The average connectivity of regular multipartite tournaments

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Abstract

The average connectivity of a digraph is the average, over all ordered pairs of vertices, of the maximum number of internally disjoint directed paths connecting these vertices. Among the results in this paper, we determine the minimum average connectivity among all orientations of the complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$ and the maximum average connectivity when all partite sets have the same order.

1 Introduction

The average connectivity and the average edge-connectivity of a graph were studied in [1, 3, 4]. These measures give a more accurate picture of the reliability of a graph than the corresponding conventional parameters. Furthermore, unlike other measures of reliability such as the toughness and integrity of a graph, which are NP-hard, they have the advantage that they can be computed efficiently.

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The concept of the average connectivity of a digraph was introduced in [5]. Let $D = (V, E)$ be a digraph with vertex set V, arc set E, order $p = |V| \geq 2$ and size $q = |E|$. Let u and v be two distinct vertices of D. The *connectivity* $\kappa(u, v)$ *from* u *to* v is the maximum number of internally disjoint directed u–v paths in D. The *total connectivity of* D is defined by $K(D) = \sum_{u,v \in V} \kappa(u, v)$. The *average connectivity* $\overline{\kappa}(D)$ of D is the average connectivity over all ordered pairs (u, v) of vertices of D, that is,

$$\overline{\kappa}(D) = \frac{1}{p(p-1)} \left(\sum_{u,v \in V} \kappa(u, v) \right) = \frac{K(D)}{p(p-1)}.$$

As is the case with the average connectivity and average edge-connectivity of a graph, the average connectivity of a digraph can be computed in polynomial time using network flow techniques. The problem of finding the maximum average connectivity among all orientations of a graph G appears to be difficult. Why this problem may be difficult, even for trees is discussed in [5]. This suggests obtaining bounds for this parameter for special classes of graphs. Let $\overline{\kappa}_{\max}(G)$ ($\overline{\kappa}_{\min}(G)$) denote the maximum (respectively, minimum) average connectivity among all orientations of G.

In [5] we show that for every tree T of order $p \geq 3$, $(2p^2 + 14p - 43)/(9p(p-1)) \leq \overline{\kappa}_{\max}(T) \leq 1/2$ and these bounds are sharp. Hence for every tree T of order $p \geq 3$, $2/9 < \overline{\kappa}_{\max}(T) \leq 1/2$. Moreover, we show in [5] that $\overline{\kappa}_{\min}(K_p) = (p+1)/6$ for $p \geq 2$, $\overline{\kappa}_{\max}(K_p) = (p-1)/2$ for odd $p \geq 3$ and $\overline{\kappa}_{\max}(K_p) = (2p^2 - 5p + 4)/(4(p-1))$ for even $p \geq 2$.

A *complete multipartite graph* is a complete k-partite graph for some $k \geq 2$. If the partite sets of a complete k-partite graph have cardinalities $n_1, n_2, \ldots, n_k$, then we denote this graph by $K_{n_1,n_2,\ldots,n_k}$. If $n_i = n$ for all i, then $K_{n_1,n_2,\ldots,n_k}$ is denoted by $K_{k(n)}$. In this paper, we determine $\overline{\kappa}_{\max}(K_{k(n)})$ for $k \geq 2$ and $n \geq 2$, and we determine $\overline{\kappa}_{\min}(K_{n_1,n_2,\ldots,n_k})$.

2 Maximum Values

In this section we establish a general upper bound on $\overline{\kappa}_{\max}(K_{n_1,n_2,\ldots,n_k})$.

Theorem 1 *Let* $G = K_{n_1,n_2,\ldots,n_k}$ *have order* p*, where* $n_1 \leq n_2 \leq \cdots \leq n_k$*. Then,*

$$\overline{\kappa}_{\max}(G) \leq \left(\sum_{1 \leq i < j \leq k} n_i n_j (p - n_j) + \sum_{i=1}^{k} \binom{n_i}{2} (p - n_i) \right) / \, p(p-1).$$

Proof. Let $V_1, V_2, \ldots, V_k$ denote the partite sets of G, where $|V_i| = n_i$ for $i = 1, 2, \ldots, k$. Let D be an orientation of G, and let u and v be two distinct vertices of D. Then, $\kappa(u, v) \leq \min\{\mathrm{od}_D\, u, \mathrm{id}_D\, v\}$, and so $\kappa(u, v) + \kappa(v, u) \leq \min\{\mathrm{od}_D\, u + \mathrm{id}_D\, u, \mathrm{od}_D\, v + \mathrm{id}_D\, v\}$. Consequently, if $u \in V_i$ and $v \in V_j$, with $i < j$, then $\kappa(u, v) + \kappa(v, u) \leq p - n_j$. This is true for all $n_i n_j$ pairs of vertices of D with one vertex in V_i

and the other in V_j. On the other hand, if $u, v \in V_i$, then $\kappa(u, v) + \kappa(v, u) \le p - n_i$. This is true for all $\binom{n_i}{2}$ pairs of vertices in V_i. Hence,

$$K(D) \le \sum_{1 \le i < j \le k} n_i n_j (p - n_j) + \sum_{i=1}^{k} \binom{n_i}{2} (p - n_i).$$

The upper bound stated in the theorem now follows since $\overline{\kappa}(D) = K(D)/p(p-1)$ and D is an arbitrary orientation of G. □

As a special case of Theorem 1, we have an upper bound on $\overline{\kappa}_{\max}(G)$ when $G = K_{k(n)}$.

Corollary 2 *For* $n \ge 2$,

$$\overline{\kappa}_{\max}(K_{k(n)}) \le \frac{n(k-1)}{2}.$$

Proof. By Theorem 1,

$$\overline{\kappa}_{\max}(K_{k(n)}) \le \left(\binom{k}{2} n^2 (nk - n) + k \binom{n}{2} (nk - n) \right) \Big/ \, nk(nk-1) = n(k-1)/2.$$

□

We show that the upper bound in Corollary 2 can be improved slightly if k is even and n is odd.

