The number of h-strongly connected digraphs with small diameter

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Abstract

Let $D_s(n;h,d=k)$ denote the number of h-strongly connected digraphs of order n and diameter equal to k. In this paper it is shown that:

i) $D_s(n;h,d=3) = 4^{\binom{n}{2}}(3/4+o(1))^n$ for every fixed $h \geq 1$;

ii) $D_s(n;h,d=4) = 4^{\binom{n}{2}}(2^{-h-2}+2^{-2}+o(1))^n$ for every fixed $h \geq 2$;

iii) $D_s(n;h,d=k) = 4^{\binom{n}{2}}((2^{h+1}-1)2^{-kh+3h-2}+o(1))^n$ for every fixed $h \geq 1$ and $k \geq 5$.

Similar asymptotic formulas hold for the number of h-connected digraphs of order n and diameter equal to k when $n \to \infty$. This extends the corresponding results for h-connected graphs given in a recent paper by the author.

1 Notation and preliminary results

All digraphs in this paper are finite, labeled, without loops or parallel directed edges. By K_n^* we denote the complete digraph of order n such that any two distinct vertices x and y are joined by two directed edges (x,y) and (y,x). For a digraph G the outdegree $d^+(x)$ of a vertex x is the number of vertices of G that are adjacent from x and the indegree $d^-(x)$ is the number of vertices of G adjacent to x. For $h \geq 2$, we say that a digraph G is h-connected (resp. h-strongly connected) if either G is a complete digraph K_{h+1}^* or else it has at least $h+2$ vertices and for any set of vertices $X \subset V(G)$, $|X| = h-1$, the digraph $G-X$ is connected (resp. strongly connected). A connected (resp. strongly connected) digraph is also said to be 1-connected (resp. 1-strongly connected). For a strongly connected digraph G the distance $d(x,y)$ from vertex x to vertex y is the length of a shortest path of the form $(x,\ldots,y)$. The eccentricity of a vertex x is $\mathrm{ecc}(x) = \max_{y\in V(G)} d(x,y)$. The diameter of G, denoted

$d(G)$ is equal to $\max_{x,y\in V(G)} d(x,y)$ if G is strongly connected and ∞ otherwise. By $D_s(n;h,d=k)$ and $D_s(n;h,d\geq k)$ (resp. $D(n;h,d=k)$ and $D(n;h,d\geq k)$) we denote the number of h-strongly connected (resp. h-connected) digraphs G of order n and diameter $d(G)=k$ and $d(G)\geq k$, respectively.
It is well known [1, p. 131] that almost all digraphs have diameter two and for every fixed integer $h\geq 1$ almost all graphs are h-connected. Also in [2] it was proved that for every fixed integer $h\geq 1$ almost all digraphs are h-strongly connected. Hence for every $h\geq 1$ we have:

$$D_s(n;h,d=2)=4^{\binom{n}{2}}(1+o(1)) \text{ and } D(n;h,d=2)=4^{\binom{n}{2}}(1+o(1)).$$

If $\lim_{n\to\infty}\frac{f(n)}{g(n)}=1$ we denote this by $f(n)\sim g(n)$, or $f(n)=g(n)(1+o(1))$. The following results will be useful in the proofs of the theorems given in the next section.

Lemma 1.1 ([4]). *The number of bipartite digraphs G whose partite sets are A, B ($A\cap B=\emptyset$, $|A|=p$, $|B|=q$) such that $d^-(x)\geq 1$ for every $x\in B$ and all edges are directed from A towards B is equal to $(2^p-1)^q$.*

Lemma 1.2 ([4]). *We have*

$$D_s(n;1,d=3)=4^{\binom{n}{2}}(3/4+o(1))^n.$$

Also we need an asymptotic evaluation of the maximum of an arithmetical function. Let

$$f(n,h;n_1,\ldots,n_k)=\binom{n}{n_1,\ldots,n_k}2^{\sum_{i=1}^{k}\binom{n_i}{2}}\prod_{i=1}^{k-1}(2^{n_i}-1)^{n_{i+1}}$$

where $n_1+\ldots+n_k=n$, $n_i\geq h$ for every $1\leq i\leq k-1$ and $n_k\geq 1$. Let us denote

$$f(n,k)=\max_D f(n,h;n_1,\ldots,n_k),$$

where D is defined by: $n_1+\ldots+n_k=n$; $n_i\geq h$ for every $1\leq i\leq k-1$ and $n_k\geq 1$.

