Classes of line graphs with small cycle double covers*

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Abstract

The *Small Cycle Double Cover Conjecture*, due to J.A. Bondy, states that every simple bridgeless graph on n vertices has a cycle double cover with at most $(n-1)$ cycles. There are a number of classes of graphs for which this conjecture is known to hold; for example, triangulations of surfaces, complete graphs, and 4-connected planar graphs. In this article, we prove that the conjecture holds for line graphs of a number of types of graphs; specifically line graphs of complete graphs, line graphs of complete bipartite graphs, and line graphs of planar graphs.

1 Introduction

A *cycle double cover* (CDC) of a graph G is a collection of cycles $\mathcal{C}$ such that every edge of G lies in precisely two cycles of $\mathcal{C}$. An obvious necessary condition for the existence of a CDC of a graph G is that the graph be bridgeless. It has been conjectured (see [14], [15]) that this condition is also sufficient. The so-called *Cycle Double Cover Conjecture* has been studied by numerous authors (see [1], [7]). In this article, we focus on a strengthening of this conjecture that involves the number of

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cycles in a CDC. A *small cycle double cover* (SCDC) of a graph G on n vertices is a CDC with at most $n-1$ cycles. Bondy [2] proposes the following.

SCDC Conjecture: Every simple bridgeless graph has a small cycle double cover.

Various classes of graphs are known to have SCDCs, including complete graphs, complete bipartite graphs, simple triangulations of surfaces, and 4-connected planar graphs (see [2], [12], [13]). Because of their structure, line graphs are a natural class of graphs to study with respect to the SCDC Conjecture. It has been shown ([11], pp. 22–26) that the line graph of any 3-connected planar graph has an SCDC. In what follows, we extend this result to all planar graphs whose line graphs are bridgeless. We also prove that the SCDC Conjecture holds for line graphs of complete graphs and line graphs of complete bipartite graphs.

Similar and further results have been independently obtained by R. Klimmek in her Ph.D. thesis [9]. The thesis contains a proof that the line graph of any 2-connected planar graph has an SCDC, and it is remarked that the result can be extended to planar graphs whose line graphs are bridgeless. By using fairly complex results about cycle covers and decompositions, Klimmek also proves that the line graph of any graph with no vertices of degree two has an SCDC. In contrast, we provide more straightforward constructions of SCDCs of line graphs of complete graphs and complete bipartite graphs.

Unless otherwise specified, we use the terminology of [4]. Graphs are *undirected* and *simple*; a multigraph is undirected and may have more than one edge between the same pair of vertices. A *bridge* in a graph is a 1-edge cut in the graph, and a *nontrivial* bridge is a bridge whose removal leaves two components, each having more than one vertex. A bridge that is *not* nontrivial is a *pendant* in the graph; i.e., a pendant is incident to a vertex of degree one.

A *cycle decomposition* of a graph G is a partition of $E(G)$ into cycles. (It is well-known that a connected graph has a cycle decomposition if and only if the graph is eulerian.)

A 2-*bridge* in a graph G is a cut vertex of degree two. Any graph not containing a 2-bridge is said to be 2-*bridge-free*. Since a 2-bridge in a graph G corresponds to a bridge in the line graph $L(G)$, an obvious necessary condition for the existence of a CDC of $L(G)$ is that G be 2-bridge-free. That this condition is also sufficient would imply the truth of the cycle double cover conjecture (see Section 2).

Certain path covers of graphs are instrumental in proving our results. A *perfect path double cover* (PPDC) of a graph H is a collection of paths, $\mathcal{P}$, such that every edge of H lies in two paths of $\mathcal{P}$, and every vertex of H occurs precisely twice as an endpoint of paths in $\mathcal{P}$. It was conjectured by Bondy [3] and proved by Li [10] that every simple graph has a PPDC. For a PPDC $\mathcal{P}$ of a graph H, we define a multigraph $M_{\mathcal{P}}(H)$, called the *associated multigraph* of $\mathcal{P}$; $M_{\mathcal{P}}(H)$ has the same vertex set as H, and for all vertices $x, y \in V(M_{\mathcal{P}}(H))$, xy is an edge of $M_{\mathcal{P}}(H)$ if and only if $\mathcal{P}$ contains a path with endpoints x and y. The PPDC $\mathcal{P}$ is called an *eulerian perfect path double cover* (EPPDC) if and only if the associated multigraph $M_{\mathcal{P}}(H)$ is a cycle. Notice that in general, $M_{\mathcal{P}}(H)$ is a 2-regular multigraph.

Let G be a graph, and let $\mathcal{C}$ be a collection of cycles of G. If $c \in \mathcal{C}$, then we denote by $ch(c)$ the *number* of chords of c in G, and by $ch(\mathcal{C})$ the number of chords of *all* the cycles of $\mathcal{C}$.

2 Preliminary Results

Let G be a connected 2-bridge-free graph, and let $L(G)$ be its line graph. Each vertex $x \in V(G)$ corresponds to a clique $K(x)$ in $L(G)$ that we call the *vertex clique for* x, and $K(x)$ is called a vertex clique of $L(G)$. The vertex cliques $\{K(x) : x \in V(G)\}$ partition the edges of $L(G)$.

For $n \geq 3$, K_n has a CDC (Lemma 1, below) and thus it seems that we can construct a CDC of a line graph by simply taking the union of CDC's of all the vertex cliques. The problem with this, of course, is that a vertex of degree two in a graph produces K_2 as a vertex clique in the line graph, and K_2 has no CDC. In fact, proving that the line graph of any 2-bridge-free graph has a CDC is as hard as proving the CDC Conjecture. To see this, let G be a bridgeless graph, and let H denote the graph obtained from G by subdividing each edge once. Clearly, H is 2-bridge-free. Now suppose that $L(H)$ has a CDC, $\mathcal{C}$, and let $c \in \mathcal{C}$, $c = v_0v_1 \ldots v_kv_0$. For $0 \leq i \leq k$, we can write $v_i = (x_iy_i)$ where $x_iy_i \in E(H)$, x_i is a vertex of G, and y_i is the vertex added to subdivide the edge x_ix_{i+1} of G. Then $c' = x_0x_1 \ldots x_kx_0$ is a cycle in G corresponding to the cycle c in $L(H)$, and we see that

$$\mathcal{C}' = \{x_0x_1 \ldots x_kx_0 \mid (x_0y_0)(x_1y_1) \ldots (x_ky_k)(x_0y_0) \in \mathcal{C}\}$$

is a CDC of G. Therefore, proving that 2-bridge-free line graphs have CDC's is at least as hard as proving the CDC Conjecture. Furthermore, Cai and Corneil [5] have shown that if a graph has a CDC, then its line graph also has a CDC. These two results together imply that a proof that line graphs of 2-bridge-free graphs have CDC's is equivalent to proving the CDC Conjecture.

Our interest centers around finding SCDCs. The simplest examples of line graphs with SCDCs arise by taking line graphs of 2-bridge-free trees. The proof of this relies on the following fact (see [2], [11]).

Lemma 1 *Every complete graph K_n on $n \geq 3$ vertices has a cycle double cover with $(n-1)$ Hamilton cycles.*

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Theorem 2 *If T is a 2-bridge-free tree, then $L(T)$ has a small cycle double cover.*

Proof: Let T be a 2-bridge-free tree on n vertices (i.e., T is a tree with no vertices of degree two). Then T has $(n-1)$ edges, so $L(T)$ is a bridgeless graph on $(n-1)$ vertices. Every vertex $x \in V(T)$ gives rise to a clique $K(x)$ in $L(T)$; the clique $K(x)$ has $d(x)$ vertices, and, when $d(x) \geq 3$, $K(x)$ has an SCDC with $(d(x)-1)$ cycles.

The union of the SCDCs of the vertex cliques of vertices of degree greater than one gives us a CDC, $\mathbf{C}$, of $L(T)$. The number of cycles in $\mathbf{C}$ is

$$|\mathbf{C}| = \sum_{x \in V(T),\ d(x) \geq 3} (d(x) - 1).$$

Observe that the sum does not change by taking the sum over all vertices, since vertices of degree one contribute nothing to the sum, and T has no vertices of degree two. Therefore,

$$\begin{aligned} |\mathbf{C}| &= \sum_{x \in V(T)} (d(x) - 1) \\ &= \sum_{x \in V(T)} d(x) - n \\ &= 2(n-1) - n \\ &= n - 2. \end{aligned}$$

Since $L(T)$ has $n-1$ vertices, $\mathbf{C}$ constitutes an SCDC of $L(T)$.

■

The proofs of our other results are also constructive, and we begin by describing the general technique used in the constructions. Let G be a graph, and $L(G)$ its line graph. Assume that G is bridgeless (the case where G is 2-bridge-free but not bridgeless requires a slight variation that will be dealt with in a later section). To construct an SCDC of $L(G)$, we require the following:

(i) a CDC of G;

(ii) a PPDC of each vertex clique of $L(G)$.

Moreover, it is necessary that the PPDC's of the vertex cliques and the CDC of G be "compatible" (the precise meaning of this will be explained in what follows).

Let x, a and b be distinct vertices of G, with a and b both adjacent to x. The pair of edges $\{ax, xb\}$ is called a *transition* at vertex x, and any cycle of G that contains vertex x induces a transition at x (consisting of the two edges of the cycle that are incident with x). Thus we see that the CDC $\mathcal{C}$ of G induces, at each vertex x of G, a *system of transitions*, denoted by $T(x)$, which is the collection of transitions at x induced by all the cycles of $\mathcal{C}$. Notice that $T(x)$ consists of $d(x)$ transitions, and that every edge incident with x is in exactly two of the transitions of $T(x)$. Note that the term "transition system" more typically refers to a *partition* of the edges incident to a vertex of a graph into classes of two elements (see [6]).

For each $x \in V(G)$, let $\mathcal{P}(x)$ denote a PPDC of the vertex clique $K(x)$ in $L(G)$, and let

$$\mathbf{P} = \bigcup_{x \in V(G)} \mathcal{P}(x).$$

Observe that every edge of $L(G)$ lies in exactly two paths of $\mathbf{P}$, so that $\mathbf{P}$ is a *path double cover* of $L(G)$.

