Balanced sampling plans with block size four excluding contiguous units

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Abstract

Constructions of balanced sampling plans excluding contiguous units, a class of designs introduced by Hedayat, Rao and Stufken, are given which provide a complete solution to this problem when $k = 4$.

1 Introduction

Consider a finite ordered population of v identifiable units, labeled as $0, 1, \ldots, v-1$. Let Λ_i denote a quantitative characteristic Λ for unit i. While the Λ_i's are unknown, they can be observed for any unit. Observation of the Λ_i's for a sample of k units at a time, for $k < v$, is performed to estimate the population total $T = \sum_{i=0}^{t-1} \Lambda_i$. In some applications, Λ_i's for contiguous units are expected to be similar. In these settings, it is natural to sample k units so that contiguous or neighboring units of the v units are less likely to appear together than units that are further apart. Hedayat, Rao and Stufken [8, 9] and Stufken [12] justify this idea in terms of the variance of the Horvitz-Thompson [10] estimator. This is the motivation for considering a special class of sampling plans [8, 9], and extensions [12]. A *balanced sampling plan excluding contiguous units* for a population of size v, with block size k, denoted BSEC(v, k, λ), is a block design with the properties that

(i) each block is a set of k different units,

(ii) each unit appears in the same number of blocks, say r,

(iii) any two contiguous units do not appear simultaneously in any of the blocks,

(iv) any two noncontiguous units appear simultaneously in the same number λ of blocks.

Throughout this paper, we assume that the units labeled as 0 and $v-1$ are also contiguous units. This assumption may not always be justified. However, when it fails to hold, balanced sampling plans excluding contiguous units can nonetheless give a considerable reduction in the variance of the Horvitz-Thompson estimator of T [11]. Hedayat, Rao and Stufken [8, 9] established the following:

Lemma 1.1 *(1) For $k \geq 3$, if a BSEC(v,k,λ) exists, then $v \geq 3k$.*

(2) For $k = 3,4$ a BSEC(v,k,λ) exists for some λ if $v \geq 3k$.

They also construct designs for $k = 5$ and $v = 23 + 3w$, for any nonnegative integer w, and some λ. An iterative method plays an important role, constructing a BSEC$(t+3, k, \lambda_2)$ from a BSEC(v,k,λ_1), where $\lambda_2 = \lambda_1(t-1)$. Colbourn and Ling [4] settle existence of BSEC$(v,3,\lambda)$s with smallest index λ.

Similarly defined designs appear in the combinatorial literature, such as cycloids [6]. In fact, a BSEC$(v,4,\lambda)$ is equivalent to a partial block design with block size four whose leave contains the pairs of a v-cycle each λ times; to recover the BSEC, relabel the elements of the partial design using $0,\ldots,v-1$ so that the leave contains pairs $\{\{i,i+1\} : 0 \leq i < v\}$ and $\{0, v-1\}$. These designs have also arisen in the study of regular packings with block size four [2], and the existence of BSEC$(v,4,1)$s has been asked in that context.

The solutions in Lemma 1.1 give values of λ that are very large, while one typically prefers designs with few blocks and hence smaller values of λ. When $k = 4$, every element x lies in $v-3$ distinct pairs of the form $\{x,y\}$ in which x and y are not contiguous. It follows that at least one of v or λ is a multiple of 3. Considering the collection of blocks in a BSEC$(v,4,\lambda)$, we find $\lambda\frac{v(v-3)}{2}$ pairs in total, six per block, so that $\lambda v(v-3) \equiv 0 \pmod{12}$. It follows that either λ is even or $v \equiv 0,3 \pmod 4$. We conclude that:

Lemma 1.2 *If a BSEC$(v,4,\lambda)$ exists, then $v \geq 12$ and*

$$\begin{aligned} v \equiv 0,3 \pmod{12} \quad &\textit{and} \quad \lambda \equiv 1,5 \pmod 6, \\ v \equiv 0 \pmod 3 \quad &\textit{and} \quad \lambda \equiv 2,4 \pmod 6, \\ v \equiv 0,3 \pmod 4 \quad &\textit{and} \quad \lambda \equiv 3 \pmod 6, \ \textit{or} \\ v \ \textit{arbitrary} \quad &\textit{and} \quad \lambda \equiv 0 \pmod 6. \end{aligned}$$