Lemma 3 *For even* $k \ge 2$ *and odd* $n \ge 3$,

$$\overline{\kappa}_{\max}(K_{k(n)}) \le \frac{n(k-1)}{2} - \frac{kn-2}{4(kn-1)}.$$

Proof. Let D be an orientation of $K_{k(n)}$ and let u be a vertex of D. Since k is even and n is odd, $(k-1)n$ is odd. It follows that either $\mathrm{od}_D\, u \le ((k-1)n-1)/2$ or $\mathrm{od}_D\, u \ge ((k-1)n+1)/2$ and $\mathrm{id}_D\, u \le ((k-1)n-1)/2$. Suppose u and v are two distinct vertices of D such that $\mathrm{od}_D\, u \le ((k-1)n-1)/2$ and $\mathrm{od}_D\, v \le ((k-1)n-1)/2$. Then $\kappa(u,v) \le \min\{\mathrm{od}_D\, u, \mathrm{id}_D\, v\} \le \mathrm{od}_D\, u$ and $\kappa(v,u) \le \min\{\mathrm{id}_D\, u, \mathrm{od}_D\, v\} \le \mathrm{od}_D\, v$. Thus, $\kappa(u,v) + \kappa(v,u) \le \mathrm{od}_D\, u + \mathrm{od}_D\, v \le (k-1)n - 1$. Similarly, if $\mathrm{id}_D\, u \le ((k-1)n-1)/2$ and $\mathrm{id}_D\, v \le ((k-1)n-1)/2$, then $\kappa(u,v) + \kappa(v,u) \le (k-1)n - 1$. Suppose now that there are m vertices of D with outdegree at most $((k-1)n-1)/2$. Then there are $kn - m$ vertices of D with outdegree at least $((k-1)n+1)/2$ and indegree at most $((k-1)n-1)/2$. It follows that

$$\begin{aligned} K(D) &\le \tbinom{m}{2}((k-1)n-1) + \tbinom{kn-m}{2}((k-1)n-1) + m(kn-m)(kn-n) \\ &= -m^2 + mkn + \tfrac{1}{2}k^3n^3 - \tfrac{1}{2}k^2n^3 - k^2n^2 + \tfrac{1}{2}kn^2 + \tfrac{1}{2}kn. \end{aligned}$$

The expression on the right hand side of the above inequality is a quadratic in m and is maximised when $m = kn/2$. Thus,

$$\begin{aligned} K(D) &\leq \tfrac{1}{2}k^3n^3 - \tfrac{1}{2}k^2n^3 - \tfrac{3}{4}k^2n^2 + \tfrac{1}{2}kn^2 + \tfrac{1}{2}kn \\ &= \tbinom{kn}{2}((k-1)n-1) + \tfrac{1}{4}k^2n^2. \end{aligned}$$

It follows that

$$\overline{\kappa}(D) \leq \frac{(k-1)n-1}{2} + \frac{k^2n^2}{4(kn)(kn-1)} = \frac{(k-1)n-1}{2} + \frac{kn}{4(kn-1)}.$$

This completes the proof of the lemma. □

Next we determine the maximum average connectivity of multipartite tournaments. We consider orientations T of the complete k-partite graph, $K_{k(n)}$, with $k \geq 2$ and $n \geq 2$ such that $\overline{\kappa}(T) = \overline{\kappa}_{\max}(K_{k(n)})$. First we determine $\overline{\kappa}_{\max}(K_{k(n)})$ for k even and n odd.

Lemma 4 *For even $k \geq 2$ and odd $n \geq 3$,*

$$\overline{\kappa}_{\max}(K_{k(n)}) = \frac{n(k-1)}{2} - \frac{kn-2}{4(kn-1)}.$$

Proof. By Lemma 3, $\overline{\kappa}_{\max}(K_{k(n)}) \leq ((k-1)n-1)/2 + kn/4(kn-1)$. Hence it suffices to show that there is an orientation T of $K_{k(n)}$ such that $\overline{\kappa}(T) = ((k-1)n-1)/2 + kn/4(kn-1)$.

Let $V_1, V_2, \ldots, V_k$ denote the partite sets of $K_{k(n)}$. For $i = 1, 2, \ldots, k$, let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$. Let $V_i^1 = \{v_{i,1}, \ldots, v_{i,(n+1)/2}\}$ and $V_i^2 = \{v_{i,(n+3)/2}, \ldots, v_{i,n}\}$ be a partition of V_i into two sets of cardinalities $(n+1)/2$ and $(n-1)/2$, respectively. Construct T from $K_{k(n)}$ by orienting for every i, $1 \leq i \leq k$, and every j, $1 \leq j \leq (k-2)/2$, the edges joining V_i and V_{i+j} from V_i to V_{i+j}, where subscripts are expressed modulo k. For $1 \leq i \leq k/2$ and for each $j \in \{1, 2\}$, we orient every edge uv with $u \in V_i^j$ and $v \in V_{i+k/2}^j$ as (u, v) and we orient every edge uv with $u \in V_i^j$ and $v \in V_{i+k/2}^{3-j}$ as (v, u).

Let X denote the set of vertices of T with indegree $((k-1)n-1)/2$, and let $Y = V(T) - X$. Then each vertex of Y has indegree $((k-1)n+1)/2$ and outdegree $((k-1)n-1)/2$. It follows from the proof of Lemma 3 that $\kappa(u, v) + \kappa(v, u) \leq (k-1)n-1$ if u and v both belong to X or both belong to Y.