The CDC $\mathcal{C}$ and the path double cover $\mathbf{P}$ are said to be *compatible* if and only if for each vertex x of G there is a bijection

$$f_x : T(x) \to \mathcal{P}(x)$$

such that for every transition $\{ax, xb\} \in T(x)$, $f_x(\{ax, xb\})$ is a path in $\mathcal{P}(x)$ with endpoints ax and xb. We call f_x a *compatibility function* between $T(x)$ and $\mathcal{P}(x)$. Suppose that x has neighbours $N(x) = \{y_1, y_2, \ldots, y_k\}$. The *transition multigraph*, denoted $M_T(x)$, has as its vertex set $N(x)$, with $y_iy_j \in E(M_T(x))$ if and only if $\{y_ix, xy_j\}$ is in $T(x)$. Observe that $M_T(x)$ is a 2-regular multigraph. The associated multigraph of the PPDC $\mathcal{P}(x)$, denoted $M_{\mathcal{P}}(x)$, has as its vertex set $\{xy_1, xy_2, \ldots xy_k\}$, but in order to simplify notation, we will denote the vertex set of $M_{\mathcal{P}}(x)$ by $N(x)$, with vertex xy_i corresponding to vertex y_i. In this case, we see that $y_iy_j \in E(M_{\mathcal{P}}(x))$ if and only if there is a path in $\mathcal{P}(x)$ with endpoints xy_i and xy_j. The following lemma provides an easy way of recognizing whether or not $\mathcal{C}$ and $\mathbf{P}$ are compatible.

Lemma 3 *Let G be a bridgeless graph, $\mathcal{C}$ a cycle double cover of G, and $\mathbf{P}$ a path double cover of $L(G)$ consisting of PPDC's of the vertex cliques of $L(G)$; i.e., $\mathbf{P} = \cup_{x \in V(G)} \mathcal{P}(x)$, where $\mathcal{P}(x)$ is a PPDC of the vertex clique $K(x)$ for each $x \in V(G)$. Then $\mathcal{C}$ and $\mathbf{P}$ are compatible if and only if, for each vertex $x \in V(G)$, the transition multigraph $M_T(x)$ is isomorphic to the associated multigraph $M_{\mathcal{P}}(x)$ of the PPDC $\mathcal{P}(x)$.*

Proof: Assume that $\mathcal{C}$ and $\mathbf{P}$ are compatible, and for each $u \in V(G)$, fix a compatibility function f_u from $T(u)$ to $\mathcal{P}(u)$. Let $x \in V(G)$ have neighbours $N(x) = \{y_1, y_2, \ldots, y_k\}$. The compatibility function f_x from $T(x)$ to $\mathcal{P}(x)$ induces a bijection

$$g_x : V(M_T(x)) \to V(M_{\mathcal{P}}(x))$$

where, for each $y_i \in V(M_T(x))$, $g_x(y_i) = y_i$. It follows immediately from the definitions of f_x and the multigraphs $M_T(x)$ and $M_{\mathcal{P}}(x)$ that $y_iy_j \in E(M_T(x))$ if and only if $y_iy_j \in E(M_{\mathcal{P}}(x))$, thus confirming that g_x is an isomorphism.

Conversely, suppose that $x \in V(G)$ and that $N(x) = \{y_1, y_2, \ldots, y_k\}$; furthermore, suppose that the multigraphs $M_T(x)$ and $M_{\mathcal{P}}(x)$ are isomorphic, and that g_x is an isomorphism between them. Then

$$g_x : V(M_T(x)) \to V(M_{\mathcal{P}}(x))$$

is a bijection such that $y_iy_j \in E(M_T(x))$ if and only if $g_x(y_i)g_x(y_j) \in E(M_{\mathcal{P}}(x))$. Recall that $\mathcal{P}(x)$ is a PPDC of a complete graph, $K(x)$, and so any permutation of the names of the vertices still results in a PPDC of $K(x)$. Thus, we may assume that the PPDC $\mathcal{P}(x)$ of $K(x)$ is chosen so that g_x is the identity function. It now follows that $\mathcal{C}$ and $\mathbf{P}$ are compatible.

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Cycle double covers $\mathcal{C}$ that are compatible with path double covers $\mathbf{P}$ play a crucial role in the proofs of our results. The following lemma provides the basis for our constructions.

Lemma 4 *Let G be a bridgeless graph, $\mathcal{C}$ a CDC of G, and $\mathbf{P}$ a path double cover of $L(G)$, consisting of PPDC's of the vertex cliques of $L(G)$, and assume that $\mathcal{C}$ and $\mathbf{P}$ are compatible. For each $u \in V(G)$, fix a compatibility function f_u from $T(u)$ to $\mathcal{P}(u)$. Let $c = v_0v_1v_2 \ldots v_{q-1}v_0$ be a cycle in $\mathcal{C}$, and for each i, $0 \leq i \leq q-1$, let $f_i = f_{v_i}$. By the definition of a compatibility function, $f_i(\{v_{i-1}v_i, v_iv_{i+1}\}) = P_i$, where P_i is a path in $\mathcal{P}(v_i)$ with endpoints $v_{i-1}v_i$ and v_iv_{i+1} (where subscripts are taken modulo q). Then*

$$E_c = \bigcup_{i=0}^{q-1} P_i,$$

is an eulerian subgraph of $L(G)$ with maximum degree at most four. Furthermore, if for each $c \in \mathcal{C}$, $D(c)$ is the set of cycles in a cycle decomposition of E_c, then

$$\mathbf{C} = \bigcup_{c \in \mathcal{C}} D(c)$$

is a cycle double cover of $L(G)$, and $|\mathbf{C}| \leq |\mathcal{C}| + ch(\mathcal{C})$ (where $ch(\mathcal{C})$ denotes the total number of chords in all cycles of $\mathcal{C}$).

Before proving this result, one additional lemma is required.

Lemma 5 *If H is a simple eulerian graph with k vertices of degree four and all remaining vertices of degree two, then H has a cycle decomposition with at most $(k+1)$ cycles.*

Proof: The proof is by induction on k, the number of vertices of degree four. The result is obvious when $k = 0$. Assume that H is a simple eulerian graph with $k \geq 1$ vertices of degree four and all remaining vertices of degree two. Let $x \in V(H)$ be an arbitrary vertex of degree four in H, and construct an Euler tour of H that starts and ends at x. This Euler tour can be represented by a sequence of vertices, with vertices of degree two each occurring once, and vertices of degree four each occurring twice, with the exception of x, which occurs three times since it is at the beginning and end of the sequence. For some vertex $y \in V(H)$ (possibly $y = x$), there will be a subsequence that starts and ends with y such that any vertex in this subsequence occurs only once (with the exception of y, which occurs twice). This subsequence corresponds to a cycle, c, in H; deleting the edges of c along with any vertices that become isolated results in an eulerian graph, H' with $k-1$ vertices of degree four, since y has degree two in H'. (An Euler tour in this graph can be obtained from the sequence we began with by taking the subsequence whose first and last vertex is y, and replacing it with just the vertex y.) By the induction hypothesis, H' has a cycle decomposition with at most k cycles; this cycle decomposition of H' along with the cycle c gives us a cycle decomposition of H with at most $k+1$ cycles, as required.

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Proof of Lemma 4: An Euler tour of E_c can be constructed by traversing, for $i = 0, 1, \ldots, (q-1)$, the path P_i from $v_{i-1}v_i$ to v_iv_{i+1} (subscripts modulo q), so E_c is certainly eulerian.

Now suppose that j and m are integers with $0 \leq j, m \leq q-1$ and $j \neq m$, and suppose that P_j and P_m have a vertex in common. We know that P_j is a path in the vertex clique $K(v_j)$ and P_m is a path in the vertex clique $K(v_m)$, so this implies v_j and v_m are adjacent in G. Since G is simple, v_jv_m is the only vertex that P_j and P_m have in common; furthermore, no other path, P_i $(i \neq j, m)$, can contain the vertex v_jv_m, for this would imply that the edge v_jv_m of G is incident to v_j, v_m and v_i, which is impossible. Thus the intersection of P_j and P_m is either

(a) a vertex of degree two, in which case $|j - m| \equiv 1 \pmod{q}$.

(b) a vertex of degree four, in which case the vertex v_jv_m corresponds to a chord v_jv_m of the cycle c in G.

To verify that

$$\mathbf{C} = \bigcup_{c \in \mathcal{C}} D(c)$$

is a CDC of $L(G)$ it is only necessary to verify that every path of $\mathbf{P}$ lies in exactly one E_c, for some $c \in \mathcal{C}$. This follows immediately from the fact that $\mathcal{C}$ and $\mathbf{P}$ are compatible, so there is a one-to-one correspondence between transitions of the cycles of $\mathcal{C}$ and the paths of $\mathbf{P}$.

Finally, we apply Lemma 5 to each E_c: since E_c is an eulerian graph with maximum degree four, and the vertices of degree four correspond to chords of c in G, it follows that E_c has a cycle decomposition into at most $1 + ch(c)$ cycles. Therefore, $|\mathbf{C}| \leq |\mathcal{C}| + ch(\mathcal{C})$.

■

The preceding lemma gives us a method for constructing a CDC, $\mathbf{C}$ of a line graph $L(G)$, provided that we have a CDC, $\mathcal{C}$, of the original graph G and a path double cover, $\mathbf{P}$ (consisting of PPDC's of the vertex cliques of $L(G)$), such that $\mathcal{C}$ and $\mathbf{P}$ are compatible. To ensure that $\mathbf{C}$ is a *small* CDC of $L(G)$, it is necessary to carefully choose the CDC $\mathcal{C}$ of G.

3 Line Graphs of Complete Graphs

The first step in proving that the line graph of K_n has an SCDC is to find an "appropriate" CDC of K_n. The following theorem provides the basis for this; the reader will observe that this theorem is simply a restatement of the result of Kirkman [8] that for all integers $n \geq 3$, there exists a Steiner Triple System on n points if and only if $n \equiv 1, 3 \pmod{6}$.

Theorem 6 *For all integers $n \geq 3$, K_n has a cycle decomposition into triangles if and only if $n \equiv 1, 3 \pmod{6}$.*

■

Two other results are required before we can proceed with the proof in the main theorem of this section. Both results concern specific types of PPDC's of complete graphs.