Since the union of a BSEC$(v,4,\lambda_1)$ and a BSEC$(v,4,\lambda_2)$ is a BSEC$(v,4,\lambda_1+\lambda_2)$, it suffices to establish the existence of BSECs for the minimum value of λ, namely for

$$\begin{aligned} v \equiv 0,3 \pmod{12} \quad &\text{and} \quad \lambda = 1, \\ v \equiv 6,9 \pmod{12} \quad &\text{and} \quad \lambda = 2, \\ v \equiv 4,7,8,11 \pmod{12} \quad &\text{and} \quad \lambda = 3, \text{ and} \\ v \equiv 1,2,5,10 \pmod{12} \quad &\text{and} \quad \lambda = 6. \end{aligned}$$

We give a uniform construction for $\lambda \in \{1,2,3,6\}$.

2 Small Orders

Our basic strategy to produce BSECs is to establish the existence of a number of designs of small order, and then to employ a recursive method to complete the proof of existence.

In order to avoid the presentation of many large designs, whenever possible we present a collection of *base blocks* for a cyclic BSEC$(v,4,\lambda)$. For each $\{a,b,c\}$ shown, the base block is $\{0,a,b,c\}$. The design itself is obtained by including, for each $\{a,b,c\}$, the v blocks $\{\{i,a+i,b+i,c+i\} : 0 \leq i < v\}$ in which elements are reduced modulo v into the range $0,\ldots,v-1$. Tables 1, 2, and 3 give base blocks for cyclic BSEC$(v,4,\lambda)$s. When 'h' is indicated with the index λ, one is to add the half orbit whose base block is $\{0,2,\frac{v}{2},\frac{v}{2}+2\}$. When 'q' is indicated with the index λ, one is to add the quarter orbit whose base block is $\{0,\frac{v}{4},\frac{2v}{4},\frac{3v}{4}\}$.

v	λ	Base Blocks
15	1	{2,5,9}
27	1	{2,6,13} {3,8,18}
39	1	{2,5,27} {4,11,30} {6,16,24}
51	1	{2,5,14} {4,19,26} {6,17,33} {8,21,31}
87	1	{2,5,9} {6,14,29} {10,34,60} {11,42,54} {13,38,59} {16,35,67} {17,39,57}
99	1	{2,5,9} {6,14,24} {11,36,65} {12,39,67} {13,43,64} {15,46,62} {17,40,66} {19,41,61}
12	2 h	{2,5,9}
21	2	{2,4,9} {3,8,14} {3,9,13}
30	2 h	{2,5,9} {3,9,19} {4,12,20} {5,11,18}
33	2	{2,4,7} {3,12,21} {4,15,20} {6,13,23} {6,14,25}
42	2 h	{2,5,8} {4,8,20} {5,14,29} {6,15,26} {7,17,30} {7,18,32}
45	2	{2,4,7} {3,7,18} {5,16,29} {6,19,28} {6,20,28} {8,18,33} {9,19,33}
54	2 h	{2,5,8} {4,8,14} {5,20,35} {7,20,36} {7,24,35} {9,21,37} {9,22,40} {10,22,33}
57	2	{2,4,7} {3,7,12} {6,20,35} {6,22,38} {8,25,36} {8,26,39} {9,23,42} {10,27,37} {11,23,44}
66	2 h	{2,5,11} {4,14,43} {7,26,48} {8,24,54} {13,34,51} {3,7,12} {6,28,52} {8,31,49} {10,37,50} {11,32,47}
69	2	{2,5,9} {6,14,27} {10,25,49} {11,28,51} {12,31,47} {2,5,11} {4,23,41} {7,33,43} {8,29,45} {12,27,47} {13,30,44}
78	2 h	{3,7,12} {6,14,41} {10,36,60} {11,33,58} {13,30,62} {15,38,59} {2,5,9} {6,14,30} {10,29,61} {11,36,56} {12,40,55} {13,34,60}
81	2	{2,5,9} {6,14,24} {11,30,53} {12,33,59} {13,38,54} {15,32,52} {2,5,9} {6,16,44} {8,32,54} {11,36,50} {12,35,52} {13,31,61} {15,34,60}