Theorem 1.3 ([5]). *The following equalities hold:*

$$f(n,h,4)=2^{\binom{n}{2}}(2^{-h-1}+2^{-1}+o(1))^n \tag{1}$$

for every $h\geq 2$;

$$f(n,h,k)=2^{\binom{n}{2}}((2^{h+1}-1)2^{-kh+3h-1}+o(1))^n \tag{2}$$

for every $h\geq 2$ and $k\geq 5$.

Note that (2) also holds for $h=1$ [3]. Moreover, for $k=4$, $f(n,h;n_1,\ldots,n_4)$ can be maximum only if $n_1=\alpha_1(n,h,4)$, $n_2=\beta_1(n,h,4)$, $n_3=h$ and $n_4=1$, where

$$\alpha_1(n,h,4)=(n-h)\frac{1}{2^h+1}-\gamma,$$

$$\beta_1(n,h,4)=(n-h)\frac{2^h}{2^h+1}-1+\gamma, \tag{3}$$

and $0 \le \gamma \le 1$.
For $k \ge 5$, $f(n,h,k) = f(n,h;h,\dots,h,\alpha_0,\beta_0,h,\dots,h,1)$, where

$$\alpha_0(n,h,k) = (n - kh + 3h)\frac{2^h-1}{2^{h+1}-1} - \gamma;$$

$$\beta_0(n,h,k) = (n - kh + 3h)\frac{2^h}{2^{h+1}-1} - 1 + \gamma, \tag{4}$$

and $0 \le \gamma \le 1$.

Notice that for $h = 1$ the explanation of the asymptotic behavior of the critical function $f(n,h,k)$, denoted by $f(n,k)$ was made by a careful analysis in [3].

Lemma 1.4 *(i) If G is an h-strongly connected digraph, $x \notin V(G)$ and x is joined by directed edges in both directions (x,y) and (y,x) with at least h distinct vertices y in G, the resulting digraph is h-strongly connected.*
(ii) If E and F are two h-strongly connected digraphs such that $V(E) \cap V(F) = \emptyset$, joined by directed edges in both directions (x_i,y_i) and (y_i,x_i) $(1 \le i \le h)$ which join h distinct vertices x_i in E $(1 \le i \le h)$ with h distinct vertices y_j in F $(1 \le j \le h)$, the resulting digraph is h-strongly connected. The property holds even if E or F is isomorphic to K_h^*.

Note that this lemma holds if h-strongly connectedness is replaced by h-connectedness.

2 Main results

We will deduce an estimation for $D_s(n;h,d=k)$ for every fixed $h \ge 2$ and $k \ge 3$ as $n \to \infty$, by considering first the case $k = 3$, when this does not depend on h.

Theorem 2.1 *We have*

$$D_s(n;h,d=3) = 4^{\binom{n}{2}}(3/4 + o(1))^n$$

for every fixed $h \ge 1$.

Proof: For $h = 1$ this property was shown in [4]. If $D(n;d \ge k)$ denotes the number of digraphs G of order n and diameter $d(G) \ge k$, from the proof of Lemma 1.3 of [4] it follows that $D(n;d\ge 4) < (n^2-n)2^{\binom{n}{2}+\binom{n-2}{2}}(2^{n-2}+(5/2)^{n-2}) = 4^{\binom{n}{2}}(5/8+o(1))^n$. Since $D_s(n;h,d\ge 4) \le D(n;d\ge 4)$ one gets

