(M,S)-optimal designs with block size three

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Abstract

The existence of (M, S)-optimal designs for block size three is completely settled. Such a design is a collection of b triples on v elements so that for every two elements the numbers of triples containing them differ by at most one, and for every two pairs of elements the numbers of triples containing them also differ by at most one.

1 Introduction

We consider the following problem. Let v and b be positive integers, $v \geq 3$, and let V be a set of v elements. Let $r = \lfloor \frac{3b}{v} \rfloor$ and $\lambda = \lfloor \frac{6b}{v(v-1)} \rfloor$. When does there exist a collection $\mathcal{B}$ of b 3-element subsets of V (*triples*) with the properties that every $x \in V$ belongs to r or $r + 1$ triples of $\mathcal{B}$, and every pair of elements $\{x, y\} \subset V$ appears in λ or $\lambda + 1$ triples of $\mathcal{B}$?

This question arises from optimality considerations for experimental designs. Shah (1960) introduced an optimality criterion for block designs. This was called *S-optimality* by Kiefer (1974) and was later generalized by Eccleston and Hedayat (1974). They introduced a new optimality criterion, *(M, S)-optimality*. Shah (1960) indicated why this criterion should lead to designs with high efficiency with respect to A, D, or E optimality. John and Mitchell (1977) conjectured that a class of S-optimal designs called regular graph designs (RGD) are useful for establishing this (M, S)-optimality. They showed that if a binary design with v treatments and b blocks has an incidence matrix $N_{v\times b}$ such that the diagonal elements of NN' are either r or $r + 1$ for some r, and the off-diagonal elements of NN' are λ or $\lambda + 1$ for some λ, then the design is (M, S)-optimal. Roy (1982) proved:

Lemma 1.1 *If $v \equiv 3 \pmod 6$ and $b \geq 0$, then there exists a (M, S)-optimal design on v points with b blocks having block size three.*

To simplify the notation, we denote by $\mathrm{MS}(v, b)$ an (M, S)-optimal design on v points having b blocks with block size three. In Theorem 2.3, we provide a necessary and sufficient condition on v and b for an $\mathrm{MS}(v, b)$ to exist.

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2 Necessary Conditions

A $(v, 3, \lambda)$ covering (packing) is a pair $(V, \mathcal{B})$, where V is a v-set of elements called *points* and $\mathcal{B}$ is a collection of 3-subsets, called *blocks*, of V, such that every 2-subset of points occurs in at least λ (at most λ)) blocks in $\mathcal{B}$. Repeated blocks in $\mathcal{B}$ are permitted. A $(v, 3, \lambda)$-design is a pair $(V, \mathcal{B})$ which is both a packing and a covering; such designs exist if and only if $v \geq 3$, $\lambda(v-1) \equiv 0 \pmod 2$, and $\lambda v(v-1) \equiv 0 \pmod 3$ (see Colbourn and Rosa (1999)). Let $\lambda_{\min}(v)$ denote the minimum positive λ for which a $(v, 3, \lambda)$-design exists.

The covering (packing) number $C_\lambda(v)$ ($D_\lambda(v)$) is the minimum (maximum) number of blocks in any $(v, 3, \lambda)$ covering (packing). Let $U_\lambda(v) = \lfloor \frac{v}{3} \lfloor \frac{\lambda(v-1)}{2} \rfloor \rfloor$ and $L_\lambda(v) = \lceil \frac{v}{3} \lceil \frac{\lambda(v-1)}{2} \rceil \rceil$.

Theorem 2.1 *(See Colbourn and Rosa (1999))*

1. *If* $v \equiv 2 \pmod 3$, $\lambda \equiv 2 \pmod 3$ *and* $\lambda(v-1) \equiv 0 \pmod 2$, *then* $C_\lambda(v) = L_\lambda(v) + 1$. *Otherwise,* $C_\lambda(v) = L_\lambda(v)$.

2. *If* $v \equiv 2 \pmod 3$ *and* $\lambda \equiv 1 \pmod 3$ *and* $\lambda(v-1) \equiv 0 \pmod 2$, *then* $D_\lambda(v) = U_\lambda(v) - 1$. *Otherwise,* $D_\lambda(v) = U_\lambda(v)$.