Table 1: Small Cyclic BSECs: $\lambda \in \{1,2\}$

When $v = 12$ and $\lambda = 1$, the number of blocks in total is only nine, and hence this BSEC cannot exist by the Fisher-type inequality in [8]. We therefore provide solu-

v	λ	Base Blocks
12	3 q	{2,4,7} {2,6,9}
16	3 q	{2,4,6} {3,6,12} {3,7,11}
19	3	{2,4,9} {2,5,13} {3,7,13} {3,8,12}
20	3 q	{2,4,6} {3,6,11} {3,8,15} {4,9,14}
23	3	{2,4,7} {2,8,14} {3,10,14} {3,11,16} {4,9,17}
28	3 q	{2,4,6} {3,6,10} {3,10,19} {5,10,21} {5,12,19} {6,13,21}
31	3	{2,4,6} {3,6,15} {3,11,20} {4,14,19} {5,12,21} {5,13,23} {6,13,20}
32	3 q	{2,4,6} {3,6,9} {4,11,21} {5,12,24} {5,13,24} {5,14,22} {7,15,23}
35	3	{2,4,6} {3,6,13} {3,12,21} {4,14,25} {5,16,24} {5,17,22} {6,15,22} {7,15,27}
40	3 q	{2,4,6} {3,6,9} {4,9,14} {5,12,28} {7,17,30} {7,22,32} {8,18,28} {8,19,30} {9,19,30}
43	3	{2,4,6} {3,6,9} {4,11,27} {5,17,27} {5,18,29} {5,19,31} {7,20,28} {7,21,30} {8,17,32} {8,18,33}
44	3 q	{2,4,6} {3,6,9} {4,9,14} {5,12,19} {7,18,33} {8,18,33} {8,19,30} {8,20,31} {9,20,33} {10,21,33}
47	3	{2,4,7} {3,7,15} {5,15,33} {6,22,35} {6,23,34} {8,24,33} {9,26,36} {2,5,15} {4,21,28} {6,20,31} {8,20,29}
52	3 q	{2,6,35} {3,31,39} {5,15,45} {9,27,41} {2,6,35} {3,31,39} {5,15,45} {9,27,41} {2,7,31} {3,35,46} {4,26,40} {8,27,42}
55	3	{2,5,13} {4,20,35} {6,21,38} {7,25,34} {10,22,36} {2,5,12} {4,13,32} {6,24,35} {8,25,41} {2,5,12} {4,13,31} {6,25,39} {8,23,34}
56	3 q	{2,5,13} {4,19,25} {7,29,47} {10,30,42} {2,5,13} {4,19,29} {6,18,42} {7,30,47} {2,5,18} {4,19,36} {6,27,34} {8,25,34} {10,21,33}
59	3	{2,5,9} {6,17,37} {8,32,46} {10,26,44} {2,5,13} {4,27,34} {6,22,39} {9,23,44} {10,28,40} {2,5,9} {6,19,41} {8,27,39} {10,21,44} {12,26,42}
67	3	{2,5,11} {4,23,43} {7,28,41} {8,29,44} {10,27,45} {12,26,42} {2,5,9} {6,16,33} {8,21,45} {11,36,55} {14,29,49} {2,5,9} {6,16,33} {8,26,45} {11,35,47} {13,38,52}
68	3 q	{2,5,11} {4,26,40} {7,31,50} {8,31,47} {10,35,48} {12,34,53} {2,5,9} {6,16,48} {8,29,54} {11,28,55} {12,35,50} {2,5,9} {6,16,48} {8,29,57} {12,30,45} {13,37,54}
71	3	{2,5,9} {6,14,27} {10,32,48} {11,31,56} {12,30,47} {2,5,9} {6,20,43} {8,30,45} {10,29,46} {11,32,44} {13,29,53} {2,5,9} {6,16,41} {8,27,51} {11,26,49} {12,29,43} {13,32,50}

