Finite field Nullstellensatz and Grassmannians

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Abstract

Let $X \subset \mathbb{P}^N$ be a projective variety defined over the Galois field $GF(q)$. Denote by $X(q)$ the set of $GF(q)$–rational points of X. Let k be an integer. We say that the pair $(X, X(q))$ satisfies the *Finite Field Nullstellensatz of order* k, (the FFN(k), for short), if every homogeneous form of degree $\leq k$ on $\mathbb{P}^N(K)$ vanishing on $X(q)$, vanishes on $X(K)$. Here, we prove the Finite Field Nullstellensatz FFN(q) for any Grassmann variety.

1 Introduction

Let p be a prime and q be a power of p. Let $GF(q)$ be the Galois field of order q and let K denote the algebraic closure of $GF(q)$. Let $X \subset \mathbb{P}^N$ be a projective variety defined over $GF(q)$. Let $X(q)$ (resp. $X(K)$) denote the set of all $GF(q)$–rational points (resp. the K–rational points) of X. Then $X(q)$ is a finite subset of $\mathbb{P}^N$. We will denote the N–dimensional projective space over $GF(q)$ by $\mathbb{P}^N(q)$.

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When X is a geometrically interesting variety, the homogeneous ideal of X (i.e. the homogeneous ideal of $X(K)$) is often known, and the knowledge of FFN(k) gives a complete description, up to degree k, of the homogeneous ideal of $X(q)$.

In this note, we will consider the case in which X is a *Grassmann variety,* and the embedding $X \subset \mathbb{P}^N$ is given by the Plücker embedding.

Recall that any Grassmannian, say G, is defined over $GF(p)$, and that the Plücker embedding of G is defined over $GF(p)$. Hence, both are defined over $GF(q)$, and so we may consider FFN(k) for the pair $(G, G(q))$ with respect to the Plücker embedding. For more details on Grassmannians, see [5], [6], [7].

The aim of this note is to prove the following theorem.

Theorem 1.1 *Let $G \subset \mathbb{P}^N$ be the Plücker embedding of any Grassmannian. Then the pair $(G, G(q))$ satisfies FFN(q).*

Remark 1.2 Theorem 1.1 is sharp because $\mathbb{P}^N(q)$ is the union of the $GF(q)$–rational points of $q+1$ hyperplanes defined over $GF(q)$. Hence, for any subvariety X of $\mathbb{P}^N$ defined over $GF(q)$, the pair $(X, X(q))$ never has property FFN($q+1$).

In dealing with Grassmann spaces, Moorhouse in [4, Theorem 2] proved that FFN($q-1$) is true for the Plücker embedding of any Grassmannian. Our theorem is stronger then Moorhouse's result and the approach is different. In particular, [4, Theorem 2] is a consequence of Theorem 1.1. In the case of the Klein quadric, that is, for the Plücker embedding of the Grassmannian of lines of $\mathbb{P}^3$, see [2]. For other results on the Finite Field Nullstellensatz, see [1].

Over the last decade, there have been numerous papers, too many to be quoted here, dealing with the p–rank of incidence matrices of classes of incidence structures, such as projective spaces, translation planes, orthogonal spaces, generalized quadrangles, unitals, designs, Hermitian varieties, Grassmann varieties. Our result is a contribution to current research on p–ranks; nevertheless, we consider our techniques a significant step toward the FFN problem for other varieties, which we feel has an independent interest.

2 The Proof of Theorem 1.1

For any integers x, n, with $0 \leq x < n$, let $G(x, n)$ be the Grassmannian of x–dimensional projective linear subspaces of $\mathbb{P}^n$. Hence $\dim(G(x,n)) = (n-x)(x+1)$. It turns out that $G(0, n) = \mathbb{P}^n$ and $G(n-1, n) = \mathbb{P}^{n*} \cong \mathbb{P}^n$.

Set $N = N(x, n) := ((n+1)!/((n-x)!(x+1)!)) - 1$. The Plücker embedding of $G(x, n)$ embeds $G(x, n)$ into $\mathbb{P}^N$.

Let $X_0, \ldots, X_n$ be homogeneous coordinates in $\mathbb{P}^n$.

We will prove Theorem 1 for $G(x, n)$ by induction on n. The case $x = 0$ just means that any homogeneous polynomial $f(X_0, \ldots, X_n)$ with $\deg(f) \leq q$ and with f vanishing at each point of $\mathbb{P}^n(q)$, is identically zero. Hence the case $x = 0$ is trivially true for any n. Now, we fix the integer x with $0 < x \leq n-2$, and we assume the result holds true for the pair $(n-1, x)$. We regard $G(x, n)$ as embedded in $\mathbb{P}^N$ by the Plücker embedding. Fix an integer $t \leq q$ and let $f \in K[X_0, \ldots, X_N]$ be a homogeneous polynomial with deg $(f) = t$ and $f(P) = 0$ for every point $P \in G(x, n)(q)$. The projective space $\mathbb{P}^n(q)$ has $a := (q^{n+1}-1)/(q-1)$ hyperplanes, and each of them defines an embedding of $G(x, n-1)(q)$ into $G(x, n)(q)$, and of $G(x, n-1)(K)$ into $G(x, n)(K)$. Let $\{H_i\}$, $1 \leq i \leq a$, be the set of all hyperplanes of $\mathbb{P}^n(q)$ and $\alpha_i : G(x, n-1) \to G(x, n)$ the associated embedding. Notice that the restriction to $G(x, n-1)$ of the Plücker embedding of $G(x, n)$ induces the Plücker embedding of $G(x, n-1)$. Hence, by the inductive assumption, we may assume that for every $1 \le i \le a$, f vanishes on every point of $\alpha_i(G(x, n-1))(K)$. It is sufficient to prove that f vanishes at a general $Q \in G(x, n)(K)$.

Any such general point Q is contained in a line $D \subset G(x, n) \subset \mathbb{P}^N$ defined over K and corresponds to fixing a $(x+1)$–dimensional linear subspace V of $\mathbb{P}^n$ and taking all x–dimensional linear subspaces of V. We choose a general such line D. By the generality of Q and of D, we may assume that $V(K)$ is a general $(x+1)$–dimensional linear subspace of $\mathbb{P}^n(K)$. Hence, we may assume that $V(K)$ intersects transversally each hyperplane $H_i(K)$, $1 \le i \le a$, that is, $\dim(V(K) \cap H_i(K)) = x$, for every $1 \le i \le a$, and that $V(K) \cap H_i(K) \neq V(K) \cap H_j(K)$, for $i \neq j$. Thus D intersects each $\alpha_i(G(x-1, n)(K))$ at a different point. Since $\deg(f) = t \le q < a$, we obtain $f|D \cong 0$. Since $Q \in D$, we obtain $f(Q) = 0$, as wanted. Now, we consider the case $x = n-1$. We have $G(n-1, n) = \mathbb{P}^{n*} \cong \mathbb{P}^n$, and the proof is as for the case $x = 0$.

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