Claim 1 *If $u, v \in X$, then $\kappa_T(u, v) + \kappa_T(v, u) = n(k-1)-1$.*

Proof. For notational convenience, we may assume that $u \in V_1$ and $u = v_{1,1}$. Suppose first that $v \in V_m$ where $2 \leq m \leq k/2$. For the case where $2 + k/2 \leq m \leq k-1$, the argument is similar. Since $v \in X$, $v \in V_m^1$. For notational convenience, we may assume that $v = v_{m,1}$. For every ℓ, $2 \leq \ell < m$, let $P_{1\ell m}$ denote the collection of n paths of the type $v_{1,1}, v_{\ell,j}, v_{m,1}$ where $1 \leq j \leq n$. Let P_{1m} be the collection of n paths consisting of the path $v_{1,1}, v_{m,1}$, the $(n-1)/2$ paths of the type $v_{1,1}, v_{m,j}, v_{m+k/2,j}, v_{1,j}, v_{m,1}$ where $2 \leq j \leq (n+1)/2$ and the $(n-1)/2$ paths of the type $v_{1,1}, v_{m,j}, v_{m+k/2,j}, v_{m,1}$ where $(n+3)/2 \leq j \leq n$. For $m+1 \leq r \leq k/2$, let P_{1rk} denote the collection of n paths of the type $v_{1,1}, v_{r,j}, v_{r+k/2,j}, v_{m,1}$ where $1 \leq j \leq n$. Let $P_{1,1+k/2}$ be the collection of $(n-1)/2$ paths of the type $v_{1,1}, v_{1+k/2,j}, v_{1,j+(n-1)/2}, v_{m,1}$ where $2 \leq j \leq (n+1)/2$. Then, $\{P_{1\ell k} \mid 2 \leq \ell < m\} \cup \{P_{1m}\} \cup \{P_{1rm} \mid m < r \leq k/2\} \cup \{P_{1,1+k/2}\}$ is a collection of $(n(k-1)-1)/2$ internally disjoint u–v paths in T. Thus, $\kappa(u,v) \geq (n(k-1)-1)/2$. On the other hand, for $m+1 \leq \ell \leq k/2$, let $P_{m\ell 1}$ be the collection of n paths of the type $v_{m,1}, v_{\ell,j}, v_{\ell+k/2,j}, v_{1,1}$ where $1 \leq j \leq n$. Let $P_{m,1+k/2}$ be the collection of $(n-1)/2$ paths of the type $v_{m,1}, v_{1+k/2,j}, v_{1,1}$ where $(n+3)/2 \leq j \leq n$. For $2+k/2 \leq r \leq m-1+k/2$, let P_{mr1} denote the collection of n paths of the type $v_{m,1}, v_{r,j}, v_{1,1}$ where $1 \leq j \leq n$. Let $P_{m,m+k/2}$ be the collection of n paths consisting of the $(n+1)/2$ paths of the type $v_{m,1}, v_{m+k/2,j}, v_{1,1}$ where $1 \leq j \leq (n+1)/2$ and the $(n-1)/2$ paths of the type $v_{m,1}, v_{1+k/2,j}, v_{m+k/2,j+(n-1)/2}, v_{1,1}$ where $2 \leq j \leq (n+1)/2$. Then, $\{P_{m\ell 1} \mid m+1 \leq \ell \leq k/2\} \cup \{P_{m,1+k/2}\} \cup \{P_{mri} \mid 2+k/2 \leq r \leq m-1+k/2\} \cup \{P_{m,m+k/2}\}$ is a collection of $(n(k-1)-1)/2$ internally disjoint v–u paths in T. Thus, $\kappa(v,u) \geq (n(k-1)-1)/2$. Consequently, $\kappa(u,v) + \kappa(v,u) \geq n(k-1) - 1$.

Suppose secondly that $v \in V_m$ where $m = 1 + k/2$. Since $v \in X$, $v \in V_m^2$. For notational convenience, we may assume that $v = v_{m,n}$. For $2 \leq \ell \leq k/2$, let $P_{1\ell m}$ denote the collection of n paths of the type $v_{1,1}, v_{\ell,j}, v_{m,n}$ where $1 \leq j \leq n$. Let P_{1m} be the collection of $(n-1)/2$ paths of the type $v_{1,1}, v_{m,j}, v_{1,j+(n-1)/2}, v_{m,n}$ where $2 \leq j \leq (n+1)/2$. Then, $\{P_{1\ell m} \mid 2 \leq \ell \leq k/2\} \cup \{P_{1m}\}$ is a collection of $(n(k-1)-1)/2$ internally disjoint u–v paths in T. On the other hand, let P_{m1} be the collection of $(n-1)/2$ paths consisting of the path $v_{m,n}, v_{1,1}$ and all paths of the type $v_{m,n}, v_{1,j}, v_{m,j}, v_{1,j+(n-1)/2}, v_{m,j+(n-1)/2}, v_{1,1}$ where $2 \leq j \leq (n-1)/2$. For $2+k/2 \leq \ell \leq k$, let $P_{m\ell 1}$ denote the collection of n paths of the type $v_{m,n}, v_{\ell,j}, v_{1,1}$ where $1 \leq j \leq n$. Then, $\{P_{m1}\} \cup \{P_{m\ell 1} \mid 2+k/2 \leq \ell \leq k\}$ is a collection of $(n(k-1)-1)/2$ internally disjoint v–u paths in T. Consequently, $\kappa(u,v)+\kappa(v,u) \geq n(k-1) - 1$.

Suppose finally that $v \in V_1$. For notational convenience, we may assume that $v = v_{1,2}$. For $2 \leq \ell \leq k/2$, let $P_{1\ell 2}$ denote the collection of n paths of the type $v_{1,1}, v_{\ell,j}, v_{\ell+k/2,j}, v_{1,2}$ where $1 \leq j \leq n$. Let P_{12} be the collection of $(n-1)/2$ paths of the type $v_{1,1}, v_{1+k/2,j}, v_{1,j+(n-1)/2}, v_{1+k/2,j+(n-1)/2}, v_{1,2}$ where $2 \leq j \leq (n+1)/2$. Then, $\{P_{1\ell 2} \mid 2 \leq \ell \leq k/2\} \cup \{P_{12}\}$ is a collection of $(n(k-1)-1)/2$ internally disjoint u–v paths in T. Thus, $\kappa(u,v) \geq (n(k-1)-1)/2$. Similarly, $\kappa(v,u) \geq (n(k-1)-1)/2$. Consequently, $\kappa(u,v) + \kappa(v,u) \geq n(k-1) - 1$.

Hence, $\kappa(u,v) + \kappa(v,u) \geq n(k-1) - 1$ for all pairs of vertices u and v both of which belong to X. However, for all such pairs u and v, $\kappa(u,v)+\kappa(v,u) \leq (k-1)n-1$ as observed earlier. Hence, $\kappa(u,v)+\kappa(v,u) = (k-1)n-1$. This completes the proof of Claim 1. □

The proof of the following claim is similar to that of Claim 1, and is therefore omitted.