Table 2: Small Cyclic BSECs: $\lambda = 3$

v	λ	Base Blocks
13	6	{2,4,8} {2,5,10} {2,6,10} {2,7,9} {3,6,9}
14	6 h	{2,4,7} {2,5,10} {2,6,10} {2,7,10} {3,6,9}
17	6	{2,4,6} {2,5,10} {2,7,12} {2,8,11} {3,7,13} {3,8,11} {3,9,13}
22	6 h	{2,4,6} {2,5,8} {2,7,14} {3,9,16} {3,10,15} {3,11,14} {4,9,17} {4,10,17} {4,12,16}
25	6	{2,4,6} {3,6,13} {3,10,19} {4,9,17} {5,10,17} {2,8,15} {3,11,17} {3,12,15} {4,11,15} {5,11,16}
26	6 h	{2,5,11} {3,12,19} {4,8,18} {2,4,9} {3,8,16} {3,9,15} {4,11,16} {2,4,8} {3,10,17} {3,11,16} {5,11,17}
29	6	{2,4,6} {3,6,12} {3,10,19} {4,11,19} {5,12,20} {5,16,21} {2,4,7} {2,9,19} {3,12,18} {3,14,18} {4,12,20} {5,12,18} {5,13,19}
34	6 h	{2,4,7} {3,13,19} {4,16,23} {5,14,26} {6,17,26} {2,4,7} {3,13,19} {4,16,24} {5,12,25} {6,15,23} {2,5,8} {4,10,22} {4,13,24} {5,13,24} {7,14,23}
37	6	{2,4,7} {3,9,22} {4,14,24} {5,14,26} {6,17,25} {7,15,28} {2,4,7} {3,15,23} {4,15,23} {5,18,24} {6,16,26} {7,16,28} {2,4,7} {3,7,18} {5,17,29} {6,14,28} {6,17,27}
38	6 h	{2,5,8} {4,8,18} {5,12,26} {6,15,28} {7,16,27} {2,4,7} {3,9,23} {4,14,25} {5,15,27} {6,18,25} {8,16,25} {2,4,7} {3,10,23} {4,16,24} {5,16,26} {6,15,29} {6,17,25}
41	6	{2,4,7} {3,7,16} {5,20,31} {6,17,29} {6,20,28} {8,17,31} {2,4,7} {3,7,16} {5,20,31} {6,18,27} {6,19,29} {8,16,27} {2,4,8} {3,14,26} {3,19,25} {5,17,26} {5,18,25} {7,17,31} {8,17,30}
46	6 h	{2,4,7} {3,7,16} {5,17,32} {6,24,30} {8,21,36} {8,25,35} {9,20,32} {2,4,7} {3,7,16} {5,17,33} {6,20,31} {6,21,29} {8,19,28} {10,22,32} {2,5,8} {4,8,21} {5,17,31} {6,20,30} {7,18,34} {7,20,31} {9,19,28}
49	6	{2,4,7} {3,7,19} {5,18,31} {6,20,34} {6,22,31} {8,20,30} {8,23,32} {10,21,38} {2,4,7} {3,7,19} {5,20,30} {6,20,31} {6,22,32} {8,21,36} {8,22,31} {9,21,32} {2,4,7} {3,7,12} {6,14,31} {6,19,35} {8,23,38} {9,22,33} {10,20,37}
50	6 h	{2,5,8} {4,8,13} {6,18,33} {7,20,36} {7,22,38} {9,26,40} {10,21,32} {2,4,7} {3,7,17} {5,20,31} {6,21,34} {6,23,31} {8,20,34} {9,20,38} {9,22,32} {2,4,7} {3,7,17} {5,20,31} {6,20,31} {6,21,33} {8,24,32} {9,21,37} {9,22,32}
53	6	{2,5,9} {6,16,33} {8,18,41} {9,21,40} {11,24,39} {2,5,39} {4,11,28} {6,18,33} {8,30,40} {2,5,39} {4,11,28} {6,18,33} {8,31,40} {2,5,39} {4,11,28} {6,18,44} {8,21,31} {2,5,39} {4,11,28} {6,26,44} {8,21,31} {2,5,39} {4,11,29} {6,15,32} {8,31,41}
58	6 h	{2,5,9} {6,27,41} {8,36,48} {11,24,43} {2,5,11} {4,20,37} {7,23,36} {8,26,38} {10,24,43} {2,5,9} {6,27,45} {8,20,44} {10,26,43} {2,5,11} {4,18,35} {7,26,37} {8,24,36} {10,23,43} {2,5,9} {6,27,45} {8,23,34} {10,30,46} {3,7,15} {5,23,37} {6,25,36} {9,25,38} {10,24,41}