We can obtain a necessary condition for the existence of a MS(v, b). Let $\overline{D}_\lambda(v)$ be $D_\lambda(v) - 1$ if $v \equiv 2 \pmod 3$, $\lambda \equiv 2 \pmod 3$, and $\lambda(v-1) \equiv 0 \pmod 2$; $\overline{D}_\lambda(v) = D_\lambda(v)$ otherwise. Let $\overline{C}_\lambda(v)$ be $C_\lambda(v) + 1$ if $v \equiv 2 \pmod 3$, $\lambda \equiv 1 \pmod 3$, and $\lambda(v-1) \equiv 0 \pmod 2$; $\overline{C}_\lambda(v) = C_\lambda(v)$ otherwise.

Lemma 2.2 *If* $\overline{D}_\lambda(v) < b < \overline{C}_\lambda(v)$ *for some positive integer* λ, *then there does not exist a MS(v, b).*

Proof: In an MS(v, b) with $D_\lambda(v) < b < C_\lambda(v)$, some pair of points is covered at least $\lambda + 1$ times since $b > D_\lambda(v)$. There also exists a pair of points covered at most $\lambda - 1$ times since $b < C_\lambda(v)$. Hence, the design is not (M, S)-optimal. To complete the proof, we must examine the four cases in which $\overline{D}_\lambda(v) < D_\lambda(v)$ or $\overline{C}_\lambda(v) > C_\lambda(v)$. For the packing cases, every maximum $(v, 3, \lambda)$-packing with $D_\lambda(v)$ blocks covers every pair λ times except for a single pair covered only $\lambda - 2$ times (this is obtained by counting). Similarly, in the covering cases every minimum $(v, 3, \lambda)$-covering with $C_\lambda(v)$ blocks covers every pair λ times except for a single pair covered $\lambda + 2$ times. □

In this paper we establish that the necessary condition implied by Lemma 2.2 is sufficient. In particular, we prove the following:

Theorem 2.3 *Let* $v \geq 3$ *and* $b > 0$. *A necessary and sufficient condition for the existence of an* $MS(v, b)$ *is that* $\overline{D}_\lambda(v) < b < \overline{C}_\lambda(v)$ *does* not *hold for any* $\lambda \geq 0$.

Necessity is proved already. In order to establish the sufficiency, we shall primarily employ recursive constructions. As a result, designs are needed for numerous small orders. Prior to constructing these, we treat one of the simpler cases.

3 Sufficiency: $v \equiv 0 \pmod 6$

We first establish a simple general result:

Lemma 3.1 *If an MS(v, b) exists, then an MS$(v, b + \lambda_{min}(v)v(v-1)/6)$ also exists. Therefore if the condition of Theorem 2.3 is sufficient for $0 \le b < \lambda_{min}(v)v(v-1)/6$, it is sufficient for all $b > 0$.*

Proof: A $(v, 3, \lambda_{\min}(v))$-design is an MS$(v, \lambda_{\min}(v)v(v-1)/6)$ design in which all replication numbers are the same, and in which all pairs occur the same number of times. Hence the union of this MS with an MS(v, b) is again an MS. □

We assume henceforth that $b < \lambda_{\min}(v)v(v-1)/6$ without further comment.

We consider $v \equiv 0 \pmod 6$. A collection $\mathcal{B}$ of triples is *resolvable* if $\mathcal{B}$ can be partitioned into *parallel classes*; in each parallel class, the triples are disjoint and contain each element of the underlying design exactly once. A resolvable maximum $(6t, 3, 1)$-packing is a *nearly Kirkman triple system* (NKTS). Rees and Stinson (1987) completed the proof that an NKTS$(6t)$ exists if and only if $t \ge 3$.

Lemma 3.2 *An MS(v, b) with $v = 6t$ exists for all $b > 0$ except when $D_\lambda(v) < b < C_\lambda(v)$ for some λ.*

Proof: The exceptions follow from Lemma 2.2. The cases when $t = 1$ and 2 follow from Lemma 4.2, so suppose that $t \ge 3$. First we handle the case when $b \le D_1(v)$. Form an NKTS(v), and order its triples so that all triples within each parallel class are listed consecutively. The first b triples under this ordering form a $(v, 3, 1)$-packing, and the numbers of triples containing each element differ by at most one since every parallel class contains every element exactly once. So the first b triples form an MS(v, b). In addition, each such MS(v, b) has the property that there is a set of $v/2$ disjoint pairs not covered by any triples. To handle the cases with $C_1(v) \le b < v(v-1)/3$, we start with a minimum $(v, 3, 1)$-covering, noting that it has $v/2$ disjoint pairs covered twice and all other pairs covered once. Then form an MS$(v, b - C_1(v))$ and add the triples of the minimum covering, aligning the two sets of $v/2$ disjoint pairs. □

