Matching generalized nested sets

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Abstract

The matching property of nested sets, which has been studied in [6], is extended to generalized nested sets, defined in [9]. Pairs of matching generalized nested sets on [m] are constructed and used for the creation of all generalized planar permutations on [m].

1. Introduction

A set S of pairwise disjoint pairs of $[2n]=\{1,2,\dots,2n\}$ such that $\bigcup_{\{\alpha,b\}\in S}\{\alpha,b\}=[2n]$ and for any $\{\alpha,b\}$, $\{c,d\}\in S$ we never have $\alpha<c<b<d$, is called nested set of pairs on [2n]. The nested sets may be regarded as non-crossing partitions, the blocks of which contain exactly two elements; they are related to nested parentheses [3] and are used in the study of planar permutations [7] and Jordan sequences [2]. The set of nested sets is studied in [8], while in [6] it is shown how elements of this set match in order to create planar permutations.

The notion of nested sets is generalized in [9], where the elements of S are not necessarily disjoint; the set S of pairs of $\mathbb{N}^*$ is called *generalized nested set* (g.n.s.) if we never have $\alpha<c<b<d$ for $\{\alpha,b\}$, $\{c,d\}\in S$. For example the set $S=\{\{1,4\},\{2,4\},\{5,6\},\{5,10\},\{6,7\},\{8,9\}\}$ is a g.n.s. on [10] (see Fig. 1).

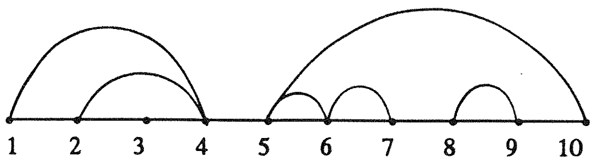

Fig.1 A generalized nested set on [10].

Generalized nested sets are characterized and constructed using a pair of finite sequences in [9].

Here, we study the matching property of g.n.s., allowing two g.n.s. to be joined in such a way that a unique cycle using elements of [m] is obtained. We construct a transformation on the set of g.n.s., which preserves the matching property and it is used for the creation of the set of g.n.s. from the nested sets. Thus, we obtain the set of generalized planar permutations (see [4]).

2. Matching generalized nested sets

If S is a g.n.s. of [m], we define the *degree* $d_S(i)$ *of i in S* to be the number of occurences of each element of $i\in[m]$ in pairs of S, and the *domain* of S, $D(S)=\{i\in[m] : d_S(i)=1 \text{ or } 2\}$. Now, if $I\subseteq[m]$, let $N(I,S)=\{i\in[m] : \exists\, j\in I$ such that $\{i,j\}\in S\}$.

We say that two g.n.s. U,L are *matching* if and only if :

$d_U(i)+d_L(i)=2$, for each $i\in D(U)\cup D(L)$, and

for every non-empty $I\subseteq D(U)\cup D(L)$, $N(I,U)\cup N(I,L)=I$ implies $I=D(U)\cup D(L)$.

The above definition suggests that the diagrams of the two matching g.n.s. can be joined in such a way that a unique cycle is generated, having U and L as its upper and lower parts (see Fig. 2).

It is easy to check that if U,L are nested sets, the matching property given here implies D(U)=D(L) and hence it coincides with the matching property given for nested sets in [8].

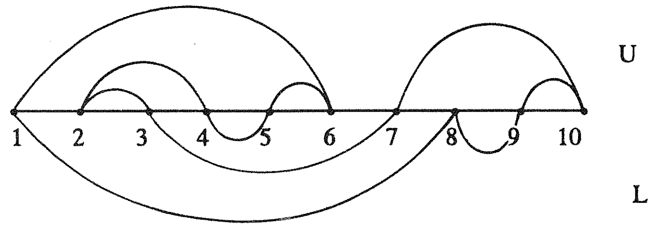

Fig. 2 The matching of two g.n.s.

Given a g.n.s. S we say that two elements $x,y \in D(S)$ are *connected* in S if there exists a sequence $t_1,t_2,\ldots,t_n$ of elements of D(S) such that $x=t_1$, $y=t_n$ and $\{t_i,t_{i+1}\} \in S$, $i=1,2,\ldots,n-1$. Moreover, if the sequence $t_1,t_2,\ldots,t_n$ is unique, we say that x,y are *uniquely connected* in S and we write xSy. Obviously this relation is transitive.

Let GN_m be the set of all g.n.s. S of [m], such that each pair of connected elements of D(S) are uniquely connected and N_m its subset consisting of nested sets. It is clear that if $S_1,S_2 \in GN_m$ with $D(S_1)= D(S_2)$, then $S_1 \subseteq S_2$ implies $S_1=S_2$.