Table 3: Small Cyclic BSECs: $\lambda = 6$

tions for $v = 12$ and $\lambda \in \{2,3\}$. Moreover, when $v = 12t$ and $\lambda = 1$, a putative cyclic solution would require both a half orbit and a quarter orbit. Since both employ the difference $6t$, no such solution can exist. In this case we resort to non-cyclic solutions. The designs in Table 4 are on $\mathbb{Z}_{v/2} \times \{0,1\}$, and are called *bicyclic*. The element (x,i) is written as x_i. When $v \equiv 0 \pmod 4$, the base block $\{0_0,(v/4)_0,0_1,(v/4)_1\}$ generates $v/4$ blocks, while others all generate $v/2$ blocks.

v	λ	Base Blocks
36	1	$\{0_0,9_0,0_1,9_1\}$ $\{0_0,1_0,3_0,8_0\}$ $\{0_0,4_0,1_1,3_1\}$ $\{0_0,6_0,10_1,14_1\}$ $\{0_0,2_1,5_1,13_1\}$ $\{0_0,6_1,7_1,12_1\}$
48	1	$\{0_0,12_0,0_1,12_1\}$ $\{0_0,1_0,2_1,8_1\}$ $\{0_0,2_0,10_0,19_1\}$ $\{0_0,3_1,20_1,22_1\}$ $\{0_0,4_0,15_0,21_0\}$ $\{0_0,4_1,13_1,14_1\}$ $\{0_0,5_0,11_1,15_1\}$ $\{0_0,5_1,18_1,21_1\}$
72	1	$\{0_0,1_1,21_0,32_1\}$ $\{0_0,2_1,12_0,21_1\}$ $\{0_0,4_1,12_1,17_0\}$ $\{0_0,5_1,25_1,28_0\}$ $\{0_0,6_1,19_1,28_1\}$ $\{0_0,8_1,20_1,22_0\}$ $\{0_0,1_0,5_0,11_0\}$ $\{0_1,1_1,3_1,7_1\}$ $\{0_0,7_0,9_0,24_1\}$ $\{0_0,13_0,16_0,7_1\}$ $\{0_0,3_1,14_1,29_1\}$ $\{0_0,18_0,0_1,18_1\}$
84	1	$\{0_0,1_1,24_1,30_0\}$ $\{0_0,3_1,14_1,27_0\}$ $\{0_0,4_1,28_1,31_0\}$ $\{0_0,6_1,16_0,33_1\}$ $\{0_0,7_1,18_0,37_1\}$ $\{0_0,8_1,23_0,34_1\}$ $\{0_0,9_1,28_0,38_1\}$ $\{0_0,10_0,30_1,35_1\}$ $\{0_0,1_0,3_0,7_0\}$ $\{0_0,5_0,13_0,22_0\}$ $\{0_0,2_1,5_1,22_1\}$ $\{0_0,12_1,16_1,26_1\}$ $\{0_1,1_1,7_1,9_1\}$ $\{0_0,21_0,0_1,21_1\}$
18	2	$\{0_0,0_1,1_1,3_1\}$ $\{0_0,0_1,2_1,5_1\}$ $\{0_0,1_0,3_0,7_0\}$ $\{0_0,1_0,4_1,5_1\}$ $\{0_0,1_1,4_0,6_1\}$