Claim 2 *If $u, v \in Y$, then $\kappa_T(u,v) + \kappa_T(v,u) = n(k-1) - 1$.*

Claim 3 *If $u \in X$ and $v \in Y$, then $\kappa_T(u,v) + \kappa_T(v,u) = n(k-1)$.*

Proof. For notational convenience, we may assume that $u \in V_1$ and $u = v_{1,1}$. Suppose first that $v \in V_m$ where $2 \le m \le k/2$. For the case where $2 + k/2 \le m \le k-1$, the argument is similar. Then, $v \in V_m^2$. For notational convenience, we may assume that $v = v_{m,n}$. For $2 \le \ell < m$, let $P_{1\ell m}$ denote the collection of n u–v paths of the type $v_{1,1}, v_{\ell,j}, v_{m,n}$ where $1 \le j \le n$. Let P_{1m} be the collection of n u–v paths consisting of the path $v_{1,1}, v_{m,n}$, the $(n+1)/2$ u–v paths of the type $v_{1,1}, v_{m,j}, v_{m+k/2,j}, v_{m,n}$ where $1 \le j \le (n+1)/2$ and the $(n-3)/2$ u–v paths of the type $v_{1,1}, v_{m,j}, v_{m+k/2,j}, v_{1,j-(n-1)/2}, v_{m,n}$ where $(n+3)/2 \le j \le n-1$. For $m+1 \le r \le k/2$, let P_{1rk} denote the collection of n u–v paths of the type $v_{1,1}, v_{r,j}, v_{r+k/2,j}, v_{m,n}$ where $1 \le j \le n$. Let $P_{1,1+k/2}$ be the collection of $(n+1)/2$ u–v paths consisting of the path $v_{1,1}, v_{1+k/2,1}, v_{m+k/2,n}, v_{1,(n+1)/2}, v_{m,n}$ and the $(n-1)/2$ u–v paths of the type $v_{1,1}, v_{1+k/2,j}, v_{1,j+(n-1)/2}, v_{m,n}$ where $2 \le j \le (n+1)/2$. Then, $\{P_{1\ell k} \mid 2 \le \ell < m\} \cup \{P_{1m}\} \cup \{P_{1rm} \mid m < r \le k/2\} \cup \{P_{1,1+k/2}\}$ is a collection of $(n(k-1)+1)/2$ internally disjoint u–v paths in T. Thus, $\kappa(u,v) \ge (n(k-1)+1)/2$. On the other hand, for $m+1 \le \ell \le k/2$, let $P_{m\ell 1}$ be the collection of n paths of the type $v_{m,n}, v_{\ell,j}, v_{\ell+k/2,j}, v_{1,1}$ where $1 \le j \le n$. Let $P_{m,1+k/2}$ be the collection of n paths consisting of the $(n-1)/2$ paths of the type $v_{m,n}, v_{1+k/2,j}, v_{1,1}$ where $(n+3)/2 \le j \le n$ and the $(n+1)/2$ paths of the type $v_{m,n}, v_{1+k/2,j}, v_{m+k/2,j}, v_{1,1}$ where $1 \le j \le (n+1)/2$. For $2 + k/2 \le r \le m - 1 + k/2$, let P_{mr1} denote the collection of n paths of the type $v_{m,n}, v_{r,j}, v_{1,1}$ where $1 \le j \le n$. Let $P_{m,m+k/2}$ be the collection of $(n-1)/2$ paths of the type $v_{m,n}, v_{m+k/2,j}, v_{1,1}$ where $(n+3)/2 \le j \le n$. Then, $\{P_{m\ell 1} \mid m+1 \le \ell \le k/2\} \cup \{P_{m,1+k/2}\} \cup \{P_{mri} \mid 2+k/2 \le r \le m-1+k/2\} \cup \{P_{m,m+k/2}\}$ is a collection of $(n(k-1)-1)/2$ internally disjoint v–u paths in T. Thus, $\kappa(v,u) \ge (n(k-1)-1)/2$. Consequently, $\kappa(u,v) + \kappa(v,u) \ge n(k-1)$.

Suppose secondly that $v \in V_m$ where $m = 1 + k/2$. By construction, $v \in V_m^1$. For notational convenience, we may assume that $v = v_{m,1}$. For $2 \le \ell \le k/2$, let $P_{1\ell m}$ denote the collection of n u–v paths of the type $v_{1,1}, v_{\ell,j}, v_{m,1}$ where $1 \le j \le n$. Let P_{1m} be the collection of $(n+1)/2$ u–v paths consisting of the path $v_{1,1}, v_{m,1}$ and the $(n-1)/2$ u–v paths of the type $v_{1,1}, v_{m,j}, v_{1,j+(n-1)/2}, v_{m,j+(n-1)/2}, v_{1,j}, v_{m,1}$ where $2 \le j \le (n+1)/2$. Then, $\{P_{1\ell m} \mid 2 \le \ell \le k/2\} \cup \{P_{1m}\}$ is a collection of $(n(k-1)+1)/2$ internally disjoint u–v paths in T. Now, let P_{m1} be the collection of $(n-1)/2$ v–u paths of the type $v_{m,1}, v_{1,j}, v_{m,j}, v_{1,1}$ where $(n+3)/2 \le j \le n$. For $2 + k/2 \le \ell \le k$, let $P_{m\ell 1}$ denote the collection of n v–u paths of the type $v_{m,1}, v_{\ell,j}, v_{1,1}$ where $1 \le j \le n$. Then, $\{P_{m1}\} \cup \{P_{m\ell 1} \mid 2 + k/2 \le \ell \le k\}$ is a collection of $(n(k-1)-1)/2$ internally disjoint v–u paths in T. Consequently, $\kappa(u,v) + \kappa(v,u) \ge n(k-1)$.

Suppose finally that $v \in V_1$. Then, $v \in V_1^2$. For notational convenience, we may assume that $v = v_{1,n}$. For $2 \le \ell \le k/2$, let $P_{1\ell n}$ denote the collection of n u–v

paths of the type $v_{1,1}, v_{\ell,j}, v_{\ell+k/2}, v_{1,n}$ where $1 \leq j \leq n$. Let P_{1n} be the collection of $(n+1)/2$ u–v paths of the type $v_{1,1}, v_{1+k/2,j}, v_{1,n}$ where $1 \leq j \leq (n+1)/2$. Then, $\{P_{1\ell n} \mid 2 \leq \ell \leq k/2\} \cup \{P_{1n}\}$ is a collection of $(n(k-1)+1)/2$ internally disjoint u–v paths in T. Thus, $\kappa(u,v) \geq (n(k-1)+1)/2$. On the other hand, for $2 \leq \ell \leq k/2$, let $P_{n\ell 1}$ denote the collection of n v–u paths of the type $v_{1,n}, v_{\ell,j}, v_{\ell+k/2}, v_{1,1}$ where $1 \leq j \leq n$. Let P_{n1} be the collection of $(n-1)/2$ v–u paths of the type $v_{1,n}, v_{1+k/2,j}, v_{1,1}$ where $(n+3)/2 \leq j \leq n$. Then, $\{P_{n\ell 1} \mid 2 \leq \ell \leq k/2\} \cup \{P_{n1}\}$ is a collection of $(n(k-1)-1)/2$ internally disjoint v–u paths in T. Thus, $\kappa(u,v)+\kappa(v,u) \geq n(k-1)/2$.