4 Sufficiency: Small orders

We employ Lemma 3.2 to establish existence of some MS(v, b)s for $v \not\equiv 0 \pmod 6$:

Lemma 4.1 *Let v, w, λ be nonnegative integers satisfying $v \equiv w \pmod 6$, $w \ne 2$, $v \ge 2w$, and $v - w \ge 18$ when $\lambda(v-1) \equiv 1 \pmod 2$. Let $r_\ell = \lceil \lambda(v-1)/2 \rceil$ and $r_h = \lfloor (\lambda+1)(v-1)/2 \rfloor$. If an MS$(w, c)$ exists for all $\overline{C}_\lambda(w) \le c \le \overline{D}_{\lambda+1}(w)$, then an MS$(v, b)$ exists whenever $\overline{C}_\lambda(v) \le b \le \overline{C}_\lambda(v) + (r_h - r_\ell)(v-w)/3 + \overline{D}_{\lambda+1}(w) - \overline{C}_\lambda(w)$.*

Proof: Let V be a v-set and $W \subset V$ be a w-set. Form a collection $\mathcal{B}$ of triples which cover every pair of elements not both from W exactly λ times, but which cover no pair inside W (see Colbourn and Rosa (1999)). The number of triples in the collection $\mathcal{B}$ is $\overline{C}_\lambda(v) - \overline{C}_\lambda(w)$. We require some notation. For each r satisfying $r_\ell \leq r \leq r_h$, let $m_{w,r}$ be the smallest c for which an MS(w, c) with minimum replication number r exists, and let $M_{w,r}$ be the largest such c.

To form an MS(v, b) with b in the required range, let $b' = b - \overline{C}_\lambda(w)$. Now choose b_1 and b_2 so that for some r, we find that $r(v-w)/3 \leq b_1 \leq (r+1)(v-w)/3$, and $m_{w,r} \leq b_2 \leq M_{w,r}$. (It is easy to verify that this can always be done.) Now add to $\mathcal{B}$ the triples of an MS(w, b_2) on W. Next form an MS$(v-w, b_1)$ on $V \setminus W$ using Lemma 3.2. When $\lambda(v-1) \equiv 0 \pmod 2$, this MS is placed arbitrarily. However, when $\lambda(v-1)$ is odd, the collection $\mathcal{B}$ covers a 1-factor of pairs on $V \setminus W$ $\lambda + 1$ times rather than λ. In this case, we employ the fact that Lemma 3.2 constructs an MS$(v-w, b_1)$ in which the uncovered pairs include a 1-factor. Aligning these two 1-factors ensures that the result is the MS(v, b) required. □

To obtain the remaining designs, we adapted a hill-climbing algorithm for triple systems (Gibbons (1996), Stinson (1985)). In particular, we enforce the limit on replication numbers as triples are added. We check, when the number of triples is as desired, that pair occurrences differ by at most one. This somewhat naive method suffices to complete the proof of the following:

Lemma 4.2 *When v and b meet the condition of Theorem 2.3 and $v \in \{4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 20, 22, 23, 28, 29, 34, 35, 40, 41, 46, 47\}$, an MS$(v, b)$ exists.*

The proof of Lemma 4.2 is by construction of many small designs (those not obtained from Lemma 4.1). Since the particular structure of the individual designs is of little interest, we do not exhibit them here; rather they are available from the authors.

5 Sufficiency: $v \equiv 1, 2, 3 \pmod 6$

We need a further definition from combinatorial design theory. A *3-frame* of type $g^n h^1$ is a triple $(V, \mathcal{G}, \mathcal{B})$ such that

1. V is a set of size $gn + h$,
2. $\mathcal{G}$ is a partition of V into n sets of size g and one set of size h, called *groups*,
3. $\mathcal{B}$ is a collection of subsets of V each of size 3 called *triples*,
4. For every two points, $x, y \in V$, either there exists a unique $G \in \mathcal{G}$ for which $\{x, y\} \subseteq G$, or there exists a unique $B \in \mathcal{B}$ such that $\{x, y\} \subset B$, but not both,
5. $\mathcal{B}$ can be partitioned into *frame parallel classes* each of which consists of disjoint triples and contains exactly the points of $V \setminus G$ for some $G \in \mathcal{G}$.

