On the decomposition of Cayley graphs into isomorphic trees

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Abstract

We prove a theorem about the decomposition of certain n-regular Cayley graphs into any tree with n edges. This result implies that the product of any r cycles of even length and the cube Q_s decomposes into copies of any tree with $2r + s$ edges.

1 Introduction

By a *decomposition* of a graph G we mean a sequence $G_1, G_2, \ldots, G_k$ of subgraphs whose edge sets partition the edge set of G. If each subgraph G_i is isomorphic to a fixed graph H, then we say that H *divides* G. Many papers and indeed a whole book [1] have been written about graph decompositions. Much investigation has been motivated by the 1963 conjecture of Ringel [9] that the complete graph on $2n + 1$ vertices can be decomposed into copies of any tree with n edges. A generalisation of Ringel's conjecture, attributed to Graham and Häggkvist (see [4]), proposes that every tree T with n edges divides every $2n$-regular graph G. When G is bipartite, Häggkvist [4] proposes an even stronger variant.

Conjecture 1 *Every tree T with n edges divides every n-regular bipartite graph G.*

In the early 90s, Fink [2], Ramras [7], and Jacobson, Truszczyński, and Tuza [6] independently discovered a theorem which verifies Conjecture 1 when G is the n-cube. The *n-cube*, denoted Q_n, is defined to be the graph with vertices all binary n-tuples and with two n-tuples adjacent when they differ in exactly one component.

It is easily seen that Q_n is n-regular, has 2^n vertices and $n2^{n-1}$ edges, and is bipartite. Indeed every edge joins an *even* vertex (one with an even number of 1s) to an *odd* vertex. The theorem referred to is the following.

Theorem 1 *Let T be any tree with n vertices. Then T divides Q_n.*

Fink and Ramras give essentially the same proof. The tree T is embedded in Q_n by having each edge move in a different coordinate. Then adding all even n-tuples to the embedded graph gives the required decomposition of Q_n. (If we think of the vertex set of Q_n as Z_2^n, then adding a fixed n-tuple is a graph automorphism.)

In his original paper Ramras considered replacing the addition of even n-tuples by the addition of an arbitrary subgroup of the automorphism group of Q_n, defining a set of edges of Q_n to be *fundamental* if applying such a subgroup to it gave a decomposition of Q_n. Later [8] he investigated fundamental sets of edges of Q_n of cardinality $2n$.

Fink went in a different direction in [3], generalizing from cubes to certain Cayley graphs. His Cayley graphs are directed, so we will start with this definition to explain his results. Of course for directed graphs to be isomorphic the edge directions must correspond. With this restriction the definition of one directed graph dividing another is analogous to that for undirected graphs. The edge directed from u to v will be denoted (u, v). Let Γ be a finite group and Δ a subset of Γ. By the *directed Cayley graph of Γ with respect to Δ* we mean the graph $G_d = G_d(\Gamma, \Delta)$ with vertex set Γ and edge set

$$\{(u, ud) : u \in \Gamma,\ d \in \Delta\}.$$

By an *oriented tree* we mean an ordinary tree in which each edge has been assigned a direction. The following is Fink's main result.

Theorem 2 *If Γ is a group with minimum generating set Δ, and if T is an oriented tree with $|\Delta|$ edges, then T divides $G_d(\Gamma, \Delta)$.*

By taking $\Gamma = Z_2^n$ and $\Delta = \{u_1, u_2, \cdots, u_n\}$, where u_i is the n-tuple with 1 in the ith coordinate and 0s elsewhere, the previous theorem can be applied to n-cubes. The resulting directed Cayley graph is Q_n with each edge replaced by two directed edges, one in each direction. By using only the edges directed from even to odd vertices Fink gets another proof of Theorem 1.

We will prove a result similar to Fink's (Theorem 2) for certain undirected Cayley graphs. Although the edges of our original tree will not be directed, we make up for this by allowing it to have $2|\Delta|$ edges instead of $|\Delta|$ in many applications. We also replace the condition that Δ be a minimal set of generators with a weaker one. Our theorem has Theorem 1 as a direct consequence.

Jacobson *et al.* define a $(p+q)$-*tree* to be a tree with a bipartite vertex partition with p vertices on one side and q on the other. They prove a more general result than Theorem 1, producing a decomposition of Q_n into a list of 2^{n-1} arbitrary $(p+q)$-trees, where $p+q = n+1$. They moreover verify Conjecture 1 when T is a *double-star* with n edges (a tree of diameter ≤ 3 with at most two vertices of degree greater than 1) and in the case when T is the path with 4 edges.