Hence, $\kappa(u,v) + \kappa(v,u) \geq n(k-1)$ for all pairs of vertices u and v of T with $u \in X$ and $v \in Y$. However, for all pairs u and v, $\kappa(u,v) + \kappa(v,u) \leq n(k-1)$ as observed earlier. Hence, $\kappa(u,v) + \kappa(v,u) = n(k-1)$. This completes the proof of Claim 3. □

We now continue with the proof of Lemma 4. Since $|X| = |Y| = kn/2$, it follows from Claims 1, 2, and 3 that

$$\begin{aligned} K(T) &= \binom{|X|}{2}((k-1)n-1) + \binom{|Y|}{2}((k-1)n-1) + |X||Y|(k-1)n \\ &= \binom{kn/2}{2}((k-1)n-1) + \binom{kn/2}{2}((k-1)n-1) + \tfrac{1}{4}k^2n^2(k-1)n \\ &= \tfrac{1}{2}k^3n^3 - \tfrac{1}{2}k^2n^3 - \tfrac{3}{4}k^2n^2 + \tfrac{1}{2}kn^2 + \tfrac{1}{2}kn \\ &= \binom{kn}{2}((k-1)n-1) + \tfrac{1}{4}k^2n^2. \end{aligned}$$

It follows that

$$\overline{\kappa}(T) = \frac{(k-1)n-1}{2} + \frac{k^2n^2}{4(kn)(kn-1)} = \frac{(k-1)n-1}{2} + \frac{kn}{4(kn-1)}.$$

This completes the proof of the theorem. □

Next we determine $\overline{\kappa}_{\max}(K_{k(n)})$ when both k and n even.

Lemma 5 *For even $k \geq 2$ and even $n \geq 2$,*

$$\overline{\kappa}_{\max}(K_{k(n)}) = \frac{n(k-1)}{2}.$$

Proof. By Corollary 2, $\overline{\kappa}_{\max}(K_{k(n)}) \leq n(k-1)/2$. Hence it suffices to show that there is an orientation T of $K_{k(n)}$ such that $\kappa(u,v) + \kappa(v,u) = n(k-1)$ for all pairs of vertices of T, and so $\overline{\kappa}(T) = n(k-1)/2$.

Let $V_1, V_2, \ldots, V_k$ denote the partite sets of $K_{k(n)}$. For $i = 1, 2, \ldots, k$, let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$. Further, let $V_i^1 = \{v_{i,1}, \ldots, v_{i,n/2}\}$ and $V_i^2 = \{v_{i,(n+2)/2}, \ldots, v_{i,n}\}$ be a partition of V_i into two sets each of cardinality $n/2$. Construct T from $K_{k(n)}$ by orienting for every i, $1 \leq i \leq k$, and every j, $1 \leq j \leq (k-2)/2$, the edges joining V_i

and V_{i+j} from V_i to V_{i+j}, where subscripts are expressed modulo k. For $1 \leq i \leq k/2$ and for each $j \in \{1, 2\}$, we orient every edge uv with $u \in V_i^j$ and $v \in V_{i+k/2}^j$ as (u, v) and we orient every edge uv with $u \in V_i^j$ and $v \in V_i^{3-j}$ as (v, u).

Let u and v be any two distinct vertices of $K_{k(n)}$. We show that $\kappa(u,v)+\kappa(v,u) \geq n(k-1)$. For notational convenience, we may assume that $u \in V_1$ and $u = v_{1,1}$.

Suppose first that $v \in V_m^1$ where $2 \leq m \leq k/2$. For the case where $2 + k/2 \leq m \leq k-1$, the argument is similar. For notational convenience, we may assume that $v = v_{m,1}$. For $2 \leq \ell < m$, let $P_{1\ell m}$ denote the collection of n u–v paths of the type $v_{1,1}, v_{\ell,j}, v_{m,1}$ where $1 \leq j \leq n$. Let P_{1m} be the collection of n u–v paths consisting of the path $v_{1,1}, v_{m,1}$, the $(n-2)/2$ u–v paths of the type $v_{1,1}, v_{m,j}, v_{m+k/2,j}, v_{1,j}, v_{m,1}$ where $2 \leq j \leq n/2$ and the $n/2$ u–v paths of the type $v_{1,1}, v_{m,j}, v_{m+k/2,j}, v_{m,1}$ where $(n+2)/2 \leq j \leq n$. For $m+1 \leq r \leq k/2$, let P_{1rk} denote the collection of n u–v paths of the type $v_{1,1}, v_{r,j}, v_{r+k/2,j}, v_{m,1}$ where $1 \leq j \leq n$. Let $P_{1,1+k/2}$ be the collection of $n/2$ u–v paths of the type $v_{1,1}, v_{1+k/2,j}, v_{1,j+n/2}, v_{m,1}$ where $1 \leq j \leq n/2$. Then, $\{P_{1\ell k} \mid 2 \leq \ell < m\} \cup \{P_{1m}\} \cup \{P_{1rm} \mid m < r \leq k/2\} \cup \{P_{1,1+k/2}\}$ is a collection of $n(k-1)/2$ internally disjoint u–v paths in T. Thus, $\kappa(u,v) \geq n(k-1)/2$. On the other hand, for $m+1 \leq \ell \leq k/2$, let $P_{m\ell 1}$ be the collection of n v–u paths of the type $v_{m,1}, v_{\ell,j}, v_{\ell+k/2,j}, v_{1,1}$ where $1 \leq j \leq n$. Let $P_{m,1+k/2}$ be the collection of n v–u paths consisting of the $n/2$ v–u paths of the type $v_{m,1}, v_{1+k/2,j}, v_{1,1}$ where $(n+2)/2 \leq j \leq n$ and the $n/2$ v–u paths of the type $v_{m,1}, v_{1+k/2,j}, v_{m+k/2,j+n/2}, v_{1,1}$ where $1 \leq j \leq n/2$. For $2+k/2 \leq r \leq m-1+k/2$, let P_{mr1} denote the collection of n v–u paths of the type $v_{m,1}, v_{r,j}, v_{1,1}$ where $1 \leq j \leq n$. Let $P_{m,m+k/2}$ be the collection of $n/2$ v–u paths of the type $v_{m,1}, v_{m+k/2,j}, v_{1,1}$ where $1 \leq j \leq n/2$. Then, $\{P_{m\ell 1} \mid m+1 \leq \ell \leq k/2\} \cup \{P_{m,1+k/2}\} \cup \{P_{mri} \mid 2+k/2 \leq r \leq m-1+k/2\} \cup \{P_{m,m+k/2}\}$ is a collection of $n(k-1)/2$ internally disjoint v–u paths in T. Thus, $\kappa(v,u) \geq n(k-1)/2$. Consequently, $\kappa(u,v) + \kappa(v,u) \geq n(k-1)$.