We define a transformation φ from GN_m to the set of all pairs (V,A), where $V \in N_m$, $A \subseteq [m]$ and $A \cap D(V)=\varnothing$ as follows :

$\varphi(U)=(V,A)$, where $D(V)=\{i \in D(U)\colon d_U(i)=1\}$, $\{x,y\} \in V$ if and only if x,y are connected in U, and $A=\{i \in D(U)\colon d_U(i)=2\}$.

Proposition 2.1 : *If $\varphi(U)=(V,A)$ and $\varphi(L)=(W,B)$, then U, L are matching g.n.s. if and only if V,W are matching nested sets and the family (A, B, D(V)) forms a partition of $D(U) \cup D(L)$.*

Proof. Suppose that U,L are matching. By the definitions of A,B and D(V) it is clear that they form a partition of $D(U) \cup D(L)$. Suppose now that $I \subseteq D(V)$, with $N(I,V) \cup N(I,W)=I$. To prove that V, W are matching it is enough to prove that $I=D(V)$.

Let $J=I\cup\{y\in A : yUi \text{ for some } i\in I\}\cup\{y\in B : yLi \text{ for some } i\in I\}$.

We first show that if xUi (similarly if xLi), then $x\in I$, for every $i\in J$, $x\in D(V)$. Indeed, if $i\in D(V)$, then $\{x,i\}\in V$ and hence $x\in N(I,V)\subseteq I$; if on the other hand $i\notin D(V)$, then $i\notin I$ and hence $i\in A\cap J$; so iUk for some $k\in I$; then xUk, i.e. $x\in I$.

We now proceed to prove that $N(J,U)\cup N(J,L)\subseteq J$. Notice first that if $x\in N(J,U)$ then $\{x,i\}\in U$, i.e. xUi for some $i\in J$. If $x\in D(V)$ then $x\in I\subseteq J$ by the previous result. If $x\notin D(V)$ (and hence $x\in A$) then either $i\in I$ and since $\{x,i\}\in U$ we have xUi, i.e. $x\in J$, or $i\in\{y\in A: yUk \text{ for some } k\in I\}$ and hence iUk for some $k\in I$, i.e. xUk for some $k\in I$, giving $x\in J$. So, in every case $N(J,U)\cup N(J,L)\subseteq J$. The proof is similar if $x\in N(J,L)$.

Since U, L are matching, the above implies that $J=D(U)\cup D(L)$ and hence $J=A\cup B\cup D(V)$. Since $I\subseteq J$, $I\subseteq D(V)$ we get $I=D(V)$.

Conversely now, suppose that V, W are matching nested sets and $(A,B,D(V))$ form a partition of $D(U)\cup D(L)$. Let $J\subseteq D(U)\cup D(L)$ and $N(J,U)\cup N(J,L)\subseteq J$. It is enough to prove that $J=D(U)\cup D(L)$.

We first show that if xUy (or xLy) and $y\in J$, then $x\in J$: since xUy, there exists a unique sequence $t_1,t_2,\ldots,t_n$ of elements of $D(U)$ such that $t_1=x$, $t_n=y$ and $\{t_i,t_{i+1}\}\in U$, $i=1,2,\ldots,n-1$. Since $y\in J$, then $t_{n-1}\in N(J,U)\subseteq J$. It is now clear that we can inductively show that $t_1=x\in J$.

Let now $I=J\cap D(V)$. Then $N(I,V)\cup N(I,W)\subseteq I$. Indeed, if $x\in N(I,V)$ (similarly, if $x\in N(I,W)$) then there exists $y\in I$ such that $\{x,y\}\in V$; so xUy and $y\in I\subseteq J$. Hence, by the above result, $x\in J$ and hence $x\in J\cap D(V)=I$.

We have proved that for $I=J\cap D(V)\subseteq D(V)=D(V)\cup D(W)$, we have that $N(I,V)\cup N(I,W)\subseteq I$ and since V,W are matching we get $I=D(V)$.

Let now $x\in A\cup B\cup D(V)$. If $x\in A$ (similarly, if $x\in B$) then there exists $y\in D(V)$

such that xUy. But then $y \in I \subseteq J$ and hence $x \in J$. If on the other hand $x \in D(V)=I$, then $x \in J$. So, $D(U) \cup D(L) = A \cup B \cup D(V) \subseteq J$ and hence $J = D(U) \cup D(L)$. ∎

In general, given a pair (V,A) such that $V \in N_m$, $A \subseteq [m]$ and $D(V) \cap A = \emptyset$, there exist several $U \in GN_m$ with $\varphi(U)=(V,A)$; (e.g. both g.n.s. of Fig. 3 give rise to the pair (V,A), where $V=\{\{1,2\},\{7,10\},\{8,9\}\}$ and $A=\{4,5,6\}$).