Later Horak, Širaň, and Wallis [5] generalized Theorem 1 in various ways. For example they show the tree in the theorem can be replaced with any graph with n edges, each block of which is an even cycle or an edge.

2 Main result

We can avoid loops in Cayley graphs, directed or not, by requiring that $1 \notin \Delta$. Defining undirected Cayley graphs presents a slight additional complication. In the directed case each element d of Δ generates $|\Gamma|$ corresponding edges (g, gd) of the Cayley graph $G_d(\Gamma, \Delta)$. The same holds in the undirected case, except when d is its own inverse, that is, d has order 2. Then the edges $\{g, gd\}$ and $\{gd, (gd)d\}$ are the same, so only $\Gamma/2$ edges result. This is why elements of order 2 get special treatment in what follows.

Let Γ be a finite group. Suppose Δ is a subset of Γ that does not contain the identity 1. By the *Cayley graph of* Γ *with respect to* Δ we mean the graph $G = G(\Gamma, \Delta)$ such that $V(G) = \Gamma$ and $E(G) = \{\{g, gd\} : g \in \Gamma,\ d \in \Delta\}$. Let Δ_2 denote the set of elements of Δ of order 2 and Δ^+ its complement in Δ. We call Δ *square-independent* if whenever $d_1, d_2, \ldots, d_t$ are distinct elements of Δ and $\prod_{i=1}^{t} d_i^{j_i} = 1$, where $j_i \in \{-2, -1, 0, 1, 2\}$ if $d_i \in \Delta^+$, and $j_i \in \{-1, 0, 1\}$ if $d_i \in \Delta_2$, $1 \le i \le t$, then all the exponents j_i are 0. This implies that if $d \in \Delta^+$, then $d^{-1} \notin \Delta^+$. As we have seen, if $d \in \Delta^+$, then there are $|\Gamma|$ edges of the form $\{g, gd\}$ in G, while if $d \in \Delta_2$ there are half this number. Thus if Δ is square-independent, then the Cayley graph $G(\Gamma, \Delta)$ has $|\Gamma|(|\Delta^+| + |\Delta_2|/2)$ edges.

We will use two well-known results in the proof of the following theorem. One is that if $h \in \Gamma$, then the map that sends each vertex g into hg is a (graph) automorphism of G. The other is that any connected graph with an even number of edges has a decomposition into paths with two edges.

Theorem 3 *Let* Γ *be a finite group, and let* Θ *be a subgroup of* Γ *with index 2. Suppose that* Δ *is a square-independent subset of* $\Gamma \backslash \Theta$ *containing* r *elements of order greater than* 2 *and* s *elements of order* 2. *Then if* T *is any tree with* $2r + s$ *edges,* T *divides* $G(\Gamma, \Delta)$.

Proof. We can remove s edges from T, leaving a tree T^* with $2r$ edges. Let T^* have the decomposition $P_1, P_2, \ldots, P_r$, where P_i is a path with 2 edges for each i. Let $\Delta^+ = \{d_1, d_2, \ldots, d_r\}$ and $\Delta_2 = \{e_1, e_2, \ldots, e_s\}$. Assign the labels $\{e_1, e_2, \ldots, e_s\}$ to the s edges of T not in T^*, and assign to the two edges of P_i the labels d_i and d_i^{-1}, $1 \le i \le r$.

Now we will assign labels to the vertices of T. Choose one vertex v_0 arbitrarily and assign it the label 1. If v is any vertex of T and $h_1, h_2, \ldots, h_m$ are the labels on the edges of the unique simple path in T from v_0 to v, in order, then give v the label $h_1 h_2^{-1} h_3 h_4^{-1} \cdots h_m^{(-1)^{m-1}}$. We claim that each vertex of T is assigned a different label by this scheme. For suppose two vertices u and v get the same label. Let x be the label on the vertex of T farthest from v_0 that is on both the path from v_0 to u and