Lemma 7 *(i) The complete graph K_{2m}, $m \geq 1$, has a PPDC whose associated multigraph consists of m digons.*
(ii) The complete graph K_{2m+1}, $m \geq 1$, has a PPDC whose associated multigraph consists of one triangle and $m-1$ digons.

Proof: Let $m \geq 1$, and let $V = \{v_0, v_1, v_2, \ldots, v_{2m-1}\}$ be the set of vertices of K_{2m}.
(i) For $0 \leq i \leq 2m-1$, define the path P_i as follows:

$$P_i = v_i v_{i+1} v_{i-1} v_{i+2} v_{i-2} \ldots v_{i+m+1} v_{i+m},$$

where the subscripts are taken modulo $2m$, and let

$$\mathcal{P} = \{P_i : 0 \leq i \leq 2m-1\}.$$

Then $\mathcal{P}$ is a PPDC of K_{2m}, and $P_i = P_{i+m}$ for $0 \leq i \leq m-1$. Thus, $\mathcal{P}$ consists of two copies of a decomposition of K_{2m} into m Hamilton paths (this is a standard construction dates back to the 19th century). It follows that the associated multigraph, $M_{\mathcal{P}}(K_{2m})$ consists of m digons.
(ii) Beginning with the graph K_{2m} and the PPDC described in (i), add a new vertex v_{2m} and join it to each vertex of V. The paths of $\mathcal{P}$ are now modified to produce a PPDC $\mathcal{Q}$ of K_{2m+1} as follows: for each path P_j, $1 \leq j \leq 2m-1$, replace the edge $v_j v_{j+1}$ by the path of length two $v_j v_{2m} v_{j+1}$. Notice that this does not affect the endpoints of these paths, so that P_j and P_{j+m}, $1 \leq j \leq m-1$, still have the same pair of endpoints and result in a digon in $M_{\mathcal{Q}}(K_{2m+1})$. The path P_0 is simply extended by adding an edge from v_0 to v_{2m}, and thus has endpoints v_{2m} and v_m. One new path, P_* must be added:

$$P_* = v_{2m} v_1 v_2 v_3 \ldots v_{2m-1} v_0.$$

The path P_* covers the edges that were "uncovered" by the modifications to the paths P_j, $1 \leq j \leq 2m-1$. Since the path P_m has endpoints v_m and v_0 as before, the paths P_0, P_m and P_* result in a triangle in $M_{\mathcal{Q}}(K_{2m+1})$, thus completing the proof in this case.

■

Recall that an *eulerian perfect path double cover* (EPPDC) of a graph is a PPDC whose associated multigraph is a single cycle.

Lemma 8 *The complete graph K_m, $m \geq 2$ has an eulerian perfect path double cover.*

Proof: Let $m \geq 2$, and let $V = \{v_0, v_1, v_2, \ldots, v_{m-1}\}$ be the set of vertices of K_m. We consider the cases m odd and m even separately.
(i) Suppose that m is odd, $m = 2k+1$. For $0 \leq i \leq m-1$, let

$$P_i = v_i v_{i+1} v_{i-1} v_{i+2} v_{i-2} \ldots v_{i+k+2} v_{i+k} v_{i+k+1},$$

where the subscripts are taken modulo m. Then

$$\mathcal{P} = \{P_i : 0 \leq i \leq m-1\}$$

is a PPDC of K_m. The edges of the associated multigraph $M_{\mathcal{P}}(K_m)$ are

$$v_0v_{k+1}, v_1v_{k+2}, v_2v_{k+3}, \ldots, v_kv_0, v_{k+1}v_1, \ldots, v_{2k}v_k,$$

and these give us the single cycle

$$v_0v_{k+1}v_1v_{k+2}v_2v_{k+3}\ldots v_kv_0.$$

Therefore, $\mathcal{P}$ is an EPPDC of K_m.

(ii) Now suppose that m is even, $m = 2k$. For $0 \le i \le k-1$, define the path P_i as follows:

$$P_i = v_iv_{i+1}v_{i-1}v_{i+2}v_{i-2}\ldots v_{i+k+1}v_{i+k},$$

where the subscripts are taken modulo m. Then

$$\mathcal{P} = \{P_i : 0 \le i \le k-1\}$$

is a decomposition of K_m into Hamilton paths, with P_i having endpoints v_i and v_{i+k}, $0 \le i \le k-1$. If we draw K_m with the vertices in the cyclic (clockwise) order

$$v_0v_kv_1v_{k+1}v_2v_{k+2}\ldots v_{k-2}v_{2k-2}v_{k-1}v_{2k-1}v_0,$$

and take the paths of the decomposition $\mathcal{P}$ and rotate each one clockwise by one vertex, we obtain a second path decomposition $\mathcal{Q}$, where path Q_i has endpoints v_{i+k} and v_{i+1}, $0 \le i \le k-2$, and Q_{k-1} has endpoints v_{2k-1} and v_0. Then is $\mathcal{S} = \mathcal{P} \cup \mathcal{Q}$ is a PPDC of K_m, and the edges of the associated multigraph $M_{\mathcal{S}}(K_m)$ are

$$v_0v_k, v_1v_{k+1}, v_2v_{k+2}, \ldots, v_{k-1}v_{2k-1}, v_kv_1, v_{k+1}v_2, \ldots, v_{2k-1}v_0.$$

These edges form the cycle

$$v_0v_kv_1v_{k+1}v_2v_{k+2}\ldots v_{k-1}v_{2k-1}v_0,$$

Therefore, $\mathcal{S}$ is an EPPDC of K_m.

■

We are now ready to prove the main result in this section.

Theorem 9 *For all $n \ge 2$, $L(K_n)$ has a small cycle double cover.*

Proof: To prove that $L(K_n)$ has an SCDC, we must show that it has a CDC with less than $n(n-1)/2$ cycles. When $n = 2$, $L(K_n)$ consists of a single vertex, and the result is trivially true. In what follows, assume that $n \ge 3$. There are a number of cases to consider.

Case 1. $n \equiv 1, 3 \pmod 6$.

By Theorem 6, we know that K_n has a cycle decomposition into triangles, and thus has a CDC, $\mathcal{C}$, consisting of two copies of this cycle decomposition. This implies that for each vertex $x \in V(K_n)$, the transition multigraph $M_T(x)$ consists of $(n-1)/2$ digons.

Since n is odd, $(n-1)$ is even, and so by Lemma 7, each vertex clique of $L(K_n)$ has a PPDC whose associated multigraph consists of $(n-1)/2$ digons. Let $\mathbf{P}$ be a path double cover of $L(K_n)$ consisting of these PPDC's of the vertex cliques. Then $\mathcal{C}$ and $\mathbf{P}$ are compatible, and we can construct a CDC, $\mathbf{C}$, of $L(K_n)$ as described in Lemma 4.

Since each $c \in \mathcal{C}$ is a triangle, the eulerian subgraph E_c of $L(K_n)$ is a cycle, and thus $|\mathbf{C}| = |\mathcal{C}|$. Therefore, the number of cycles in $\mathbf{C}$ is $n(n-1)/3$; since $L(K_n)$ has $n(n-1)/2$ vertices, it is clear that $\mathbf{C}$ is an SCDC of $L(K_n)$.

Case 2. $n \equiv 2, 4 \pmod 6$.
In this case, $n-1 \equiv 1, 3 \pmod 6$, and so using the construction in Case 1, K_{n-1} has a CDC, $\mathcal{C}'$, with $(n-1)(n-2)/3$ triangles, such that for each $x \in V(K_{n-1})$, the transition multigraphs $M_T(x)$ consists of $(n-2)/2$ digons.

Let $V(K_{n-1}) = \{v_0, v_1, \ldots, v_{n-2}\}$, and construct K_n by adding the vertex v_{n-1} and joining it to each vertex of K_{n-1}. Without loss of generality, we may assume that the vertices of K_{n-1} are labeled so that $T_0 = v_0v_1v_2v_0$ is a triangle in $\mathcal{C}'$. Furthermore, there exist $(n-4)/2$ distinct triangles $T_i \in \mathcal{C}'$, $1 \le i \le \frac{n-4}{2}$ (that are also distinct from T_0), such that $v_{2i+1}v_{2i+2} \in E(T_i)$.

The CDC $\mathcal{C}'$ of K_{n-1} can be modified to be a CDC $\mathcal{C}$ of K_n as follows:

(i) Replace the edges v_0v_1 and v_1v_2 of T_0 with the edges v_0v_{n-1} and $v_{n-1}v_2$.
(ii) For $i = 1, \ldots, \frac{n-4}{2}$, replace the edge $v_{2i+1}v_{2i+2}$ of T_i with the edges $v_{2i+1}v_{n-1}$ and $v_{n-1}v_{2i+2}$ (so that T_i is now a 4-cycle).
(iii) Add the triangles $v_0v_{n-1}v_1$, $v_1v_{n-1}v_2$, and for $i = 1, \ldots, \frac{n-4}{2}$, add the triangles $v_{2i+1}v_{n-1}v_{2i+2}$.

One can verify that this is a CDC of K_n with $[(n-1)(n-2)/3] + 2$ triangles and $(n-4)/2$ four-cycles. Furthermore, for each vertex $x \in V(K_n)$, the transition multigraph $M_T(x)$ consists of one triangle and $(n-4)/2$ digons.

Since n is even, $n-1$ is odd, so by Lemma 7 each vertex clique of $L(K_n)$ has a PPDC whose associated multigraph consists of one triangle and $(n-4)/2$ digons. Let $\mathbf{P}$ be a path double cover of $L(K_n)$ consisting of these PPDC's of the vertex cliques. Then $\mathcal{C}$ and $\mathbf{P}$ are compatible, and we can construct a CDC, $\mathbf{C}$, of $L(K_n)$ as described in Lemma 4.