Suppose secondly that $v \in V_m$ where $m = 1 + k/2$. For notational convenience, we may assume that $v = v_{m,1}$. For $2 \leq \ell \leq k/2$, let $P_{1\ell m}$ denote the collection of n paths of the type $v_{1,1}, v_{\ell,j}, v_{m,1}$ where $1 \leq j \leq n$. Let P_{1m} be the collection of $n/2$ paths consisting of the path $v_{1,1}, v_{m,1}$ and the $(n-2)/2$ paths of the type $v_{1,1}, v_{m,j}, v_{1,j+n/2}, v_{m,j+n/2}, v_{1,j}, v_{m,1}$ where $2 \leq j \leq n/2$. Then, $\{P_{1\ell m} \mid 2 \leq \ell \leq k/2\} \cup \{P_{1m}\}$ is a collection of $n(k-1)/2$ internally disjoint u–v paths in T. On the other hand, let P_{m1} be the collection of $n/2$ paths of the type $v_{m,1}, v_{1,j}, v_{m,j}, v_{1,1}$ where $(n+2)/2 \leq j \leq n$. For $2 + k/2 \leq \ell \leq k$, let $P_{m\ell 1}$ denote the collection of n paths of the type $v_{m,1}, v_{\ell,j}, v_{1,1}$ where $1 \leq j \leq n$. Then, $\{P_{m1}\} \cup \{P_{m\ell 1} \mid 2 + k/2 \leq \ell \leq k\}$ is a collection of $n(k-1)/2$ internally disjoint v–u paths in T. Consequently, $\kappa(u,v) + \kappa(v,u) \geq n(k-1)$.

Suppose finally that $v \in V_1$. For notational convenience, we may assume that $v = v_{1,2}$. For $2 \leq \ell \leq k/2$, let $P_{1\ell 2}$ denote the collection of n paths of the type $v_{1,1}, v_{\ell,j}, v_{\ell+k/2,j}, v_{1,2}$ where $1 \leq j \leq n$. Let P_{12} be the collection of $n/2$ paths of the type $v_{1,1}, v_{1+k/2,j}, v_{1,j+n/2}, v_{1+k/2,j+n/2}, v_{1,2}$ where $1 \leq j \leq n/2$. Then, $\{P_{1\ell 2} \mid 2 \leq \ell \leq k/2\} \cup \{P_{12}\}$ is a collection of $n(k-1)/2$ internally disjoint u–v paths in T. Thus, $\kappa(u,v) \geq n(k-1)/2$. Similarly, $\kappa(v,u) \geq n(k-1)/2$. Consequently,

$\kappa(u,v) + \kappa(v,u) \geq n(k-1)$.

Hence for all pairs u and v of vertices of T, $\kappa(u,v)+\kappa(v,u) \geq n(k-1)$. However, as shown in the proof of Theorem 1, $\kappa(u,v) + \kappa(v,u) \leq n(k-1)$. Consequently, $\kappa(u,v) + \kappa(v,u) = n(k-1)$. Since this is true for all $\binom{nk}{2}$ pairs of vertices of T, it follows that $\overline{\kappa}(T) = n(k-1)/2$. □

Next we determine $\overline{\kappa}_{\max}(K_{k(n)})$ for k odd.

Lemma 6 *For odd* $k \geq 3$,

$$\overline{\kappa}_{\max}(K_{k(n)}) = \frac{n(k-1)}{2}.$$

Proof. By Corollary 2, $\overline{\kappa}_{\max}(K_{k(n)}) \leq n(k-1)/2$. Hence it suffices to show that there is an orientation T of $K_{k(n)}$ such that $\kappa(u,v) + \kappa(v,u) = n(k-1)$ for all pairs of vertices of T, and so $\overline{\kappa}(T) = n(k-1)/2$.

Let $V_1, V_2, \ldots, V_k$ denote the partite sets of $K_{k(n)}$. For $i = 1, 2, \ldots, k$, let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$. Construct T from $K_{k(n)}$ by orienting for every i, $1 \leq i \leq k$, and every j, $1 \leq j \leq (k-1)/2$, the edges joining V_i and V_{i+j} from V_i to V_{i+j}, where subscripts are expressed modulo k.

Let u and v be any two distinct vertices of $K_{k(n)}$. We show that $\kappa(u,v)+\kappa(v,u) = n(k-1)$. For notational convenience, we may assume that $u \in V_1$ and $u = v_{1,1}$.