Table 4: Small Bicyclic BSECs

We close with a solution for BSEC(24,4,1). We were unable to find a bicyclic solution, but found a solution with an automorphism of order three on $\{0,1,2,3,4,5,6,7\} \times \mathbb{Z}_3$. Its starter blocks are $\{0_0,0_1,1_0,2_0\}$, $\{0_0,2_1,3_0,4_0\}$, $\{0_0,3_1,5_0,5_1\}$, $\{0_0,3_2,6_0,7_0\}$, $\{0_0,4_1,5_2,6_2\}$, $\{0_0,4_2,6_1,7_2\}$, $\{1_0,1_1,4_0,6_0\}$, $\{1_0,2_2,5_2,6_1\}$, $\{1_0,3_2,4_1,5_0\}$, $\{1_0,3_0,3_1,7_0\}$, $\{1_0,5_1,7_1,7_2\}$, $\{2_0,2_1,5_2,7_1\}$, $\{2_0,3_1,6_0,6_1\}$, $\{2_0,4_0,4_1,7_2\}$, and the remaining 28 blocks are obtained by adding 1 and 2 modulo 3 to the subscripts.

3 Indices One and Two

We employ some recursive constructions. We require a definition. An *incomplete transversal design of order n* and *block size k* with *holes* of sizes $h_1, h_2, \ldots, h_l$, or $TD(k,n) - \sum_{i=1}^{l} TD(k,h_i)$, is a quadruple $(X, \mathcal{H}, \mathcal{G}, \mathcal{B})$ with the following properties. X is a set of kn elements. $\mathcal{G}=\{G_1, G_2, \ldots, G_k\}$ is a partition of X into k sets each of size n; each element of the partition is a *group*. $\mathcal{H}=\{H_1, H_2, \ldots, H_l\}$ is a set of pairwise disjoint subsets of X, with the property that $|H_j \cap G_i| = h_j$ for $1 \leq j \leq l$ and $1 \leq i \leq k$; each H_j is a *hole*. $\mathcal{B}$ is a set of k-subsets of X, with the property that each $B \in \mathcal{B}$ satisfies $|B \cap G_i| = 1$ for each $1 \leq i \leq k$; sets in $\mathcal{B}$ are *blocks*. Finally, each unordered pair of elements from X is either in a hole or group together, or in exactly one block of $\mathcal{B}$.

Lemma 3.1 *If a BSEC$(m,4,\lambda)$ exists, then a BSEC$(4m,4,\lambda)$ exists.*

Proof: There exists a TD$(4, m)$-TD$(4, 4)$ (see [1], for example). Replicate every block λ times. Let $\{x_{ij} : 1 \leq i, j \leq 4\}$ be the points in the hole, with $\{x_{ij} : 1 \leq j \leq 4\}$ in the ith group. For $1 \leq i \leq 4$, place a BSEC$(m, 4, \lambda)$ on the points of the ith group so that the pairs $\{x_{i1}, x_{i3}\}$, $\{x_{i2}, x_{i3}\}$, and $\{x_{i2}, x_{i4}\}$ appear in the leave. To fill the hole, form sixteen blocks of the form $\{\{i, i+2, i+6, i+13\} : 0 \leq i \leq 16\}$ with arithmetic modulo 16. Relabel the points of these 16 blocks using the mapping:

0	8	9	1	3	11	10	2	4	12	13	5	7	15	14	6
x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	x_{41}	x_{42}	x_{43}	x_{44}

and place the sixteen blocks obtained, λ times each, on the sixteen points of the hole. The resulting design is the required BSEC. □

Lemma 3.2 *Let $\lambda \in \{1, 2\}$. Let $m \geq 4$, and $m \equiv 0, 1 \pmod 4$ when $\lambda = 1$. Let $x = 1$ or $4 \leq x \leq m$, and $x \equiv 0, 1 \pmod 4$ when $\lambda = 1$. Then if a BSEC$(3m, 4, \lambda)$ exists and a BSEC$(3x, 4, \lambda)$ exists, so does a BSEC$(12m + 3x, \lambda)$.*

Proof: Let $(X, \mathcal{G}, H, \mathcal{B})$ be TD$(5, m)$-TD(5,1). Delete $m - x$ points from G_5, but do not delete the point in $H \cap G_5$. Let $\mathcal{G} = \{G_1, G_2, G_3, G_4, G_5\}$ be the five groups of the resulting design.