$$D_s(n;h,d\ge 4) < 4^{\binom{n}{2}}(5/8+o(1))^n. \tag{5}$$

Let $A_{ij}^{(k)}$, respectively $H_{ij}^{(k)}$, denote the set of digraphs (respectively h-strongly connected digraphs) having vertex set $\{1,\dots,n\}$ such that $d(i,j) \ge k$. In [4] it was shown that $|A_{ij}^{(3)}| = 3^{n-2}\cdot 2^{\binom{n}{2}+\binom{n-2}{2}}$. Since $|H_{ij}^{(3)}| \le |A_{ij}^{(3)}|$ we get

$$|H_{ij}^{(3)}| \le 4^{\binom{n}{2}}(3/4+o(1))^n. \tag{6}$$

Now a sufficiently large subset of $H_{ij}^{(3)}$ can be constructed as follows:
Consider an h-strongly connected digraph F with vertex set $\{1, \ldots, n\} \backslash \{i, j\}$ and nonadjacent vertices i and j such that the sets of neighbors $N(i)$, $N(j) \subset V(F)$ satisfy: $|N(i)| = |N(j)| = h$ and $N(i) \cap N(j) = \emptyset$. Vertices i and j are joined by directed edges in both directions with all vertices in $N(i)$ and $N(j)$, respectively. For every vertex $k \in V(F) \backslash \{N(i) \cup N(j)\}$ we suppose that the condition: $(i, k) \in E(G)$ implies $(k, j) \notin E(G)$ is fulfilled, where G denotes the digraph obtained on this way. By Lemma 1.4, G is h-strongly connected and the distance $d(i, j) \geq 3$. This implies that for every fixed choice of the subdigraph induced by $\{i, j\}$, for every $k \in V(F) \backslash \{N(i) \cup N(j)\}$ the subdigraph induced by $\{i, j, k\}$ can be chosen in exactly 12 ways. Hence $|H_{i,j}^{(3)}| \geq 12^{n-2h-2} D_s(n-2, h)$, where $D_s(n, h)$ denotes the number of h-strongly connected digraphs of order n. Since almost all digraphs of order n are h-strongly connected as $n \to \infty$, it follows that $D_s(n-2, h) \sim 4^{\binom{n-2}{2}}$, which implies $|H_{ij}^{(3)}| \geq 4^{\binom{n}{2}}(3/4 + o(1))^n$. Consequently,

$$|H_{ij}^{(3)}| = 4^{\binom{n}{2}}(3/4 + o(1))^n$$

for every $1 \leq i, j \leq n$ and $i \neq j$. Because $D_s(n; h, d \geq 3) = |\bigcup_{\substack{1 \leq i,j \leq n \\ i \neq j}} H_{ij}^{(3)}|$ and

$$|H_{i_0 j_0}^{(3)}| \leq |\bigcup_{\substack{1 \leq i,j \leq n \\ i \neq j}} H_{ij}^{(3)}| \leq (n^2 - n)|H_{i_0 j_0}^{(3)}|$$

one deduces that

$$D_s(n, h, d \geq 3) = 4^{\binom{n}{2}}(3/4 + o(1))^n. \tag{7}$$

Since $D_s(n; h, d = 3) = D_s(n; h, d \geq 3) - D_s(n; h, d \geq 4)$, the conclusion follows from (5) and (7). □

Because any h-strongly connected digraph is also h-connected, we get:

Corollary 2.2 *The following equality holds for every fixed $h \geq 1$:*

$$D(n; h, d = 3) = 4^{\binom{n}{2}}(3/4 + o(1))^n$$

Theorem 2.3 *We have:*

$$(i)\ D_s(n; h, d = 4) = 4^{\binom{n}{2}}(2^{-h-2} + 2^{-2} + o(1))^n$$

for every fixed $h \geq 2$;

$$(ii)\ D_s(n; h, d = k) = 4^{\binom{n}{2}}((2^{h+1} - 1)2^{-kh+3h-2} + o(1))^n$$

for every fixed $h \geq 1$ and $k \geq 5$.