For each triangle $c \in \mathcal{C}$, the eulerian subgraph E_c in $L(K_n)$ is a cycle. Since each 4-cycle $f \in \mathcal{C}$ has two chords, the eulerian subgraph E_f in $L(K_n)$ has exactly two vertices of degree four. By Lemma 5, E_f has a cycle decomposition with at most three cycles. Therefore

$$\begin{aligned} |\mathbf{C}| &\le \frac{(n-1)(n-2)}{3} + 2 + 3 \cdot \frac{n-4}{2} \\ &= \frac{1}{6}(2n^2 + 3n - 20) \\ &< \frac{n(n-1)}{2}, \end{aligned}$$

for all $n \ge 0$, and thus $\mathbf{C}$ is an SCDC of $L(K_n)$.

Case 3. $n \equiv 0 \pmod 6$.
In this case, $n-3 \equiv 3 \pmod 6$, and so using the construction in Case 1, K_{n-3} has a CDC, $\mathcal{C}'$, consisting of $(n-3)(n-4)/3$ triangles, such that for each $x \in V(K_{n-3})$, the transition multigraph $M_T(x)$ consists of $(n-4)/2$ digons.

Let $V(K_{n-3}) = \{v_0, v_1, \ldots, v_{n-4}\}$, and construct K_n by adding the vertices v_{n-3}, v_{n-2}, v_{n-1}, and joining each of these to all vertices of K_{n-3} and to each other. The CDC $\mathcal{C}'$ of K_{n-3} can be modified to be a CDC $\mathcal{C}$ of K_n as follows:

(i) Add two copies of the triangle $v_{n-3}v_{n-2}v_{n-1}v_{n-3}$.

(ii) Add the 6-cycle $v_0v_{n-1}v_1v_{n-3}v_2v_{n-2}v_0$, and the three 4-cycles $v_0v_{n-3}v_1v_{n-2}v_0$, $v_1v_{n-2}v_2v_{n-1}v_1$, and $v_2v_{n-1}v_0v_{n-3}v_2$ (this constitutes a CDC of a $K_{3,3}$ with one 6-cycles and three 4-cycles).

(iii) For $i = 1, \ldots, \frac{n-6}{2}$, add the three 4-cycles $v_{2i+1}v_{n-3}v_{2i+2}v_{n-2}v_{2i+1}$, $v_{2i+1}v_{n-2}v_{2i+2}v_{n-1}v_{2i+1}$ and $v_{2i+1}v_{n-1}v_{2i+2}v_{n-3}v_{2i+1}$ (this constitutes a CDC of a $K_{2,3}$ with three 4-cycles).

We see that $\mathcal{C}$ is a CDC of K_n with $2 + (n-3)(n-4)/3$ triangles, one 6-cycle, and $3 + 3(n-6)/2$ four-cycles. Furthermore, for each vertex $x \in V(K_n)$, the transition multigraph $M_T(x)$ consists of one triangle and $(n-4)/2$ digons.

Since n is even, $n-1$ is odd, so by Lemma 7 each vertex clique of $L(K_n)$ has a PPDC whose associated multigraph consists of one triangle and $(n-4)/2$ digons. Let $\mathbf{P}$ be a path double cover of $L(K_n)$ consisting of these PPDC's of the vertex cliques. Then $\mathcal{C}$ and $\mathbf{P}$ are compatible, and we can construct a CDC, $\mathbf{C}$, of $L(K_n)$ as described in Lemma 4.

For each triangle $c \in \mathcal{C}$, the eulerian subgraph E_c in $L(K_n)$ is a cycle. Since each 4-cycle $f \in \mathcal{C}$ has two chords, the eulerian subgraph E_f in $L(K_n)$ has exactly two vertices of degree four. By Lemma 5, E_f has a cycle decomposition with at most three cycles. Finally, the 6-cycle $g \in \mathcal{C}$ has nine chords, and thus the eulerian subgraph E_g has exactly nine vertices of degree four. It follows from Lemma 5 that E_g has a cycle decomposition into at most ten cycles. Therefore

$$
\begin{aligned}
|\mathbf{C}| &\le 2 + \frac{(n-3)(n-4)}{3} + 3\left(3 + \frac{3(n-6)}{2}\right) + 10 \\
&= \frac{1}{6}(2n^2 + 13n - 12) \\
&< \frac{n(n-1)}{2},
\end{aligned}
$$

only if $n \ge 16$, and thus $\mathbf{C}$ is an SCDC of $L(K_n)$ whenever $n \ge 16$. Since $n \equiv 0 \pmod 6$, the cases $n = 6$ and $n = 12$ require special treatment (see Case 5).

Case 4. $n \equiv 5 \pmod 6$.
In this case, $n-2 \equiv 3 \pmod 6$, and so using the construction in Case 1, K_{n-2} has a CDC, $\mathcal{C}'$, consisting of $(n-2)(n-3)/3$ triangles, such that for each $x \in V(K_{n-2})$, the transition multigraph $M_T(x)$ consists of $(n-3)/2$ digons.

Let $V(K_{n-2}) = \{v_0, v_1, \ldots, v_{n-3}\}$, and construct K_n by adding the vertices v_{n-2}, v_{n-1}, and joining each of these to all vertices of K_{n-2} and to each other. The CDC $\mathcal{C}'$ of K_{n-2} can be modified to be a CDC $\mathcal{C}$ of K_n as follows:

(i) Add two copies of the triangle $v_0v_{n-2}v_{n-1}v_0$.

(ii) For $i = 0, 1, \ldots, \frac{n-5}{2}$, add two copies of the 4-cycle $v_{2i+1}v_{n-2}v_{2i+2}v_{n-1}v_{2i+1}$.

We see that $\mathcal{C}$ is a CDC of K_n (in fact, $\mathcal{C}$ is simply two copies of a cycle decomposition of K_n), and consists of $2 + (n-2)(n-3)/3$ triangles and $(n-3)$ four-cycles. Furthermore, for each vertex $x \in V(K_n)$, the transition multigraph $M_T(x)$ consists of $(n-1)/2$ digons.

Since n is odd, $n-1$ is even, so by Lemma 7 each vertex clique of $L(K_n)$ has a PPDC whose associated multigraph consists of $(n-1)/2$ digons. Let $\mathbf{P}$ be a path double cover of $L(K_n)$ consisting of these PPDC's of the vertex cliques. Then $\mathcal{C}$ and $\mathbf{P}$ are compatible, and we can construct a CDC, $\mathbf{C}$, of $L(K_n)$ as described in Lemma 4.

For each triangle $c \in \mathcal{C}$, the eulerian subgraph E_c in $L(K_n)$ is a cycle. Since each 4-cycle $f \in \mathcal{C}$ has two chords, the eulerian subgraph E_f in $L(K_n)$ has exactly two vertices of degree four. Therefore

$$\begin{aligned}|\mathbf{C}| &\leq 2 + \frac{(n-2)(n-3)}{3} + 3(n-3)\\ &= \frac{1}{3}(n^2+4n-15)\\ &< \frac{n(n-1)}{2},\end{aligned}$$

only if $n \geq 6$, and thus $\mathbf{C}$ is an SCDC of $L(K_n)$ whenever $n \geq 6$. Since $n \equiv 5 \pmod 6$, the case $n = 5$ and requires special treatment.

Case 5. Special cases: $n = 5, 6$ and 12.
Suppose $n = 5$, and let $V(K_5) = \{v_0, v_1, v_2, v_3, v_4\}$. Then $V(L(K_5)) = \{x_{i,j} | v_iv_j \in E(K_5)\}$. One can verify that the following four cycles in $L(K_n)$ constitute a cycle decomposition of $L(K_n)$:

$$\begin{aligned}c_0 &= x_{0,1}x_{1,2}x_{2,3}x_{3,4}x_{0,4}x_{0,1}\\ c_1 &= x_{1,4}x_{1,3}x_{0,3}x_{0,2}x_{2,4}x_{1,4}\\ c_2 &= x_{0,1}x_{0,2}x_{1,2}x_{1,3}x_{2,3}x_{2,4}x_{3,4}x_{0,3}x_{0,4}x_{1,4}x_{0,1}\\ c_3 &= x_{0,1}x_{1,3}x_{3,4}x_{1,4}x_{1,2}x_{2,4}x_{0,4}x_{0,2}x_{2,3}x_{0,3}x_{0,1}\end{aligned}$$

Let $\mathbf{C}$ denote the collection of cycles obtained by taking two copies of each c_i, $0 \leq i \leq 3$. Then $\mathbf{C}$ is a CDC of $L(K_5)$ with eight cycles; since $L(K_5)$ has ten vertices, $\mathbf{C}$ is an SCDC.

When $n = 6$, there is a CDC, $\mathcal{C}$, of K_6 with triangles as follows. Let $V(K_6) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, and take the collection of cycles consisting of $v_0v_1v_2v_0$, $v_0v_2v_3v_0$, $v_0v_3v_4v_0$, $v_0v_4v_5v_0$, $v_0v_5v_1v_0$, $v_1v_2v_4v_1$, $v_2v_3v_5v_2$, $v_3v_4v_1v_3$, $v_4v_5v_2v_4$, $v_5v_1v_3v_5$. Again, one can verify that $\mathcal{C}$ is a CDC of K_6 with triangles, and that the transition multigraph $M_T(v_i)$, $0 \leq i \leq 5$ is a cycle of length five. By Lemma 8, there is a PPDC of $K(v_i)$, $0 \leq i \leq 5$ so that the resulting path double cover, $\mathbf{P}$, of $L(K_n)$ and the CDC $\mathcal{C}$ are compatible. Thus a CDC, $\mathbf{C}$, of $L(K_n)$ can be constructed as in the proof of Lemma 4. Each triangle in $\mathcal{C}$ yields a cycle in $\mathbf{C}$, and so $|\mathbf{C}| = |\mathcal{C}| = 10$. Since $10 \le \binom{6}{2} - 1$, $\mathbf{C}$ is an SCDC of $L(K_6)$, as required.

