$, $P_0 = \{\{1, 3\}, \{4, 10\}, \{5, 13\}, \{7, 21\}, \{11, 26\}, \{19, 44\}, \{20, 49\}, \{22, 57\}\}$, $R = \{\{714, 867\}, \{260, 402\}, \{381, 483\}, \{792, 1040\}, \{544, 883\}, \{531, 873\}, \{717, 1049\}, \{408, 951\}, \{119, 1028\}, \{352, 439\}, \{257, 651\}, \{106, 1054\}, \{357, 891\}, \{472, 908\}, \{139, 202\}, \{66, 374\}, \{743, 947\}, \{167, 899\}, \{17, 366\}, \{43, 221\}, \{53, 649\}, \{383, 776\}, \{262, 847\}, \{231, 390\}, \{38, 809\}, \{442, 516\}, \{389, 796\}, \{98, 412\}, \{171, 358\}, \{664, 816\}, \{181, 833\}, \{386, 522\}, \{263, 701\}, \{779, 868\}, \{477, 672\}, \{71, 542\}, \{302, 428\}, \{79, 687\}, \{599, 1068\}, \{1044, 1077\}, \{836, 1037\}, \{253, 524\}, \{519, 646\}, \{306, 782\}, \{76, 888\}, \{193, 686\}, \{396, 527\}, \{769, 981\}, \{316, 641\}, \{34, 233\}, \{114, 558\}, \{508, 975\}, \{826, 929\}, \{219, 455\}\}$. □

The following lemma is a combination of Theorem 2 and Lemma 7 in [2]. Here, we need a new concept. A *maximum packing with triangles*, MPT(n), is an ordered triple $(\mathcal{P}, \mathcal{T}, \mathcal{L})$, where $\mathcal{P}$ is the vertex set of K_n, $\mathcal{T}$ is a collection of edge disjoint triangles from the edge set of K_n with $|\mathcal{T}|$ as large as possible, and $\mathcal{L}$ is the collection of edges in K_n not belonging to any of the triangles of $\mathcal{T}$. The collection of edges $\mathcal{L}$ is called the *leave*.

Lemma 4.3 *There exists a GS*$(2,3,n,13)$ *for any prime power* $n \equiv 1$ $(mod\ 6)$ *and* $n \geq 61$.

Proof. Apply Theorem 2 and Lemma 7 in [2]; it suffices to show that there exists a MPT(13) $= (\mathcal{P}, \mathcal{T}, \mathcal{L})$ with r partial parallel classes such that $r \leq 10$, which is listed below.

$\mathcal{P} = \{1, 2, \cdots, 13\}, \mathcal{T} = \bigcup_{i=1}^{9} P_i$, $\mathcal{L} = \emptyset$ is an empty set.
$P_1 = \{\{1,2,5\},\{3,4,7\},\{8,9,12\}\}$; $P_2 = \{\{1,3,8\},\{2,4,9\},\{5,7,12\}\}$;
$P_3 = \{\{2,3,6\},\{4,5,8\},\{9,10,13\}\}$; $P_4 = \{\{3,5,10\},\{4,6,11\},\{7,9,1\}\}$;
$P_5 = \{\{5,6,9\},\{7,8,11\},\{12,13,3\}\}$; $P_6 = \{\{6,8,13\},\{9,11,3\},\{10,12,4\}\}$;
$P_7 = \{\{6,7,10\},\{11,12,2\},\{13,1,4\}\}$; $P_8 = \{\{8,10,2\},\{11,13,5\},\{12,1,6\}\}$;
$P_9 = \{\{10,11,1\},\{13,2,7\}\}$. ▯

Lemma 4.4 *There exists a GS*$(2,3,v,13)$ *for all* $v \in F_3 = \{v : 43 \leq v \leq 117$ *and* $v \equiv 1,3,7,9$ $(mod\ 18)\}$.

Proof. From Lemmas 4.1 and 4.2, we have a GS$(2,3,n,13)$ for all $n \in G = \{n : 15 \leq n \leq 39$ and $n \equiv 1,3 \pmod 6\}$. Apply Lemma 2.10 with $n \in G$; we get a GS$(2,3,v,13)$ for all $v \in F_3$. This completes the proof. ▯

Lemma 4.5 *There exists a GS*$(2,3,v,13)$ *for* $v \in F_4 = \{105\}$.

Proof. From Lemma 4.1, we have a GS$(2,3,15,13)$. From Lemma 5 of [2], we have a 3–GDD(13^7) with the property that all blocks of the design can be partitioned into t sets $S_0, S_1, \cdots, S_{t-1}$ such that $t \leq 13$ and the minimum distance in S_r, $0 \leq r \leq t-1$, is 3. Apply Lemma 2.5 with $m = 7$ and $n = 15$; we get a GS$(2,3,105,13)$. ▯

Now, we are in a position to prove Theorem 1.4.

Proof of Theorem 1.4: From Theorem 1.3 of [6], we need only to consider the values v, such that $v \in E$. Lemma 4.3 provides a GS$(2,3,v,13)$ for all $v \in F_5 = \{67, 103, 121\}$. It is readily checked that the union of F_i, for $1 \leq i \leq 5$, is the same as E. The conclusion then follows from the above lemmas of this section. ▯

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(Received 23 May 2000)