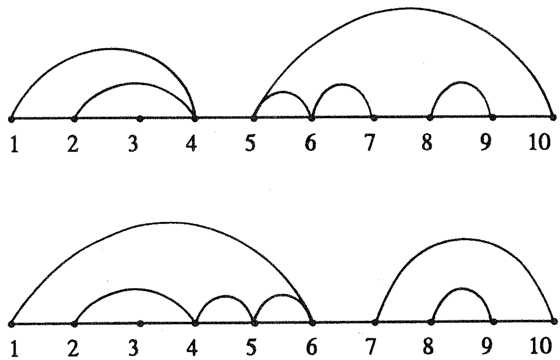

Fig. 3 Two g.n.s. with the same image.

Our aim is to determine the set of all $U \in GN_m$ with $\varphi(U)=(V,A)$. For this, we introduce two preliminary constructions.

Construction 1 : Given a pair $V=\{x,y\} \subseteq [m]$ with $x<y$ and a set $K \subseteq [m]$, such that $V \cap K = \emptyset$, we define $T(V,K)=\{S \in GN_m : D(S)=V \cup K,\ d_S(x)=d_S(y)=1$ and $d_S(z)=2$ for each $z \in K\}$.

Clearly, for each $S \in T(V,K)$, any two elements of $V \cup K$ are connected in S.

An inductive construction of the set $T(V,K)$ is obtained as follows :

Let $K_V=\{z \in K: x<z<y\}$ and $n=|K_V| \cdot |K \setminus K_V|$.

If $n=0$ then either $K_V=\emptyset$ or $K_V=K$. In both cases let S be the set of all pairs $\{z,w\}$ such that z,w are consecutive elements of $V \cup K$ and $\{z,w\} \neq V$; (here, the elements $\min(V \cup K)$, $\max(V \cup K)$ are assumed to be consecutive).

It is easy to show that $T(V,K)=\{S\}$.

Now, assuming that $n>0$ and $T(V,K)$ is constructed for every pair V and set K such that $|K_V|\cdot|K\setminus K_V|<n$, we will construct the set $T(V,K)$, where $|K_V|\cdot|K\setminus K_V|=n$.

Indeed, let z_1 (resp. z_2) be the left (resp. right) neighbour of x in $V\cup K$. If we set $V_1=\{y,z_1\}$ and $V_2=\{y,z_2\}$ then by the induction hypothesis the sets $T(V_1,K\setminus\{z_1\})$ and $T(V_2,K\setminus\{z_2\})$ can be constructed. Furthermore, by adding to each $S\in T(V_i,K\setminus\{z_i\})$ the pair $\{x,z_i\}$, we introduce the set $T^*(V_i,K\setminus\{z_i\})$, $i=1,2$.

It is clear that $T(V,K)=T^*(V_1,K\setminus\{z_1\})\cup T^*(V_2,K\setminus\{z_2\})$.

The previous discussion also helps for the evaluation of the cardinal number of $T(V,K)$.

Indeed, since $|T(V,K)|=|T(V',K')|$ whenever $|K|=|K'|$ and $|K_V|=|K'_{V'}|$, the function $f(r,s)=|T(V,K)|$ where $r=|K_V|$ and $r+s=|K|$ is well defined for $r,s\in[|K|]$.

Clearly, by the above construction we obtain that for every $r,s\in[|K|]$:

$$f(r,0)=f(0,s)=1 \quad\text{and}\quad f(r,s)=|T(V,K)|=|T^*(V_1,K\setminus\{z_1\})|+|T^*(V_2,K\setminus\{z_2\})|$$
$$=|T(V_1,K\setminus\{z_1\})|+|T(V_2,K\setminus\{z_2\})|=f(r,s-1)+f(r-1,s)$$

Then, using a generating function and a standard procedure, it is easy to show that

$$f(r,s)=\binom{r+s}{r} \quad\text{thus obtaining that}\quad |T(V,K)|=\binom{|K|}{|K_V|}$$

Now in order to proceed to the second construction, let B be a subset of a totally ordered set X, $\mathcal{C}$ a noncrossing partition of B (see [1]) and $\Sigma=\mathcal{C}\cup(X\setminus B)$. We determine a relation "$\prec$" on Σ as follows :

$\alpha\prec\beta$ if and only if $\beta\in\mathcal{C}$ and there exist x,y consecutive elements of β such that $x<\min\alpha\le\max\alpha<y$.