Finally, we consider the case $n = 12$. Let G_i, $0 \le i \le 2$ be a complete graph on four vertices, with $V(G_i) = \{v_{4i}, v_{4i+1}, v_{4i+2}, v_{4i+3}\}$. We construct K_{12} by taking the disjoint union of G_0, G_1 and G_2, and then adding the edges of a complete 3-partite graph, H, whose parts are $V(G_0)$, $V(G_1)$ and $V(G_2)$. Each G_i, $0 \le i \le 2$, has a CDC with triangles: simply take the cycles $v_{4i}v_{4i+1}v_{4i+2}v_{4i}$, $v_{4i}v_{4i+1}v_{4i+3}v_{4i}$, $v_{4i}v_{4i+2}v_{4i+3}v_{4i}$ and $v_{4i+1}v_{4i+2}v_{4i+3}v_{4i+1}$. The edges of H can be decomposed into 16 triangles:

$v_0v_4v_8v_0$	$v_0v_5v_9v_0$	$v_0v_6v_{10}v_0$	$v_0v_7v_{11}v_0$
$v_1v_4v_9v_1$	$v_1v_5v_{10}v_1$	$v_1v_6v_{11}v_1$	$v_1v_7v_8v_1$
$v_2v_4v_{10}v_2$	$v_2v_5v_{11}v_2$	$v_2v_6v_8v_2$	$v_2v_7v_9v_2$
$v_3v_4v_{11}v_3$	$v_3v_5v_8v_3$	$v_3v_6v_9v_3$	$v_3v_7v_{10}v_3$

One can verify that two copies of this cycle decomposition of H, along with the CDC's of G_0, G_1, and G_2 gives us a CDC, $\mathcal{C}$, of K_{12} with 44 triangles. Furthermore, for each v_i, $0 \le i \le 11$, the transition multigraph $M_T(v_i)$ consists of one triangle and four digons. By Lemma 7, $K(v_i)$, $0 \le i \le 11$, has a PPDC so that the resulting path double cover, $\mathbf{P}$, of $L(K_n)$ is compatible with the CDC $\mathcal{C}$. Thus a CDC, $\mathbf{C}$, of $L(K_n)$ can be constructed as in the proof of Lemma 4, with each triangle in $\mathcal{C}$ yielding a cycle in $\mathbf{C}$, and hence $|\mathbf{C}| = |\mathcal{C}| = 44$. Since $44 \le \binom{12}{2} - 1$, $\mathbf{C}$ is an SCDC of $L(K_{12})$, as required.

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4 Line Graphs of Complete Bipartite Graphs

The technique used in the previous section to prove the SCDC Conjecture for line graphs of complete graphs can also be applied to line graphs of complete bipartite graphs.

Theorem 10 *The line graph of every complete bipartite graph $K_{m,n}$ with $m, n \ge 1$ has a small cycle double cover, except when $\{m, n\} = \{1, 2\}$.*

Proof: To prove that $L(K_{m,n})$ has an SCDC, we must show that it has a CDC with at most $mn - 1$ cycles. If m or n is equal to one, say, without loss of generality that $m = 1$, then $K_{1,n}$ is a tree; furthermore, if $n \ne 2$, then this tree is 2-bridge-free, and the result follows from Theorem 2. Thus, from now on we will assume that $m \ge 2$ and $n \ge 2$. Let $K_{m,n}$ have bipartition (X, Y) with $|X| = m$ and $|Y| = n$. We consider two cases: either at least one of m and n is even, or both m and n are odd.

Case 1. First assume that at least one of m and n is even; without loss of generality, suppose that m is even, say $m = 2p$ for some integer $p \ge 1$. Let $X = \{x_0, x_1, x_2, \ldots, x_{2p-1}\}$ and $Y = \{y_0, y_1, y_2, \ldots, y_{n-1}\}$. For each integer i, $0 \le i \le p-1$

and for each integer j, $0 \le j \le n-1$, define the 4-cycle $C_{i,j} = x_{2i}y_jx_{2i+1}y_{j+1}x_{2i}$, where the subscripts for the y's are taken modulo n. One can verify that

$$\mathcal{C} = \{C_{i,j} | 0 \le i \le p-1; 0 \le j \le n-1\}$$

is a CDC of $K_{m,n}$ with $mn/2$ cycles of length four. For each vertex $x \in X$, the transition multigraph $M_T(x)$ that is induced by the cycles of $\mathcal{C}$ is a cycle of length $n = d(x)$, and for each vertex $y \in Y$, the transition multigraph $M_T(y)$ that is induced by the cycles of $\mathcal{C}$ consists of $d(y)/2 = p$ digons.

For each vertex $x \in X$, let $\mathcal{P}(x)$ be an EPPDC of K_n, and for each $y \in Y$, let $\mathcal{P}(y)$ be a PPDC of K_{2p} whose associated multigraph consists of p digons; the existence of these is guaranteed by Lemmas 7 and 8. Set

$$\mathbf{P} = (\bigcup_{x \in X} \mathcal{P}(x)) \cup (\bigcup_{y \in Y} \mathcal{P}(y));$$

then $\mathbf{P}$ is a path double cover of $L(K_{m,n})$, and it follows from Lemma 3 that $\mathcal{C}$ and $\mathbf{P}$ are compatible. We can now apply Lemma 4 to obtain a CDC, $\mathbf{C}$, of $L(K_{m,n})$, with $|\mathbf{C}| \le |\mathcal{C}| + ch(\mathcal{C})$. However, since $K_{m,n}$ is bipartite and the cycles of $\mathcal{C}$ are all of length four, it follows that $ch(\mathcal{C}) = 0$, and thus $|\mathbf{C}| = |\mathcal{C}| = mn/2 \le mn - 1$. Therefore, $\mathbf{C}$ is an SCDC of $L(K_{m,n})$.

Case 2. We may now assume that both m and n are odd, with $m, n \ge 3$; then $m = 2p+1$ and $n = 2q+1$ for some $p, q \ge 1$. Let $X = \{x_0, x_1, x_2, \ldots, x_{2p}\}$ and $Y = \{y_0, y_1, y_2, \ldots, y_{2q}\}$. Partition X into two sets, $X' = \{x_{2p-2}, x_{2p-1}, x_{2p}\}$ and $X'' = X \backslash X'$; similarly, partition Y into $Y' = \{y_{2q-2}, y_{2q-1}, y_{2q}\}$ and $Y'' = Y \backslash Y'$. Then $K_{m,n}$ is the union of four edge-disjoint bipartite graphs: (i) a $K_{3,3}$ with bipartition (X', Y'); (ii) a $K_{3,2q-2}$ with bipartition (X', Y''); (iii) a $K_{2p-2,3}$ with bipartition (X'', Y'); (iv) a $K_{2p-2,2q-2}$ with bipartition (X'', Y''). A CDC $\mathcal{C}$ of $K_{m,n}$ can be constructed by taking CDC's of each of these four graphs.

For the graph $K_{3,3}$ with bipartition (X', Y'), the collection $\mathcal{C}_1$ consisting of the cycles

$$x_{2p-2}y_{2q-2}x_{2p-1}y_{2q-1}x_{2p-2} \qquad x_{2p-1}y_{2q-1}x_{2p}y_{2q}x_{2p-1}$$
$$x_{2p}y_{2q}x_{2p-2}y_{2q-2}x_{2p} \qquad x_{2p-2}y_{2q-1}x_{2p}y_{2q-2}x_{2p-1}y_{2q}x_{2p-2}$$

is a CDC with three 4-cycles and one 6-cycle.

For the $K_{3,2q-2}$ with bipartition (X', Y''), let $\mathcal{C}_2$ be the collection of cycles consisting of, for $j = 0, 1, \ldots, q-2$,

$$y_{2j}x_{2p-2}y_{2j+1}x_{2p-1}y_{2j}$$
$$y_{2j}x_{2p-1}y_{2j+1}x_{2p}y_{2j}$$
$$y_{2j}x_{2p}y_{2j+1}x_{2p-2}y_{2j}.$$

Then $\mathcal{C}_2$ is a CDC with $3(q-1)$ cycles of length four.

Similarly, for the $K_{2p-2,3}$ with bipartition (X'', Y'), let $\mathcal{C}_3$ be the collection of cycles consisting of, for $i = 0, 1, \ldots, p-2$,

$$x_{2i}y_{2q-2}x_{2i+1}y_{2q-1}x_{2i}$$
$$x_{2i}y_{2q-1}x_{2i+1}y_{2q}x_{2i}$$
$$x_{2i}y_{2q}x_{2i+1}y_{2q-2}x_{2i}.$$

Then $\mathcal{C}_3$ is a CDC with $3(p-1)$ cycles of length four.

Finally, for the $K_{2p-2,2q-2}$ with bipartition (X'', Y''), let

$$\mathcal{C}^* = \{x_{2i}y_{2j}x_{2i+1}y_{2j+1}x_{2i} \ : \ 0 \le i \le p-2, 0 \le j \le q-2\}.$$

Then $\mathcal{C}^*$ is a cycle decomposition with $(p-1)(q-1)$ cycles of length four, and two copies of this, which we will call $\mathcal{C}_4$, constitutes a CDC with $2(p-1)(q-1)$ cycles of length four.

The union of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$ thus gives us a CDC, $\mathcal{C}$. of $K_{m,n}$ with $2pq+p+q-1 = (mn-3)/2$ cycles of length four and one cycle of length six. Furthermore, for each vertex $u \in V(K_{m,n})$, the transition multigraph $M_T(u)$ induced by the CDC $\mathcal{C}$ consists of one triangle and $(d(u)-3)/2$ digons.

For each vertex $x \in X$, let $\mathcal{P}(x)$ be a PPDC of K_n such that the associated multigraph $M_{\mathcal{P}}(x)$ consists of one triangle and $(n-3)/2$ digons, and for each $y \in Y$, let $\mathcal{P}(y)$ be a PPDC of K_m whose associated multigraph $M_{\mathcal{P}}(y)$ consists of one triangle and $(m-3)/2$ digons. Such PPDC's exist by Lemma 7. Set

$$\mathbf{P} = (\bigcup_{x \in X} \mathcal{P}(x)) \cup (\bigcup_{y \in Y} \mathcal{P}(y));$$

then $\mathbf{P}$ is a path double cover of $L(K_{m,n})$, and it follows from Lemma 3 that $\mathcal{C}$ and $\mathbf{P}$ are compatible. We can now apply Lemma 4 to obtain a CDC, $\mathbf{C}$, of $L(K_{m,n})$, with $|\mathbf{C}| \le |\mathcal{C}| + ch(\mathcal{C})$. Since $K_{m,n}$ is bipartite, the only chords of $\mathcal{C}$ occur in the single cycle of length six, which has three chords, and thus

$$|\mathbf{C}| \le |\mathcal{C}| + 3 = \frac{mn-1}{2} + 3 = \frac{mn+5}{2} \le mn - 1$$

whenever $mn \ge 7$. Since $m, n \ge 3$, this condition is satisfied, and therefore $\mathbf{C}$ is an SCDC of $L(K_{m,n})$.