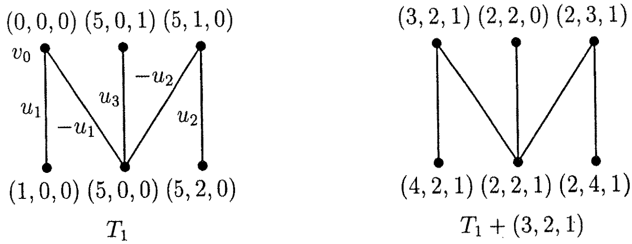

Figure 1: The tree T_1 and one of its translates

the path from v_0 to v. Then the common label on u and v has the form

$$xg_1^{j_1}g_2^{j_2}\cdots g_m^{j_m} = xh_1^{k_1}h_2^{k_2}\cdots h_n^{k_n},$$

where $g_1,\ldots,g_m,h_1,\ldots,h_n$ are elements of Δ, distinct except that perhaps $g_1 = h_1 \in \Delta^+$ and $j_1 = -k_1 = \pm 1$, j_i (respectively k_i) $\in \{-1,0,1\}$ if g_i (respectively h_i) $\in \Delta_2$, and j_i (respectively k_i) $\in \{-2,-1,0,1,2\}$ if g_i (respectively h_i) $\in \Delta^+$. But then

$$g_m^{-j_m}g_{m-1}^{-j_{m-1}}\cdots g_1^{-j_1}h_1^{k_1}h_2^{k_2}\cdots h_n^{k_n} = 1,$$

contradicting the assumption that Δ is square-independent.

Now we can use the vertex labels of T to identify it with an isomorphic subgraph T_1 in G because the labels of the endpoints of each edge of T are of the form g, gd, where $d \in \Delta$. We claim that the subgraphs hT_1, $h \in \Theta$ form a decomposition of G. Note that these subgraphs account for $(2r+s)|\Theta|/2$ edges, which is the number of edges in G. Thus it suffices to show that if g and h are distinct elements of Θ, then gT_1 and hT_1 have no common edge. Suppose that $g\{x, xd\} = h\{y, ye\}$ with d and e in Δ.

Case 1 $gx = hy$ and $gxd = hye$

Then $d = e$. Now the edges $\{x, xd\}$ and $\{y, yd\}$ of T_1 are either identical or at least have a common vertex. Since $x = y$ implies $g = h$ we assume $xd = y$. Then $gx = hy = hxd$, so $d = x^{-1}h^{-1}gx$. This is a contradiction because Θ, with index 2, is a normal subgroup of Γ, and so the latter element is in Θ.

Case 2 $gx = hye$ and $gxd = hy$

Then $gx = hye = gxde$ and so $de = 1$. Thus $d = e \in \Delta_2$. But there is only one edge in T with label d, so $\{x, xd\} = \{y, yd\}$. If $x = y$, then $gx = hxd$, and again $d = x^{-1}h^{-1}gx \in \Theta$. But $x = yd$ leads to $gx = hyd = hx$, and $g = h$. ∎

An example of the above proof is shown in Figure 1. Here we have taken $\Gamma = Z_6 \times Z_8 \times Z_2$, $\Theta = \{(x_1, x_2, x_3) \in \Gamma : x_1 + x_2 + x_3 \equiv 0 \pmod 2\}$, and $\Delta = \{u_1 =$

$(1,0,0), u_2 = (0,1,0), u_3 = (0,0,1)\}$. Note that u_1 and u_2 are in Δ^+, while $u_3 \in \Delta_2$. On the left is shown a labeling leading to the tree T_1 embedded in $G(\Gamma, \Delta)$. On the right is shown the translation of T_1 by $(3,2,1) \in \Theta$.

3 An application to product graphs

Let G_1 and G_2 be graphs. We define the *product* of G_1 and G_2, denoted $G_1 \times G_2$, to be the graph with vertex set $V(G_1) \times V(G_2)$ and with vertices (u,v) and (x,y) adjacent exactly when $u = x$ and v and y are adjacent in G_2 or u and x are adjacent in G_1 and $v = y$. Graph products provide an alternate definition of the cube, since Q_n is the product of K_2 with itself n times.

Theorem 4 *Let $C_{m_1}, C_{m_2}, \ldots, C_{m_r}$ be cycles with an even number of edges, let s be a nonnegative integer, and set $G = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_r} \times Q_s$. Then if T is any tree with $2r+s$ edges, T divides G.*

Proof. Let Γ be the group $Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r} \times Z_2^s$. Let Θ be the kernal of the natural homomorphism from Γ onto Z_2 given by $(x_1, x_2, \ldots, x_{r+s}) \to x_1 + x_2 + \ldots x_{r+s}$. Let u_i be the element of Γ has ith component 1 and zeros otherwise, and take $\Delta^+ = \{u_1, u_2, \ldots, u_r\}$ and $\Delta_2 = \{u_{r+1}, u_{r+2}, \ldots, u_{r+s}\}$. It may be checked that Δ is square-independent. Then G is the Cayley graph $G(\Gamma, \Delta)$ and the previous theorem applies. ∎

Figure 1 illustrates part of a tree decomposition of $C_6 \times C_8 \times Q_1$.

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