The BSEC$(12m + 3x, 4, \lambda)$ to be constructed has elements $(G_1 \cup G_2 \cup G_3 \cup G_4 \cup G_5) \times \{0, 1, 2\}$. For each block $B \in \mathcal{B}$, place on $B \times \{0, 1, 2\}$ the blocks of a 4-GDD of type $3^{|B|}$ and index λ so that $\{(x, 0), (x, 1), (x, 2)\}$ is a group for each $x \in B$.

Next, for each group G_i, $i = 1, 2, 3, 4, 5$, place the blocks of a BSEC$(3|G_i|, 4, \lambda)$ on $G_i \times \{0, 1, 2\}$, so that for every $g \in G_i$, the pairs $\{(g, 0), (g, 1)\}$ and $\{(g, 0), (g, 2)\}$ appear in the leave of the BSEC. Finally, on $H \times \{0, 1, 2\}$, place the blocks of a BSEC(15,4,λ) so that $\{(g, 0), (g, 2)\}$ and $\{(g, 1), (g, 2)\}$ appear in the leave. It is routine to check that the result is the required BSEC. □

Lemma 3.3 *A BSEC$(v, 4, 1)$ exists whenever $v \geq 12$ and $v \equiv 0, 3 \pmod{12}$ except when $v = 12$.*

Proof: Solutions are given in §2 when v = 15, 24, 27, 36, 39, 48, 51, 72, 84, 87, 99. Apply Lemma 3.2 to handle all remaining values with $v \equiv 3 \pmod{12}$. Now Lemma 3.1 handles all values of $v \equiv 0 \pmod{48}$ with $v \geq 96$, and all values of $v \equiv 12 \pmod{48}$ with $v \geq 60$. Lemma 3.2 with $x = 8$ handles all values of $v \equiv 24, 36 \pmod{48}$ with $v \geq 120$. □

Lemma 3.4 *A BSEC$(v, 4, 2)$ exists whenever $v \geq 12$ and $v \equiv 0 \pmod 3$.*

Proof: Lemma 3.3 handles all values with $v \equiv 0, 3 \pmod{12}$ except for $v = 12$. Solutions are given in §2 when v = 12, 18, 21, 30, 33, 42, 45, 54, 57, 66, 69, 78, 81. For $v = 93$, form a {4,5}-GDD of type $6^5 7^1$ containing a block of size 5, by deleting four points of a block in a TD(5,7). Employ this design on 31 points, giving weight 3 as in Lemma 3.2, to produce a BSEC(93,4,2). For the remaining cases, apply Lemma 3.2. □

4 Indices Three and Six

Lemma 4.1 *Let $\lambda \in \{3,6\}$. Let $m \geq 12$ and $x = 0$, or $m \geq x \geq 12$, where $m, x \equiv 0,3 \pmod 4$ when $\lambda = 3$. If a BSEC$(m,4,\lambda)$exists, and either $x = 0$ or a BSEC$(x,4,\lambda)$ also exists, then a BSEC$(4m+x,4,\lambda)$ exists.*

Proof: Form a TD$(5,m)$-TD(5,2), which exists by the main theorem in [1]. Select one group and delete $m - x$ elements of this group; when $x > 0$, these do not include the two elements of the group in the hole. Now replace each block of size four by three copies of the block, and replace each block of size five by the blocks of a 4-GDD of type 1^5 and index three (this is all possible 4-sets on 5 elements). Now on each nonempty group, place a BSEC whose leave includes the pair of elements in the hole. It remains only to fill the hole. Let $\{x_i, y_i\}$ be the elements in the intersection of the hole and the ith group. When $x > 0$, place the blocks of a 4-GDD of type 2^5 and index 3 so that its groups align on $\{x_1,y_2\}$, $\{x_2,y_3\}$, $\{x_3,y_4\}$, $\{x_4,y_5\}$, $\{x_5,y_1\}$. When $x = 0$, instead use type 2^4 and index 3 placing the groups on $\{x_1,y_2\}$, $\{x_2,y_3\}$, $\{x_3,y_4\}$, $\{x_4,y_1\}$.