Suppose first that $v \in V_m$ where $2 \leq m \leq (k+1)/2$. For the case where $(k+3)/2 \leq m \leq k-1$, the argument is similar. For notational convenience, we may assume that $v = v_{m,1}$. For $2 \leq \ell < m$, let $P_{1\ell m}$ denote the collection of n paths of the type $v_{1,1}, v_{\ell,j}, v_{m,1}$ where $1 \leq j \leq n$. Let P_{1m} be the collection of n paths consisting of the path $v_{1,1}, v_{m,1}$ and the $n-1$ paths of the type $v_{1,1}, v_{m,j}, v_{m+(k-1)/2,j}, v_{1,j}, v_{m,1}$ where $2 \leq j \leq n$. For $m+1 \leq r \leq (k+1)/2$, let P_{1rk} denote the collection of n paths of the type $v_{1,1}, v_{r,j}, v_{r+(k-1)/2,j}, v_{m,1}$ where $1 \leq j \leq n$. Then, $\{P_{1\ell k} \mid 2 \leq \ell < m\} \cup \{P_{1m}\} \cup \{P_{1rm} \mid m+1 \leq r \leq (k+1)/2\}$ is a collection of $n(k-1)/2$ internally disjoint u–v paths in T. Thus, $\kappa(u,v) \geq n(k-1)/2$. On the other hand, for $m+1 \leq \ell \leq (k+1)/2$, let $P_{m\ell 1}$ be the collection of n paths of the type $v_{m,1}, v_{\ell,j}, v_{\ell+(k-1)/2,j}, v_{1,1}$ where $1 \leq j \leq n$. For $(k+3)/2 \leq r \leq m+(k-1)/2$, let P_{mr1} denote the collection of n paths of the type $v_{m,1}, v_{r,j}, v_{1,1}$ where $1 \leq j \leq n$. Then, $\{P_{m\ell 1} \mid m+1 \leq \ell \leq (k+1)/2\} \cup \{P_{mri} \mid (k+3)/2 \leq r \leq m+(k-1)/2\}$ is a collection of $n(k-1)/2$ internally disjoint v–u paths in T. Thus, $\kappa(v,u) \geq n(k-1)/2$. Consequently, $\kappa(u,v) + \kappa(v,u) \geq n(k-1)$.

Suppose secondly that $v \in V_1$. For notational convenience, we may assume that $v = v_{1,2}$. For $2 \leq \ell \leq (k+1)/2$, let $P_{1\ell 2}$ denote the collection of n paths of the type $v_{1,1}, v_{\ell,j}, v_{\ell+(k-1)/2,j}, v_{1,2}$ where $1 \leq j \leq n$. Then, $\{P_{1\ell 2} \mid 2 \leq \ell \leq (k+1)/2\}$ is a collection of $n(k-1)/2$ internally disjoint u–v paths in T. Thus, $\kappa(u,v) \geq n(k-1)/2$. Similarly, $\kappa(v,u) \geq n(k-1)/2$. Consequently, $\kappa(u,v) + \kappa(v,u) \geq n(k-1)$.

Hence for all pairs u and v of vertices of T, $\kappa(u,v)+\kappa(v,u) \geq n(k-1)$. However, as shown in the proof of Theorem 1, $\kappa(u,v) + \kappa(v,u) \leq n(k-1)$. Consequently,

$\kappa(u,v)+\kappa(v,u)=n(k-1)$. Since this is true for all $\binom{nk}{2}$ pairs of vertices of T, it follows that $\overline{\kappa}(T)=n(k-1)/2$. □

Lemmas 4, 5, and 6 imply the following result.

Theorem 7 *For integers $k\geq 2$ and $n\geq 2$,*

$$\overline{\kappa}_{\max}(K_{k(n)})=\begin{cases}\frac{n(k-1)}{2} & \text{if } k \text{ is odd or if } k \text{ and } n \text{ are even}\\[2ex] \frac{n(k-1)}{2}-\frac{kn-2}{4(kn-1)} & \text{if } k \text{ is even and } n \text{ is odd}\end{cases}$$

As a special case of Theorem 7, we have the following result.

Corollary 8

$$\overline{\kappa}_{\max}(K_{n,n})=\begin{cases}\frac{n}{2} & \text{if } n \text{ is even}\\[2ex] \frac{n}{2}-\frac{n-1}{2(2n-1)} & \text{if } n \text{ is odd}\end{cases}$$

3 Minimum Values

We now turn our attention to the problem of finding the minimum average connectivity among all orientations of the complete multipartite graph. We begin with the following result in [5].

Theorem 9 (Henning and Oellermann [5]) *If G is a graph of order p and size q, then $\overline{\kappa}_{\min}(G)\geq q/p(p-1)$. Moreover, equality holds if and only if G is bipartite.*

As a special case of Theorem 9, we have the following result.

Corollary 10

$$\overline{\kappa}_{\min}(K_{m,n})=\frac{mn}{(m+n)(m+n-1)}.$$

For a digraph D and an (ordered) pair u,v of vertices of D, let $\kappa_{\geq 2}(u,v)$ be the maximum number of internally disjoint u–v paths of D having length at least 2. Let

$$K_{\geq 2}(D)=\sum_{u,v\in V}\kappa_{\geq 2}(u,v).$$

Then the total connectivity is given by $K(D)=q(D)+K_{\geq 2}(D)$, where $q(D)$ is the number of arcs in D. So, $K(D)\geq q(D)$ for any digraph D.

We now determine $\overline{\kappa}_{\min}(K_{n_1,n_2,\ldots,n_k})$. For this purpose, let $V_1,V_2,\ldots,V_k$ be the partite sets of $K_{n_1,n_2,\ldots,n_k}$ where $|V_i|=n_i$. An orientation of $K_{n_1,n_2,\ldots,n_k}$ is a *transitive orientation*, denoted $K^{\rightarrow}_{n_1,n_2,\ldots,n_k}$, if for every i and j, $1\leq i<j\leq k$, the arcs between V_i and V_j are directed from V_i to V_j.

Lemma 11 *If* $p = n_1 + n_2 + \cdots + n_k$ *, then*

$$\overline{\kappa}(K^{\rightarrow}_{n_1,n_2,\ldots,n_k}) = \left(\sum_{1\le i<j\le k} n_i n_j + \sum_{1\le i<j<t\le k} n_i n_j n_t \right) / \, p(p-1).$$

Proof. Since

$$q(T) = \sum_{1\le i<j\le k} n_i n_j$$

and

$$\begin{aligned} K_{\geq 2}(T) &= n_1n_2n_3 + n_1(n_2+n_3)n_4 + \cdots + n_1(n_2+n_3+\cdots+n_{k-1})n_k \\ &\quad + n_2n_3n_4 + n_2(n_3+n_4)n_5 + \cdots + n_2(n_3+n_4+\cdots+n_{k-1})n_k \\ &\quad + \cdots + n_{k-2}n_{k-1}n_k \\ &= \textstyle\sum_{1\le i<j<t\le k} n_i n_j n_t, \end{aligned}$$

we have

$$K(T) = \sum_{1\le i<j\le k} n_i n_j + \sum_{1\le i<j<t\le k} n_i n_j n_t. \qquad \square$$

Lemma 12 $\overline{\kappa}_{\min}(K_{n_1,n_2,\ldots,n_k}) \geq \overline{\kappa}(K^{\rightarrow}_{n_1,n_2,\ldots,n_k})$.