■

5 Line Graphs of Planar Graphs

One of the keys to constructing SCDCs for the line graphs of complete graphs and line graphs of complete bipartite graphs is the existence of CDC's of complete graphs and complete bipartite graphs for which we can exactly describe the transition multigraphs. Planar graphs provide another class of graphs with CDC's for which we can exactly describe the transition multigraphs.

Theorem 11 *If G is a 2-bridge-free planar graph, then $L(G)$ has a small cycle double cover.*

The basic technique that we will use is the same as that used for line graphs of complete graphs and complete bipartite graphs, but requires some modification to allow for bridges and cut vertices in the graph. (An important part of our previous constructions was a CDC of a graph G, which exists only if G is bridgeless.)

One preliminary observation is that it suffices to prove Theorem 11 for connected graphs; the next lemma allows us to further restrict the graphs we must consider.

Lemma 12 *Suppose G is a connected 2-bridge-free graph, and suppose that $xy \in E(G)$ is a nontrivial bridge in G. Let H_1 and H_2 be the two components of $G - \{xy\}$; without loss of generality, $x \in V(H_1)$ and $y \in V(H_2)$. Define $G_1 = H_1 \cup \{y\} \cup \{xy\}$ and $G_2 = H_2 \cup \{x\} \cup \{xy\}$. If $L(G_1)$ and $L(G_2)$ both have small cycle double covers, then $L(G)$ has a small cycle double cover.*

Proof: The definitions of G_1 and G_2 ensure that $L(G) = L(G_1) \cup L(G_2)$, with $L(G_1) \cap L(G_2)$ consisting of the single vertex in $L(G)$ that corresponds to the bridge xy of G.

Let m denote the number of edges in G. Suppose that $L(G_j)$ has m_j vertices, $j = 1, 2$; then $m_1 + m_2 = m + 1$, where m is the number of *vertices* in $L(G)$ (since G has m edges). Since $L(G_j)$ has an SCDC, there is a CDC, $\mathcal{C}_j$, of $L(G_j)$ with $|\mathcal{C}_j| \leq (m_j - 1)$, $j = 1, 2$.

The structure of $L(G)$ guarantees that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is a CDC of $L(G)$, and

$$\begin{aligned} |\mathcal{C}| &= |\mathcal{C}_1| + |\mathcal{C}_2| \\ &\leq (m_1 - 1) + (m_2 - 1) \\ &= m - 1. \end{aligned}$$

Therefore, $\mathcal{C}$ is an SCDC of $L(G)$.

■

As a consequence of this lemma and our previous comment, it suffices to prove Theorem 11 for connected graphs whose only bridges are pendants. Notice that such a graph is either a tree (in particular, a star), or it is a graph in which any vertex of degree greater than one has at least two *non-pendant* incident edges. Since we have already proved this result for trees (see Theorem 2), we need only consider the second case. Finally, observe that we need only prove the theorem for plane graphs (i.e., planar graphs embedded in the plane).

Let G be a connected plane graph with blocks $G_1, G_2, \ldots G_p$. For $1 \leq i \leq p$, if $|V(G_i)| \geq 3$, then we define $\mathcal{F}_i$ to be the set of facial cycles of G_i; if $|V(G_i)| < 3$, then $\mathcal{F}_i = \emptyset$. The *facial cycle double cover* (FCDC) of G is defined as

$$\mathcal{F} = \bigcup_{i=1}^{p} \mathcal{F}_i.$$

Observe that $\mathcal{F}$ is a collection of cycles such that every edge of G that is *not* a bridge lies in two of the cycles, and any edge of G that is a bridge lies in none of the cycles. Also, observe that if G is 2-connected, then the number of cycles in $\mathcal{F}$ is simply the number of faces of G.

Let G be a connected, 2-bridge-free plane graph with no non-trivial bridges, and let $\mathcal{F}$ denote the FCDC of G. As is the case for a CDC of a bridgeless graph, the FCDC $\mathcal{F}$ of G induces, at each vertex x of G, a system of transitions, $T(x)$. If x is incident to k pendants, then $T(x)$ consists of $d(x) - k$ transitions, no transition containing a pendant incident with x, and containing every other edge incident to x in two of the transitions. The transition multigraph, $M_T(x)$ is defined as before.

To construct a SCDC of $L(G)$, we require, for each vertex $x \in V(G)$ with $d(x) \geq 2$, a double cover of the edges of the vertex clique $K(x)$ with paths and cycles, such that the paths are compatible with the FCDC, $\mathcal{F}$, and such that the total number of cycles is not "too large".

For each $x \in V(G)$ with $d(x) \geq 2$, let $\mathcal{Z}(x)$ be a *path and cycle double cover* (PCDC) of $K(x)$: a collection of paths and cycles of $K(x)$ such that every edge of $K(x)$ lies in two elements of $\mathcal{Z}(x)$. Note that if $d(x) = 1$, then $K(x)$ has no edges, and hence no PCDC of $K(x)$ is required. The associated multigraph of $\mathcal{Z}(x)$, denoted $M_{\mathcal{Z}}(x)$, is defined as before. Define

$$\mathbf{Z} = \bigcup_{x \in V(G), d(x) \geq 2} \mathcal{Z}(x).$$

Then $\mathbf{Z}$ is a path and cycle double cover of $L(G)$. The FCDC $\mathcal{F}$ and the PCDC $\mathbf{Z}$ are compatible if and only if for each vertex x of G there is a bijection

$$f_x : T(x) \to \mathcal{Z}(x)$$

such that for every transition $\{ax, xb\} \in T(x)$, $f_x(\{ax, xb\})$ is a path in $\mathcal{Z}(x)$ with endpoints ax and xb. This is analogous to our earlier definition of compatibility for CDC's and PPDC's, and the following lemma is analogous to Lemma 3, providing an easy tool for checking compatibility.

Lemma 13 *Let G be a connected plane graph with no non-trivial bridges, $\mathcal{F}$ the facial cycle double cover of G, and*

$$\mathbf{Z} = \bigcup_{x \in V(G), d(x) \geq 2} \mathcal{Z}(x),$$

where $\mathcal{Z}(x)$ is a PCDC of the vertex clique $K(x)$ for each $x \in V(G)$. Then $\mathcal{F}$ and $\mathbf{Z}$ are compatible if and only if, for each vertex $x \in V(G)$, the transition multigraph $M_T(x)$ is isomorphic to the associated multigraph $M_{\mathcal{Z}}(x)$ of $\mathcal{Z}(x)$.

■

The next lemma is analogous to Lemma 4, and details how a FCDC $\mathcal{F}$ of G that is compatible with a PCDC $\mathbf{Z}$ of $L(G)$ can be used to construct a CDC of $L(G)$. For $\mathbf{Z}$ a PCDC of $L(G)$, let $\mathcal{C}(\mathbf{Z})$ denote the cycles of $\mathbf{Z}$.

Lemma 14 *Let G be a 2-bridge-free plane graph with no nontrivial bridges, $\mathcal{F}$ the FCDC of G, and $\mathbf{Z}$ a PCDC of $L(G)$, such that $\mathcal{F}$ and $\mathbf{Z}$ are compatible. For each $u \in V(G)$, fix a compatibility function f_u from $T(u)$ to $\mathcal{Z}(u)$. Let $c = v_0v_1v_2 \ldots v_{q-1}v_0$ be a cycle in $\mathcal{F}$, and for each i, $0 \leq i \leq q-1$, let $f_i = f_{v_i}$. By the definition of a compatibility function, $f_i(\{v_{i-1}v_i, v_iv_{i+1}\}) = P_i$, where P_i is a path in $\mathcal{Z}(v_i)$ with endpoints $v_{i-1}v_i$ and v_iv_{i+1} (where subscripts are taken modulo q). Then*

$$E_c = \bigcup_{i=0}^{q-1} P_i,$$

is an eulerian subgraph of $L(G)$ with maximum degree at most four. Furthermore, if for each $c \in \mathcal{F}$, $D(c)$ is the set of cycles in a cycle decomposition of E_c, then

$$\mathbf{C} = \left(\bigcup_{c \in \mathcal{F}} D(c) \right) \cup \mathcal{C}(\mathbf{Z})$$

is a cycle double cover of $L(G)$, and $|\mathbf{C}| \leq |\mathcal{F}| + ch(\mathcal{F}) + |\mathcal{C}(\mathbf{Z})|$.

∎

For a vertex $x \in V(G)$, the transition multigraph, $M_T(x)$, has one of the following three forms, depending on whether or not x is a cut vertex, and on whether or not x has any incident pendants.

(1) If x is not a cut vertex of G, then $M_T(x)$ is a single cycle of length $d(x)$.

(2) If x is a cut vertex so that $G - \{x\}$ has one nontrivial component and $k > 0$ trivial components, then $M_T(x)$ consists of a single cycle of length $d(x) - k$, and k isolated vertices

(3) If x is a cut vertex such that $G - \{x\}$ has $q \geq 2$ nontrivial components $X_1, X_2, \ldots, X_q$, with $k_i \geq 2$ edges from x to X_i, $1 \leq i \leq q$, and $d(x) - (k_1 + k_2 + \cdots + k_q)$ trivial components, then $M_T(x)$ consists of q cycles with lengths $k_1, k_2, \ldots, k_q$, and $d(x) - (k_1 + k_2 + \cdots + k_q)$ isolated vertices.

In each of these cases, we must construct a PCDC, $\mathcal{Z}(x)$, of the complete graph on $d(x)$ vertices so that $M_{\mathcal{Z}}(x)$ is isomorphic to $M_T(x)$. This motivates the following definition.