It is permitted to choose $h = 0$, in which case the type is written as g^n. A 3-frame of type 6^n exists whenever $n \geq 4$ (see Abel and Furino (1996)). We use such frames to treat three further congruence classes.

Lemma 5.1 *Let* $v \equiv 1, 2, 3 \pmod 6$. *Then an MS*(v, b) *exists for all* $b > 0$ *except when* $v \equiv 2 \pmod 6$ *and either*

1. $\lambda \equiv 2 \pmod 6$ *and* $D_\lambda(v) \leq b < C_\lambda(v)$, *or*
2. $\lambda \equiv 4 \pmod 6$ *and* $D_\lambda(v) < b \leq C_\lambda(v)$.

Proof: If $v < 25$, Lemma 4.2 gives the solutions. Suppose then that $v \geq 25$. We first treat the cases with $b \leq D_1(v)$. Let $n = \lfloor v/6 \rfloor$. Form a 3-frame $(V, \mathcal{G}, \mathcal{B})$ of type 6^n, having groups $G_1, \ldots, G_n$ whose frame parallel classes are R_{ij} for $0 \leq i < n$ and $0 \leq j < 3$. The frame parallel classes R_{i0}, R_{i1} and R_{i2} are precisely those which cover all elements except those of group G_i.

We employ certain MS(v, b)s as well for small values of v. For $v = 7$, if $b \leq 5$, we select the first b blocks of $a01$, 234, 025, 135, $a45$ to form an MS$(7, b)$. If $b = 6$ or 7, we select the first b blocks of $a01$, $a23$, $a45$, 024, 035, 125, 134. When $v = 8$, if $b \leq 6$, select the first b blocks of 012, $a45$, $b03$, $b15$, 234, $a14$ to form an MS$(8, b)$. If $b = 7$ or 8, select the first b blocks of 012, 345, $a24$, $b15$, $a05$, $b23$, $a13$, $b04$. When $v = 9$ and $b \leq 12$, select the first b blocks of $a03$, $b14$, $c25$, $a04$, $b15$, $c23$, $a05$, $b13$, $c24$, 012, 345, abc to form an MS$(9, b)$.

Start with the 3-frame and append elements $\{a\}$, $\{a, b\}$, or $\{a, b, c\}$ so that v mod 6 new elements are added. Start with the MS$(v, 0)$ having no blocks. We add one block at a time. For each group G_i in turn, we add the blocks of R_{i0} one at a time. Once all are added, we place the blocks of the MS(7,j), MS(8,j), or MS(9,j) constructed for $0 \leq j \leq 3$, identifying the elements 0, 1, 2, 3, 4, 5 with the elements of G_i. Then we add in turn all blocks in R_{i1}. To continue, we instead place the blocks of the MS(7,j), MS(8,j), or MS(9,j) constructed for $4 \leq j \leq 6$. Then we add one at a time the triples of R_{i2}. To complete the handling of this group, we instead place the blocks of the MS(7,j), MS(8,j), or MS(9,j) constructed for $7 \leq j \leq \ell$, where $\ell = 7$, 8, or 9 depending upon the number of points. After a group is processed in this way, all elements have the same replication number. So we can continue to process the next group in exactly the same way until all groups are handled. This process reaches $D_1(v)$ triples when $v \equiv 1, 2 \pmod 6$. When $v \equiv 3 \pmod 6$, we complete the process by adding in turn all of the missing triples on each of the groups, and finally adding the missing triple $\{a, b, c\}$.