The leave is then a single cycle as required. □

Lemma 4.2 *Let $\lambda \in \{3,6\}$. Let $m \geq 6$ when $\lambda = 6$; and $m \geq 8$, $m \neq 10$, and $m \equiv 0 \pmod 2$ when $\lambda = 3$. Let $3m-1 \geq x \geq 12$, and $x \equiv 0,3 \pmod 4$ when $\lambda = 3$. If a BSEC$(2m,4,\lambda)$ and a BSEC$(x,4,\lambda)$ both exist, so does a BSEC$(8m+x,4,\lambda)$.*

Proof: We employ 4-GDDs of type $2^4\sigma^1$ when $\sigma = 0,1,2,3$, for index 3. The GDDs with $\sigma = 0,2$ are in [7]. When $\sigma = 1$, develop starter blocks {0,1,2,3} and $\{0,3,6,\infty\}$ modulo 8 to get the 4-GDD. When $\sigma = 3$, instead use the starter blocks $\{0,1,3,\infty_1\}$, $\{0,1,3,\infty_2\}$, $\{0,1,3,\infty_3\}$.

Form a TD$(5,m)$-TD(5,1) of index $\frac{\lambda}{3}$. Give weight 2 to every point in the first four groups. Give weight 2 to the point of the hole in the fifth group. Give the remaining $m-1$ points weights $w_2, \ldots, w_m$ so that $w_i \in \{0,1,2,3\}$ when $2 \leq i \leq m$ and $2 + \sum_{i=2}^{m} w_i = x$. Now for each block B, employ a 4-GDD of type $2^4\sigma^1$ where σ is the weight of the point in the fifth group. The remainder of the proof parallels Lemma 4.1. □

We use these two constructions to settle existence of BSEC$(v,4,\lambda)$s with $\lambda \equiv 0 \pmod 3$.

Lemma 4.3 *A BSEC$(v,4,3)$ exists whenever $v \equiv 0,3 \pmod 4$ and $v \geq 12$.*

Proof: A BSEC$(v,4,1)$ is given for $v \equiv 0,3 \pmod{12}$ except when $v = 12$. BSEC$(v,4,3)$s are given in §2 for $v =$ 12, 16, 19, 20, 23, 28, 31, 32, 35, 40, 43, 44, 47, 52, 55, 56, 59, 67, 68, and 71. Lemma 4.2 with $m = 8$ and $x = 19$ handles $v = 83$. A variant of Lemma 4.2 with $m = 10$, using a TD(5,10)-TD(5,2)-TD(5,1), with $x = 23$ handles $v = 103$. For the remaining values of v, write $v = 4m + x$ with $x = 0$ or $12 \leq x \leq m$ and apply Lemma 4.1. □

Lemma 4.4 *A BSEC$(v,4,6)$ exists whenever $v \geq 12$.*

Proof: A BSEC$(v,4,3)$ when $v \equiv 0,3 \pmod 4$ exists by Lemma 4.3. A BSEC$(v,4,2)$ when $v \equiv 0 \pmod 3$ exists by Lemma 3.4. It remains to treat cases with $v \equiv 1,2,5,10 \pmod{12}$. BSEC$(v,4,6)$s are given in §2 for v = 13, 14, 17, 22, 25, 26, 29, 34, 37, 38, 41, 46, 49, 50, 53, 58. Apply Lemma 4.2 with $m = 6$ and $x \in \{13,14\}$ to handle $v \in \{61,62\}$. For the remaining values of v, write $v = 4m + x$ with $12 \le x \le m$ and apply Lemma 4.1. □

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