For $q \geq 1$, let $k_1, k_2, \ldots, k_q$ be integers with $k_i \geq 2$, $1 \leq i \leq q$. A $(k_1, k_2, \ldots, k_q)$-*path-and-cycle double cover* ($(k_1, k_2, \ldots, k_q)$-PCDC), $\mathcal{Z}$, of the complete graph on m vertices, K_m, is a collection of $k_1+k_2+\cdots+k_q$ paths and $m-(k_1+k_2+\cdots+k_q)+(q-1)$ cycles such that

(a) every edge of K_m lies in exactly two elements of $\mathcal{Z}$;

(b) for $1 \leq i \leq q$, there exists $X_i \subseteq V(K_m)$ with $|X_i| = k_i$ such that the X_i are pairwise disjoint, and there exists $\mathcal{Z}_i \subseteq \mathcal{Z}$, with $|\mathcal{Z}_i| = |X_i|$, such that every vertex of X_i is the endpoint of precisely two paths of $\mathcal{Z}_i$.

If $q = 1$, then we write k_1-PCDC instead of (k_1)-PCDC. Notice that if $q = 1$ and $k_1 = m$, then $\mathcal{Z}$ is simply a PPDC of K_m; i.e., an m-PCDC of K_m is a PPDC of K_m.

Lemma 15 *Let $m \geq 2$, and let $k_1, k_2, \ldots, k_q$, $q \geq 1$, be integers with $k_i \geq 2$, $1 \leq i \leq q$, and $k_1 + k_2 + \cdots k_q \leq m$. Then K_m has a $(k_1, k_2, \ldots, k_q)$-PCDC.*

Proof: Let $V(K_m) = \{v_0, v_1, v_2, \ldots, v_{m-2}, v_\infty\}$. For each j, $0 \leq j \leq m-2$, define the cycle C_j as follows:

$$C_j = \begin{cases} v_j v_{j+1} v_{j+m-2} v_{j+2} v_{j+m-3} \cdots v_{j+(m-2)/2} v_{j+m/2} v_\infty v_j & \text{if } m \text{ is even;} \\ v_j v_{j+1} v_{j+m-2} v_{j+2} v_{j+m-3} \cdots v_{j+(m+1)/2} v_{j+(m-1)/2} v_\infty v_j & \text{if } m \text{ is odd,} \end{cases}$$

where the subscripts are taken modulo $m-1$. One can verify that the collection of cycles $\mathbf{C} = \{C_j : 0 \leq j \leq m-2\}$ is a CDC of K_m with $m-1$ Hamilton cycles.

Let $r = k_1 + k_2 + \cdots + k_q$. Define $k_0 = 0$; for all j, $1 \leq j \leq q$, let

$$\alpha(j) = \sum_{i=0}^{j-1} k_i,$$

and then set

$$X_j = \{v_{\alpha(j)}, v_{\alpha(j)+1}, v_{\alpha(j)+2}, \ldots, v_{\alpha(j)+k_j-1}\},$$

with one exception: in case $r = m$, set

$$X_q = \{v_{\alpha(q)}, v_{\alpha(q)+1}, v_{\alpha(q)+2}, \ldots, v_{\alpha(q)+k_q-2}, v_\infty\}.$$

Notice that $|X_j| = k_j$, and that the X_j are pairwise disjoint.

We modify the cycles of **C** to obtain a $(k_1, k_2, \ldots, k_q)$-PCDC, $\mathcal{Z}$, as follows: for each j, $1 \leq j \leq q$, and for each i, $0 \leq i \leq k_j - 2$,

- let $P_{\alpha(j)+i} = C_{\alpha(j)+i} - \{v_{\alpha(j)+i}v_{\alpha(j)+i+1}\}$;
- in the case where $r = m$, then for $j = q$ and $i = k_q - 2$, let $P_{\alpha(q)+k_q-2} = C_{\alpha(q)+k_q-2} - \{v_{\alpha(q)+k_q-2}v_\infty\}$;
- let $P_{\alpha(j)+k_j-1} = v_{\alpha(j)}v_{\alpha(j)+1}v_{\alpha(j)+2} \cdots v_{\alpha(j)+k_j-2}v_{\alpha(j)+k_j-1}$;
- in the case where $r = m$, then for $j = q$ and $i = k_q - 2$, let $P_{\alpha(q)+k_q-1} = v_{\alpha(q)}v_{\alpha(q)+1}v_{\alpha(q)+2} \cdots v_{\alpha(q)+k_q-2}v_\infty$.

Then $\mathcal{Z}_j = \{P_{\alpha(j)}, P_{\alpha(j)+1}, P_{\alpha(j)+2}, \ldots, P_{\alpha(j)+k_j-1}\}$ is a collection of paths that covers the same edges, with the same multiplicities, as the cycles $\{C_{\alpha(j)}, C_{\alpha(j)+1}, C_{\alpha(j)+2}, \ldots, C_{\alpha(j)+k_j-2}\}$. Also, $|\mathcal{Z}_j| = |X_j| = k_j$, and every vertex of X_j is the endpoint of exactly two paths of $\mathcal{Z}_j$. It is a straightforward exercise to verify that $\mathcal{Z}$ does consist of r paths and $m - r + (q-1)$ cycles, and is thus a $(k_1, k_2, \ldots, k_q)$-PCDC, $\mathcal{Z}$, as required.

■

The next result follows immediately from the construction described in Lemma 15.

Corollary 16 *Let $m \geq 2$, and let $k_1, k_2, \ldots, k_q$, $q \geq 1$, be integers with $k_i \geq 2$, $1 \leq i \leq q$, and $k_1 + k_2 + \cdots k_q \leq m$. Then K_m has a $(k_1, k_2, \ldots, k_q)$-PCDC, $\mathcal{Z}$, such that the associated multigraph, $M_{\mathcal{Z}}(K_m)$, consists of q cycles of lengths $k_1, k_2, \ldots, k_q$, and $m - (k_1 + k_2 + \ldots + k_q)$ isolated vertices.*

■

This corollary ensures that we can find a PCDC **Z** of $L(G)$ that is compatible with the FCDC $\mathcal{F}$ of G. The next result guarantees that the CDC of $L(G)$ that we construct using Lemma 14 has the required number of cycles.

Lemma 17 *If G is a connected bridgeless plane graph with $m > 0$ edges and b blocks, then the facial cycle double cover of G, $\mathcal{F}$, is a cycle double cover with the property that*

$$|\mathcal{F}| + ch(\mathcal{F}) + b \leq m.$$

Proof: Let G be a connected bridgeless plane graph with n vertices, m edges, and b blocks. It follows immediately from the definition of the FCDC that $\mathcal{F}$ is a CDC of G.

The proof of the rest of the result is by induction on the number of vertices, n. When $n = 3$, G is a cycle of length three (so $m = 3$), and two copies of this cycle constitute the FCDC $\mathcal{F}$ of G. In this case, the cycles of $\mathcal{F}$ are chordless, and $b = 1$, so $|\mathcal{F}| + ch(\mathcal{F}) + b = 2 + 0 + 1 = 3 = m$, and hence the result holds.

Suppose now that G is a connected bridgeless plane graph with $n \geq 4$ vertices, b blocks, and m edges, and that the result holds for all connected bridgeless planar graphs on less than n vertices. There are three of cases to consider.

Case 1. Suppose that G has a cut vertex, x. Then there exist connected subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \{x\}$. Let m_i and b_i, respectively, denote the number of edges and blocks in G_i, $i = 1, 2$; then $m_1 + m_2 = m$ and $b_1 + b_2 = b$. Since $|V(G_i)| < n$, $i = 1, 2$, we may apply the induction hypothesis, and so the FCDC $\mathcal{F}_i$ of G_i has the property that $|\mathcal{F}_i| + ch(\mathcal{F}_i) + b_i \leq m_i$. We see that the FCDC $\mathcal{F}$ of G is simply the union of $\mathcal{F}_1$ and $\mathcal{F}_2$, and that $ch(\mathcal{F}) = ch(\mathcal{F}_1) + ch(\mathcal{F}_2)$. Therefore,

$$\begin{aligned}
|\mathcal{F}| + ch(\mathcal{F}) + b &= |\mathcal{F}_1| + |\mathcal{F}_2| + ch(\mathcal{F}_1) + ch(\mathcal{F}_2) + b_1 + b_2 \\
&= (|\mathcal{F}_1| + ch(\mathcal{F}_1) + b_1) + (|\mathcal{F}_2| + ch(\mathcal{F}_2) + b_2) \\
&\leq m_1 + m_2 = m.
\end{aligned}$$

Case 2. Suppose that G is 2-connected, but that no 2-vertex-cut of G is an edge. In this case, the FCDC $\mathcal{F}$ of G is simply the CDC of G by facial cycles; since no 2-vertex-cut of G is an edge, the facial cycles are chordless, and thus $ch(\mathcal{F}) = 0$; also, $b = 1$. Therefore,

$$\begin{aligned}
|\mathcal{F}| + ch(\mathcal{F}) + b &= f(G) + 1 \\
&= m - n + 3, \text{ by Euler's formula,} \\
&\leq m, \text{ since } n \geq 3.
\end{aligned}$$

Case 3. Finally, suppose that G is 2-connected, but that G has a vertex cut $\{x, y\}$ such that xy is an edge of G. In this case, $b = 1$ and there exist 2-connected subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$, and $G_1 \cap G_2$ consists of the vertices x, y, and the edge xy. Without loss of generality, we may assume that G is embedded so that xy is an edge of the outer (infinite) face of G_1 and of G_2. Let m_i denote the number of edges of G_i, $i = 1, 2$; then $m_1 + m_2 = m + 1$. Also, if b_i is the number of blocks of G_i, then $b_i = 1$, $i = 1, 2$. Let $\mathcal{F}_i$ denote the FCDC of G_i, $i = 1, 2$, and let $C_i \in \mathcal{F}_i$ be the cycle corresponding to the outer face of G_i.