Since $(v, 3, 1)$-designs exist whenever $v \equiv 1, 3 \pmod 6$, this completes the construction of MS(v, b)s for all $b > 0$ in these cases. However, the situation when $v \equiv 2 \pmod 6$ requires further examination. We consider the situation when $C_1(v) \leq b < D_2(v)$ in some detail. We form on $V \cup \{a, b\}$ a minimum $(v, 3, 1)$-covering which contains a minimum (8,3,1)-covering on $G_1 \cup \{a, b\}$, and in which all other pairs occurring in two triples lie within groups $G_2, \ldots, G_n$ (such a covering exists; see Colbourn and Rosa (1999)). This covering is an MS$(v, C_1(v))$. We begin by processing the group G_1. First we add, one at a time, the triples of R_{10}. Then we replace the (8,3,1)-covering, which is an MS(8,11), in turn by the blocks of an MS$(8,j)$ for $j = 12, 13, 14$, obtained by taking the first j blocks of the sequence 013, 124, a23, 345, ab4, a05, b15, b02, 014, a12, 235, b34, ab0, 245. Then add in turn each of the blocks of R_{11}. Next replace the blocks of the MS(8,14) by those of an MS(8,j) for $j = 15, 16$ obtained by taking the first j blocks of the sequence 013, 124, a23, 345, ab4, a05, b15, b02, 014, a12, 235, b34, a04, b25, a15, b03. For the moment, we do *not* complete the processing of group G_1. Rather, since the MS(v, b) thus far produced has all elements in exactly the same number of triples, we detour to handle the remaining groups. Each remaining group is processed in turn, using the original method but subject to the condition that a spanning set of four uncovered pairs in the MS$(8, j)$ employed is aligned on $\{a, b\}$ together with the three pairs covered twice in the group by the $(v, 3, 1)$-covering. Once all remaining groups are treated in this way, the MS(v, b) produced is a $(v, 3, 2)$-packing in which the pairs covered only once form $(v - 8)/3$ triangles and two 4-gons. Add each triangle as a triple in turn to handle all cases with $b \leq D_2(v) - 2$. An MS$(v, D_2(v) - 1)$ is then obtained by replacing the MS(8,16) by an MS(8,17) which is easily produced (Colbourn and Rosa (1987)).

The cases when $C_\lambda(v) \leq b \leq D_{\lambda+1}(v)$ for $\lambda = 2, 3, 4, 5$ are quite similar, and we omit the details. (The essential requirement is to produce the solutions for $v = 8$. The MS$(8, j)$s for $20 \leq j \leq 26$ can be obtained as follows. Form an MS(8,20) with blocks 013, 124, 235, 034, 145, 025, a01, a12, a23, a34, a45, a05, b02, b13, b24, b35, b04, b15, ab0 and ab1. Then add 1, 2, 3, 4, or 5 triples from 345, 123, 024, ab5, and 015 to treat the cases with $21 \leq j \leq 25$. An MS(8,26) is obtained by developing the base blocks $0_0 1_0 1_1$, $0_0 1_0 3_1$, $0_0 2_0 2_1$, $0_1 1_1 0_0$, $0_1 1_1 3_0$, and $0_1 2_1 1_0$ modulo $(4, -)$, and then adding the triples $0_0 1_0 2_0$ and $0_1 1_1 2_1$. To produce MS$(8, j)$s with $30 \leq b \leq 56$, we employ the fact that the solutions with $0 \leq j \leq 26$ contain no repeated blocks. Hence, taking all triples *not* in one of the MS$(8, j)$s with $0 \leq j \leq 26$ yields an MS$(8, 56 - j)$.) When $b \geq v(v - 1)$, an MS(v, b) is obtained as the union of an MS$(v, b - v(v - 1))$ and a $(v, 3, 6)$-design, which completes the proof. □

6 Sufficiency: $v \equiv 4, 5 \pmod 6$

When $v \equiv 4, 5 \pmod 6$, the idea of Lemma 5.1 does not apply directly since the MS$(10, j)$s and MS$(11, j)$s required cannot have the needed subset of 4 or 5 elements no two of which appear in a triple. Nevertheless, a variant of the method using more complicated frames does treat these situations. Ling and Colbourn (1997) collect together frames which contain at most one group of size six, which we use here:

Lemma 6.1 *(See Ling and Colbourn (1997)) There exist 3-frames of the following types:*

1. 12^n *for* $n \geq 4$*;*

2. $30^n(6x)^1$ *for* $n \geq 4$ *and* $0 \leq x \leq n - 1$*;*

t	Blocks
2	{0,4,5}
3	{0,7,8}, {0,2,6}
4	{0,2,7}, {0,3,9}, {0,10,11}
5	{0,13,14}, {0,2,6}, {0,3,11}, {0,7,12}
6	{0,4,14}, {0,2,15}, {0,3,9}, {0,5,12}, {0,16,17}
7	{0,19,20}, {0,3,14}, {0,4,16}, {0,5,18}, {0,6,15}, {0,7,17}

Table 1: Small Packings

3. $12^4 6^1$, $12^4 18^1$, $12^6 6^1$, 18^5, $12^5 24^1 30^1$, *and* $18^5 36^1$.