Proof. We proceed by induction on $k \geq 2$. The result is obvious when $k = 2$. Assume that the result holds for k-partite tournaments. Consider now any complete $(k+1)$-partite graph $G = K_{n_1,n_2,\ldots,n_{k+1}}$. Let $V_1, V_2, \ldots, V_{k+1}$ be the partite sets of G where $|V_i| = n_i$. Let D be an orientation of G such that $\overline{\kappa}(D) = \overline{\kappa}_{\min}(G)$. Let T_1 be the orientation of G obtained from D by reorienting all arcs of D that are directed from vertices in V_{k+1} to vertices in V_j $(1 \le j \le k)$ (if any) so that they are directed from vertices in V_j to those in V_{k+1}. Then all vertices of V_{k+1} have outdegree 0 in T_1.

We now show that $K(T_1) \le K(D)$, from which it clearly follows that $\overline{\kappa}(T_1) \le \overline{\kappa}(D)$. Let $u \in V_{k+1}$, and let I_u be the vertices adjacent to u in D and O_u the vertices adjacent from u in D.

For each $x \in I_u$ and each $y \in O_u$, x, u, y is a path of length 2 which gets counted once in $\kappa_D(x,y)$ and hence gets counted once in $K_{\geq 2}(D)$. Let $\mathcal{P}_u$ be the collection of these paths that get counted in $K_{\geq 2}(D)$. So for each $u \in V_{k+1}$, there are at least $|I_u| \cdot |O_u|$ paths of length 2 which each get counted once in $K_{\geq 2}(D)$. As these paths no longer exist in T_1, they do not get counted in $K_{\geq 2}(T_1)$. Hence there are at least $\sum_{u\in V_{k+1}} |I_u| \cdot |O_u|$ paths of length 2 which each get counted once in $K_{\geq 2}(D)$ but do not get counted in $K_{\geq 2}(T_1)$.

For every $x \in I_u$ and every $y \in O_u$, there is at most one x–u containing the edge yu in T_1 that is counted in $K_{\geq 2}(T_1)$. Let $\mathcal{Q}_u$ be the collection of these paths that get counted in $K_{\geq 2}(T_1)$. So for each $u \in V_{k+1}$, there are at most $|I_u| \cdot |O_u|$ paths of length 2 from vertices in $D - V_{k+1}$ to vertices of V_{k+1}, that get counted in $K_{\geq 2}(T_1)$ but do not get counted in $K_{\geq 2}(D)$. Hence there are at most $\sum_{u\in V_{k+1}} |I_u| \cdot |O_u|$ paths of length 2 from vertices in $D - V_{k+1}$ to vertices of V_{k+1}, that get counted in $K_{\geq 2}(T_1)$ but do not get counted in $K_{\geq 2}(D)$.

All paths of length at least 2 in D, which are not in $\cup_{u \in V_{k+1}} \mathcal{P}_u$, but that were counted in $K_{\geq 2}(D)$ either still exist in T_1 (if they did not contain internal vertices of V_{k+1}) or they no longer exist in T_1 if they do contain internal vertices from V_{k+1}. As no paths of length at least 2 other than those in $\cup_{u \in V_{k+1}} \mathcal{Q}_u$, get counted in $K_{\geq 2}(T_1)$ if they are not also counted in $K_{\geq 2}(D)$ it now follows that $K_{\geq 2}(T_1) \leq K_{\geq 2}(D)$. Hence, $\overline{\kappa}(T_1) \leq \overline{\kappa}(D)$. By our choice of D, $\overline{\kappa}(T_1) = \overline{\kappa}(D)$.

Let $D' = D - V_{k+1}$. Then, D' is an orientation of $K_{n_1, n_2, \ldots, n_k}$. Moreover, $K_{\geq 2}(T_1) = K_{\geq 2}(D') + q(D')n_{k+1}$. (Note that there are exactly $q(D')n_{k+1}$ paths of length at least 2 with one end in D' and the other in V_{k+1} that get counted in $K_{\geq 2}(T_1)$. All other paths of length at least 2 that get counted in $K_{\geq 2}(T_1)$ have both ends in D'.)

By the inductive hypothesis, $\overline{\kappa}(D') \geq \overline{\kappa}(T')$ where T' is a transitive orientation of $K_{n_1, n_2, \ldots, n_k}$. So, $K(D') \geq K(T')$. Since $K_1(D') = K_1(T') = q(D')$, it follows that $K_{\geq 2}(D') \geq K_{\geq 2}(T')$. Hence, $K_{\geq 2}(T_1) \geq K_{\geq 2}(T') + q(D')n_{k+1}$.

Let T be the transitive orientation of G obtained from $T' \cup V_{k+1}$ by orienting all the edges of G between vertices of T' and V_{k+1} from vertices of T' to vertices of V_{k+1}. Then,

$$K_{\geq 2}(T) = K_{\geq 2}(T') + q(T')n_{k+1} = K_{\geq 2}(T') + q(D')n_{k+1} \leq K_{\geq 2}(T_1).$$

So, $K(T) \leq K(T_1) = K(D)$. By our choice of D, $K(T) = K(D)$. This completes the proof of Lemma 12. □

The results of lemmas 11 and 12 can be summarized as follows.

Theorem 13

$$\overline{\kappa}_{\min}(K_{n_1, n_2, \ldots, n_k}) = \left(\sum_{1 \leq i < j \leq k} n_i n_j + \sum_{1 \leq i < j < t \leq k} n_i n_j n_t \right) \Big/ \, p(p-1) = \overline{\kappa}(T),$$

where T is a transitive orientation of $K_{n_1, n_2, \ldots, n_k}$ and $p = n_1 + n_2 + \cdots + n_k$.

We have yet to establish whether the transitive orientation of $K_{n_1, n_2, \ldots, n_k}$ is the only orientation that achieves $\overline{\kappa}_{\min}(K_{n_1, n_2, \ldots, n_k})$.

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