The FCDC $\mathcal{F}$ of G can be described as follows:

$$\mathcal{F} = (\mathcal{F}_1 - \{C_1\}) \cup (\mathcal{F}_2 - \{C_2\}) \cup \{C_1 \Delta C_2\},$$

where $C_1 \Delta C_2$ denotes the symmetric difference of C_1 and C_2. Thus we have $|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| - 1$. Also, xy is now a chord of the cycle $C_1 \Delta C_2$ of $\mathcal{F}$, and hence

$ch(\mathcal{F}) = ch(\mathcal{F}_1) + ch(\mathcal{F}_2) + 1$. Since $|V(G_i)| < n$, $i = 1, 2$, we apply the induction hypothesis to obtain $|\mathcal{F}_i| + ch(\mathcal{F}_i) \leq m_i - 1$. Therefore,

$$\begin{aligned} |\mathcal{F}| + ch(\mathcal{F}) + b &= (|\mathcal{F}_1| + |\mathcal{F}_2| - 1) + (ch(\mathcal{F}_1) + ch(\mathcal{F}_2) + 1) + 1 \\ &= (|\mathcal{F}_1| + ch(\mathcal{F}_1)) + (|\mathcal{F}_2| + ch(\mathcal{F}_2)) + 1 \\ &\leq (m_1 - 1) + (m_2 - 1) + 1 \\ &= m. \end{aligned}$$

This completes the proof of the Lemma.

■

The next corollary follows immediately from this result.

Corollary 18 *If G is a connected plane graph with no non-trivial bridges, having $m > 0$ edges and b blocks, then the facial cycle double cover of G, $\mathcal{F}$, has the property that*

$$|\mathcal{F}| + ch(\mathcal{F}) + b \leq m.$$

■

Proof of Theorem 11: It suffices to prove this result for connected plane graphs. In addition, Lemma 12 ensures that we need only prove the theorem for graphs with no nontrivial bridges; i.e., graphs whose only bridges are pendants. Thus, let G be a connected 2-bridge-free plane graph with no nontrivial bridges.

Let m denote the number of edges of G, and let $\mathcal{F}$ denote the FCDC of G. Let $x \in V(G)$, $d(x) \geq 2$; as remarked earlier, the associated multigraph $M_T(x)$ of the transitions $T(x)$ induced by $\mathcal{F}$ is one of the following.

(i) $M_T(x)$ is a single cycle of length $d(x)$, provided that x is not a cut vertex of G.

(ii) $M_T(x)$ consists of a single cycle of length $d(x) - k$, and k isolated vertices, provided x is a cut vertex so that $G - \{x\}$ has one nontrivial component and $k > 0$ trivial components.

(iii) $M_T(x)$ consists of q cycles with lengths $k_1, k_2, \ldots, k_q$, and $d(x) - (k_1 + k_2 + \cdots + k_q)$ isolated vertices, provided that x is a cut vertex such that $G - \{x\}$ has $q \geq 2$ nontrivial components $X_1, X_2, \ldots, X_q$, with $k_i \geq 2$ edges from x to X_i, $1 \leq i \leq q$, and $d(x) - (k_1 + k_2 + \cdots + k_q)$ trivial components.

In the first case, (i), let $\mathcal{Z}(x)$ be an EPPDC of $K(x)$; this exists by Lemma 8, and also by Corollary 16, and thus ensures that $M_{\mathcal{Z}}(x)$ is isomorphic to $M_T(x)$. In the second case, (ii), let $\mathcal{Z}(x)$ be a k-PCDC of $K(x)$ with the property that $M_{\mathcal{Z}}(x)$ is isomorphic to $M_T(x)$; the existence of such a PCDC is guaranteed by Corollary 16. Finally, in the third case, (iii), let $\mathcal{Z}(x)$ be a $(k_1, k_2, \ldots, k_q)$-PCDC of $K(x)$ with the property that $M_{\mathcal{Z}}(x)$ and $M_T(x)$ be isomorphic; again, such a PCDC exists by Corollary 16.

We now define

$$\mathbf{Z} = \bigcup_{x \in V(G), d(x) \geq 2} \mathcal{Z}(x).$$

Since each $\mathcal{Z}(x)$ is a PCDC of $K(x)$, it follows that $\mathbf{Z}$ is a PCDC of $L(G)$. By Lemma 13, $\mathcal{F}$ and $\mathbf{Z}$ are compatible, and thus, we can apply Lemma 14 to construct a CDC, $\mathbf{C}$, of $L(G)$. It also follows from Lemma 14 that

$$|\mathbf{C}| \leq |\mathcal{F}| + ch(\mathcal{F}) + |\mathcal{C}(\mathbf{Z})|,$$

where $ch(\mathcal{F})$ is the number of chords of the cycles of $\mathcal{F}$, and $|\mathcal{C}(\mathbf{Z})|$ is the number of cycles in $\mathbf{Z}$.

To show that $\mathbf{C}$ is, in fact, an SCDC of $L(G)$, we must first evaluate $|\mathcal{C}(\mathbf{Z})|$. Let $x \in V(G)$, $d(x) \geq 2$, and denote by r_x the number of *non-pendant* edges incident to x and by q_x the number of nontrivial components of $G - \{x\}$. It follows from the definition of a $(k_1, k_2, \ldots, k_q)$-PCDC that $\mathcal{Z}(x)$ consists of r_x paths and $d(x) - 1 - r_x + q_x$ cycles. Therefore, the number of cycles in $\mathbf{Z}$ is simply

$$\begin{aligned}
|\mathcal{C}(\mathbf{Z})| &= \sum_{x \in V(G), d(x) \geq 2} (d(x) - r_x + (q_x - 1)) \\
&= \sum_{x \in V(G), d(x) \geq 2} (d(x) - r_x) + \sum_{x \in V(G), d(x) \geq 2} (q_x - 1).
\end{aligned}$$

Since $d(x) - r_x$ is the number of pendants incident to x, it follows that

$$\sum_{x \in V(G), d(x) \geq 2} (d(x) - r_x)$$

is simply the total number of pendant edges in the graph G.

To evaluate

$$\sum_{x \in V(G), d(x) \geq 2} (q_x - 1),$$

first observe that q_x does not change if the pendants incident with x are deleted. Therefore, first delete all pendant edges, along with the degree one vertices incident with those pendants from the graph G. What remains is a bridgeless graph, G' with b' blocks, each block corresponding to a block of G that is not a pendant. The blocks of G' form a tree, T, with b' edges corresponding to the blocks of G' and $b'+1$ vertices corresponding to the cut vertices of G'. If x is not a cut vertex of G (and hence of G'), or a cut vertex so that $G - \{x\}$ has just one nontrivial components, then $q_x = 1$, and x contributes nothing to the sum. However, if x is a cut vertex of G such that $G - \{x\}$ has at least two nontrivial components (and hence is a cut vertex of G'), then q_x is equal to the degree of x in the tree T. This implies that

$$\sum_{x \in V(G), d(x) \geq 2} (q_x - 1) = \sum_{x \in V(T)} (q_x - 1),$$

and,

$$\sum_{x \in V(T)} q_x = 2b'.$$

Therefore, since T is a tree,

$$\begin{aligned}
\sum_{x \in V(T)} (q_x - 1) &= 2b' - (b' + 1) \\
&= b' - 1.
\end{aligned}$$

It now follows that

$$\sum_{x \in V(G), d(x) \geq 2} (q_x - 1) = b' - 1.$$

But b' is just the number of blocks of G that are not pendants (single edges), and thus

$$\sum_{x \in V(G), d(x) \geq 2} (d(x) - r_x + (q_x - 1)) = b - 1,$$

where b is the number of blocks in G. Therefore, $|\mathcal{C}(\mathbf{Z})| = b - 1$.

It now follows from Lemma 14 that the CDC $\mathbf{C}$ of $L(G)$ has at most $|\mathcal{F}| + ch(\mathcal{F}) + (b-1)$ cycles. However, by Corollary 18, $|\mathcal{F}| + ch(\mathcal{F}) + b \leq m$, and thus

$$|\mathbf{C}| = |\mathcal{F}| + ch(\mathcal{F}) + (b - 1) \leq m - 1.$$

Therefore, $\mathbf{C}$ is an SCDC of $L(G)$.

■

References

[1] B. Alspach, L. Goddyn and C.-Q. Zhang, Graphs with the circuit cover property, *Trans. Amer. Math. Soc.* **344** (1994) 131–154.

[2] J.A. Bondy, Small cycle double covers of graphs, in *Cycles and Rays* (G. Hahn, G. Sabidussi and R. Woodrow, eds.), NATO ASI ser. C, Kluwer Academic Publishers, Dordrecht, 1990, pp. 21–40.

[3] J.A. Bondy, Perfect path double covers of graphs, *J. Graph Theory* **14** (1990) 259–272.

[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Elsevier, Amsterdam, 1976).

[5] L. Cai and D. Corneil, On cycle double covers of line graphs, *Discrete Math.* **102** (1992) 103–106.

[6] H. Fleischner, *Eulerian Graphs and Related Topics, Part 1, Volume 1*, Ann. of Discrete Math. **45**, North-Holland, Amsterdam, 1990, p. III.40.

[7] F. Jaeger, A survey of the cycle double cover conjecture, in *Cycles in Graphs* (B.R. Alspach and C.D. Godsil, eds.), Ann. of Discrete Math. **27**, North-Holland, Amsterdam, 1985, pp. 1–12.

[8] T.P. Kirkman, On a problem in combinations, *Cambridge and Dublin Math. J.* **2** (1847) 191–204.

[9] R. Klimmek, *Small Cycle Decompositions and Small Cycle Double Covers of Graphs*, Ph.D. Thesis, Freien Universität Berlin, 1997.

[10] H. Li, Perfect path double covers in every simple graph, *J. Graph Theory* **14** (1990) 645–650.

[11] K. Seyffarth, *Cycle and Path Covers of Graphs*, Ph.D. Thesis, University of Waterloo, 1989.

[12] K. Seyffarth, Maximal planar graphs of diameter two, *J. Graph Theory* **13** (1989) 619–648.

[13] K. Seyffarth, Small cycle double covers of 4-connected planar graphs, *Combinatorica* **13** (1993) 477–482.

[14] P.D. Seymour, Sums of circuits, in *Graph Theory and Related Topics* (J.A. Bondy and U.S.R. Murty, eds.), Academic Press, New York, 1979, pp. 341–355.

[15] G. Szekeres, Polyhedral decompositions of cubic graphs, *Bull. Austral. Math. Soc.* **8** (1973) 367–387.

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