Proof: For $h = 1$, (ii) was proved in [4]. Let $h \geq 2$, $k \geq 4$ and G be an h-strongly connected digraph of order n. If $x \in V(G)$ has $\mathrm{ecc}(x) = k$, then

$$V_1(x) \cup \ldots \cup V_k(x)$$

is a partition of $V(G) \backslash \{x\}$, where $V_i(x) = \{y \mid y \in V(G) \text{ and } d(x,y) = i\}$ for $1 \leq i \leq k$. It follows that there are directed edges from x towards all vertices of $V_1(x)$ and for every $2 \leq i \leq k$ and any vertex $z \in V_i(x)$ there exists a directed edge (t, z), where $t \in V_{i-1}(x)$. Also the h-strongly connectedness of G implies that $|V_i(x)| \geq h$ for every $i = 1, \ldots, k-1$. Let n_i be the number of vertices in $V_i(x)$, $1 \leq i \leq k$. By Lemma 1.1 one deduces
$|\{G \mid G \text{ is } h\text{-strongly connected}, V(G) = \{1, \ldots, n\} \text{ and } \mathrm{ecc}(x) = k\}|$
$\leq \sum_{\substack{n_1+\ldots+n_k=n-1 \\ n_1,\ldots,n_{k-1}\geq h, n_k \geq 1}} \binom{n-1}{n_1,\ldots,n_k} 4^{\sum_{i=1}^k \binom{n_i}{2}} \prod_{i=1}^{k-1}(2^{n_i}-1)^{n_{i+1}} \prod_{i=1}^{k} 2^{n_i(n_{i-1}+\ldots+1)}$
$= 2^{\binom{n}{2}} \sum_{\substack{n_1+\ldots+n_k=n-1 \\ n_1,\ldots,n_k \geq 1}} f(n-1; n_1, \ldots, n_k)$
because

$$2^{\sum_{i=1}^k \binom{n_i}{2}} \prod_{i=1}^{k} 2^{n_i(n_{i-1}+\ldots+1)} = 2^{\binom{n}{2}}. \tag{8}$$

Furthermore

$$\sum_{\substack{n_1+\ldots+n_k \\ n_1,\ldots,n_{k-1}\geq h, n_k\geq 1}} f(n-1; n_1, \ldots, n_k) \leq \binom{n-2}{k-1} f(n-1, k)$$

since the number of compositions $n - 1 = n_1 + \ldots + n_k$ having k positive terms equals $\binom{n-2}{k-1}$. Hence $D_s(n; h, d = k) \leq |\bigcup_{x \in V(G)} \{G \mid G \text{ is } h\text{-strongly connected}, V(G) = \{1, \ldots, n\} \text{ and } \mathrm{ecc}(x) = k\}| \leq n 2^{\binom{n}{2}} \binom{n-2}{k-1} f(n-1, h, k)$ and this expression equals $4^{\binom{n}{2}}(2^{-h-2} + 2^{-2} + o(1))^n$ for $k = 4$ and $4^{\binom{n}{2}}((2^{h+1}-1)2^{-kh+3h-2} + o(1))^n$ for $k \geq 5$ by Theorem 1.3. The proof of the theorem is by double inequality. We shall consider two cases: I $k \geq 5$ and II $k = 4$.