In particular, for every number v of elements satisfying $v \equiv 0 \pmod 6$ and $v \geq 48$, there is a 3-frame having all group sizes a multiple of 6 and having at most one group of size 6.

We shall also need some small packings. Table 1 gives, for each $2 \leq t \leq 7$, a packing by triples on $6t$ points as follows. First, for each $0 \leq i < 6t$, we add i to each element of each triple, reducing modulo $6t$, to produce $6t$ blocks from each. The pairs which appear in none of the resulting $6t(t-1)$ triples can be partitioned into five classes so that each element appears in exactly one pair in each class (i.e. the classes are *1-factors*), as a consequence of the lemma of Stern and Lenz (1980). We can therefore extend the set of $6t(t-1)$ triples to a packing on $6t+5$ elements, by adding five new 'infinite' elements, and adjoining each to the pairs of one of the 1-factors to form triples. The result is a $(6t+5, 3, 1)$ packing, but it is not an (M,S)-optimal design because the five infinite elements appear two fewer times than do the remaining elements. To remedy this, consider the first block shown for each packing in Table 1. Each has the property that the three elements of the block are distinct modulo 3. Hence adding $3i$, $3i+1$, or $3i+2$ for $0 \leq i < 2t$ yields a parallel class on the elements $0, \ldots, 6t-1$. Removing two of these parallel classes therefore produces an MS$(6t+5, t(6t+5))$. Instead deleting one infinite point and removing one parallel class yields an MS$(6t+4, t(6t+4))$.

Theorem 6.2 *An MS(v, b) exists whenever $v \equiv 4, 5 \pmod 6$ and v, b meet the condition in Theorem 2.3.*

Proof: When $v \leq 47$, see Lemma 4.2. For $v \geq 52$, write $v = 6t + s$. Form a frame on $6t$ elements with groups $G_1, \ldots, G_n$, so that each group except possibly the last has size at least 12, and all groups have sizes which are a multiple of six (use Lemma 6.1). We process the groups (and their associated frame parallel classes) in turn for $i = 1, \ldots, n$. To process group G_i, we proceed as follows. Let m_j (M_j) be the minimum (maximum, respectively) number c of blocks in an MS$(|G_i| + s, c)$ with replication numbers all at least j (at most j, respectively). To handle the jth frame parallel class ($j = 1, \ldots, |G_i|/2$), first remove the blocks of the MS$(|G_i| + s, M_{j-1})$, and add instead the blocks of the MS($|G_i| + s, c$)s in turn for $m_{j-1} \leq c \leq M_j$. Then add each of the blocks of the jth frame parallel class one at a time. Once all frame parallel classes for G_i are so handled, when $i < n$ replace the MS($|G_i| + s, M_{|G_i|/2}$) by the MS($|G_i| + s, (|G_i|)(|G_i| + s)/6$) formed earlier, with no pairs covered on the s additional elements.

When $v \equiv 4 \pmod 6$, we can now add the omitted frame parallel classes in each of the MS($|G_i| + s, M_{|G_i|/2}$)s used. Then the MS($|G_n| + 4, M_{|G_n|/2}$) can be replaced in turn by MS($|G_n| + 4, c$)s for $M_{|G_n|/2} < c \leq \lfloor (|G_n| + 4)(|G_n| + 2)/6 \rfloor$. This handles all cases when pairs appear in zero or one triples. To continue to larger numbers of blocks, we simply place an appropriate $(v, 3, 1)$-covering first, and then add the blocks of a packing.

When $v \equiv 5 \pmod 6$, the completion is similar except that two omitted frame parallel classes are to be added for each group; we leave the tedious details to the reader. □

We have established that the conditions of Lemma 2.2 are sufficient for the existence of an MS(v, b) for all $v \geq 3$ and $b > 0$, completing the proof of Theorem 2.3.

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