Case I. In order to produce a suitable lower bound for $D(n; h, d = k)$ in the case $k \geq 5$ we shall generate a large class of h-strongly connected digraphs of order n and diameter equal to k as follows: Let $x \in \{1, \ldots, n\}$ be a fixed vertex and $X_1 \cup \ldots \cup X_k$ be a partition of $\{1, \ldots, n\} \backslash \{x\}$ such that $|X_1| = |X_2| = \ldots = |X_{k-4}| = h$, $|X_{k-3}| = \alpha_0$, $|X_{k-2}| = \beta_0$, $|X_{k-1}| = h$ and $|X_k| = 1$, where $\alpha_0 = \alpha_0(n-1, h, k)$ and $\beta_0 = \beta_0(n-1, h, k)$ are given by (4). Vertex x is joined by directed edges in both directions with all vertices of X_1 and the unique vertex of X_k is joined by directed edges in both directions with all vertices of X_{k-1}. Let us denote $X_i = \{x_i^1, \ldots, x_i^h\}$ for every $1 \leq i \leq k-4$ and $i = k-1$. We choose an h-element subset $Y_{k-3} = \{x_{k-3}^1, \ldots, x_{k-3}^h\} \subset X_{k-3}$ and an h-element subset $\{x_{k-2}^1, \ldots, x_{k-2}^h\} \subset X_{k-2}$. Now for every $1 \leq i \leq k-2$ we join vertex x_i^j with x_{i+1}^j by directed edges (x_i^j, x_{i+1}^j) and (x_{i+1}^j, x_i^j) for every $j = 1, \ldots, h$. Every $X_1, X_2, \ldots, X_{k-4}$ and X_{k-1} induces a subdigraph isomorphic to K_h^* and subdigraphs induced by X_{k-3} and X_{k-2} are h-strongly connected and have diameter equal to two. Also for any vertex $u \in X_{k-3}$

there exists at least one directed edge (s,u), where $s \in X_{k-4}$ and for any vertex $v \in X_{k-2}$ there exists at least one directed edge (t,v), where $t \in X_{k-3}$. If G denotes a digraph generated by this procedure, it is easy to see that $|V(G)| = n$, $\mathrm{ecc}(x) = k$ and $d(G) = k$; by Lemma 1.4 it follows that G is h-strongly connected. The number of directed edges oriented from classes X_j towards classes X_i where $i < j$ is a function $\varphi(k,h)$ which does not depend on n.

The number of digraphs generated in this way is greater than or equal to $\binom{n-1}{\alpha_0}\binom{n-1-\alpha_0}{\beta_0}2^{\binom{n}{2}-\varphi(k,h)-\binom{\alpha_0}{2}-\binom{\beta_0}{2}}D_s(\alpha_0;h,d=2)D_s(\beta_0;h,d=2)(2^h-1)^{\alpha_0-h}(2^{\alpha_0}-1)^{\beta_0-h}2^{h(h-1)}2^{h(\alpha_0-1)}2^{h(\beta_0-1)}$ by Lemma 1.1 and (8). Indeed, each vertex $z \in X_{k-3} \setminus \{x^1_{k-3},\ldots,x^h_{k-3}\}$ must have at least one incoming edge from some vertex in X_{k-4}, hence there are 2^h-1 choices for the set of incoming edges to any such vertex. If $z = x^i_{k-3}$ $(1 \le i \le h)$, there exists the directed edge (x^i_{k-4}, x^i_{k-3}); hence there are 2^{h-1} choices for the set of incoming edges to any vertex in $\{x^1_{k-3},\ldots,x^h_{k-3}\}$. So the number of choices for the set of incoming edges to X_{k-3} is equal to $(2^h-1)^{\alpha_0-h}2^{h(h-1)}$. In a similar way we find the number of choices for the set of incoming edges to X_{k-2} and X_{k-1}. Since $D_s(\alpha;h,d=2) \sim 4^{\binom{\alpha}{2}}$ as $\alpha \to \infty$, this expression is equal to

$$2^{\binom{n}{2}}f(n-1,h,k)(1+o(1))^n = 4^{\binom{n}{2}}((2^{h+1}-1)2^{-kh+3h-2}+o(1))^n$$

by Theorem 1.3. Hence $D_s(n;h,d=k) \ge 4^{\binom{n}{2}}((2^{h+1}-1)2^{-kh+3h-2}+o(1))^n$ and the proof is complete in this case.

Case II. If $k = 4$ the construction is somewhat similar to the case $k \ge 5$:
We consider a partition $X_1 \cup X_2 \cup X_3 \cup X_4$ of $\{1,\ldots,n\} \setminus \{x\}$ such that $|X_1| = \alpha_1(n-1,h,4)$, $|X_2| = \beta_1(n-1,h,4)$ (given by (3)), $|X_3| = h$ and $|X_4| = 1$. Let $X_4 = \{w\}$.
We choose any vertex $t \in X_2$ and join t with x by a directed edge (t,x). By choosing $Y_1 \subset X_1$ and $Y_2 \subset X_2$ the remaining adjacencies are defined as for the case $k \ge 5$. Let us denote the set of h-strongly connected digraphs of order n produced in this way by $\mathcal{G}$. If $G \in \mathcal{G}$, we have $d(x,w) = 4$; also $d(u,v) \le 4$ for every $u,v \in V(G)$ unless $u \in X_1$ and $v = w$, when we have only $d(u,w) \le 5$. If $G \in \mathcal{G}$ has $d(G) = 5$ we define the digraph $\varphi(G)$ deduced from G by deleting directed edges joining w in both directions with vertices of X_3 and replacing them by directed edges joining w in both directions with the h vertices of $Y_2 \subset X_2$. We have $d_{\varphi(G)}(x,w) = 3$. If $u \in X_1$ has $d_G(u,w) = 5$ then $d_G(u,Y_2) = 3$, which implies $d_{\varphi(G)} = 4$, hence $\varphi(G)$ has diameter equal to four. If the vertex w in X_4 is fixed, the ordered partition $X_1 \cup X_2 \cup X_3$ can be generated in

$$\binom{n-2}{\alpha_1}\binom{n-2-\alpha_1}{\beta_1} = \frac{(n-1)!}{\alpha_1!\beta_1!}(1+o(1))^n$$

ways. In this case φ is injective and for every $F,G \in \mathcal{G}$ we have $\varphi(G) \ne F$ since $d_F(x,w) = 4$ but $d_{\varphi(G)}(x,w) = 3$.
Hence we can generate a class consisting of $|\mathcal{G}|$ h-strongly connected digraphs of order n and diameter equal to four as follows: we choose a digraph $G \in \mathcal{G}$ if $d(G) = 4$; otherwise we choose the digraph $\varphi(G)$.

It follows that the number of digraphs generated in this way is equal to $|\mathcal{G}| = \frac{(n-1)!}{\alpha_1!\beta_1!}2^{\binom{n}{2}-\varphi(4,h)-\binom{\alpha_1}{2}-\binom{\beta_1}{2}}D_s(\alpha_1;h,d=2)D_s(\beta_1;h,d=2)(2^{\alpha_1}-1)^{\beta_1-h}2^{h(\alpha_1-1)}$ $2^{h(\beta_1-1)}(1+o(1))^n$, where $\varphi(k,h)$ was defined in the case $k \geq 5$. As for the case I the last expression is equal to

$$2^{\binom{n}{2}}f(n-1,h,4)(1+o(1))^n = 4^{\binom{n}{2}}(2^{-h-2}+2^{-2}+o(1))^n$$

which concludes the proof. □

Corollary 2.4 *Equalities (i) and (ii) also hold for the numbers* $D(n;h,d=4)$ *and* $D(n;h,d=k)$ *of* h*-connected digraphs* G *of order* n *and diameter* $d(G)=4$*, respectively* $d(G)=k\geq 5$.

Corollary 2.5 *For every fixed* $h \geq 1$ *and* $k \geq 2$ *we have*

$$lim_{n\to\infty}\frac{D_s(n;h,d=k)}{D_s(n;h,d=k+1)} = lim_{n\to\infty}\frac{D(n;h,d=k)}{D(n;h,d=k+1)} = \infty.$$

Corollary 2.6 *The following equalities*

$$lim_{n\to\infty}\frac{D_s(n;h,d=k)}{D_s(n;h+1,d=k)} = lim_{n\to\infty}\frac{D(n;h,d=k)}{D(n;h+1,d=k)} = \infty$$

hold for every fixed $h \geq 1$ *and* $k \geq 4$.

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(Received 16/11/2000)