This relation is transitive and it is used for the following definition :

Given $\beta\in\mathcal{C}$ and $\alpha\in\Sigma$, we say that β is the *father* of α if and only if $\alpha\prec\beta$ and there is no $\gamma\in B$ with $\alpha\prec\gamma\prec\beta$.

Furthermore $c,d\in\Sigma$ are called *brothers* if they both have no father or they have the same father β and there is no element of the father which lies between them (i.e. there is no $x\in\beta$ such that $\min(c\cup d)<x<\max(c\cup d)$).

Construction 2 : Given a totally ordered set X and $B \subseteq X$, we denote by $R(B,X)$ the set of all noncrossing partitions C of X, such that each member of C contains exactly one element of B.

The set $R(B,X)$ is determined by induction on $n=|X\setminus B|$ as follows : If $n=0$ and hence $X=B$, then $R(B,X)=\{C_0\}$, where $C_0=\{\{\beta\};\ \beta\in B\}$. Now assuming that $n>0$ and the set $R(B,X)$ is determined for each X with $|X\setminus B|=n-1$, we will determine the set $R(B,X)$, when $|X\setminus B|=n$. Indeed, given $\xi\in X\setminus B$ and $C\in R(B,X\setminus\{\xi\})$, let T be the set which contains the father and the brothers of ξ in $C\cup\{\xi\}$. For each $\gamma\in T$, we obtain through C a noncrossing partition C_γ of X by adding ξ to γ. It is clear that $C_\gamma\in R(B,X)$ and $R(B,X) = \bigcup_{C\in R(B,X\setminus\{\xi\})}\{C_\gamma:\ \gamma\in T\}$.

We now come to determine the set of all g.n.s. which are mapped under the transformation φ to a pair (V,A).

Main Construction : Let $\Sigma=V\cup A$ and $\mathcal{L}$ be the partition of Σ generated by the equivalence relation $*$ with :

$$\alpha * b \quad \text{iff } \alpha=b \quad \text{or} \quad \alpha,b \quad \text{are brothers}$$

If we add to each member X of $\mathcal{L}$ the common father, if it exists, we obtain a set $\overline{X}$ on which the natural ordering of [m] gives a total ordering, defined as follows :

$$\alpha\leq b \quad \text{if and only if } \max \alpha \leq \max b$$

If we choose a family $C=(C_X)$, $X\in\mathcal{L}$, where $C_X\in R(\overline{X}\cap V,\overline{X})$, we define for each $v\in V$ a set $A_v(C)=\{\alpha\in A:\alpha,v$ belong to the same block of some $C_X\}$. It is clear that if $\alpha\in A_v(C)$, then v is either the father or a brother of α. Moreover, since each member of C contains an element of V, we deduce that the family $(A_v(C))$, $v\in V$ is a partition of A.

Now, for each $v\in V$ we choose $S_v\in T(v,A_v(C))$ and we set $U=\bigcup_{v\in V} S_v$.

Clearly, since for $v_1,v_2\in V$ with $v_1\neq v_2$ we obtain $D(S_{v_1})\cap D(S_{v_2})=\varnothing$, we conclude that $S_{v_1}\cap S_{v_2}=\varnothing$. Furthermore, we have that :

$$d_u(x)=\begin{cases} 0 \text{ , if } x\notin D(V)\cup A \\ 1 \text{ , if } x\in D(V) \\ 2 \text{ , if } x\in A \end{cases}$$

The following two propositions accomplish the generation of all $U\in GN_m$, which are mapped to a given pair (V,A).

Proposition 2.2 : *Every set U obtained by a pair (V,A) according to the main construction is a g.n.s. in GN_m, which is mapped under the basic transformation φ to the pair (V,A).*

Proof. We first show that U is a g.n.s. Indeed, if this is not true, there exist $v_1,v_2\in V$ with $v_1\neq v_2$ and $\{\alpha,b\}\in S_{v_1}$, $\{c,d\}\in S_{v_2}$ such that $\alpha<c<b<d$.

Without loss of generality, we may assume that $\alpha,b\in A_{v_1}(\mathcal{C})$ and $c,d\in A_{v_2}(\mathcal{C})$, since the cases where some of α,b,c,d belong to the corresponding v_i, $i=1,2$, are treated similarly.

Then, for α,b we have two possible cases :

(I) α,b are brothers.

(II)The father of one is a brother of the other.

For case I, we assume that α,b have a common father $v=(x,y)$; let X be the set of all the children of v; (in the case that α,b have no father, we work similarly).

We consider two subcases :

(I_1) If $d>y$, then v is a brother of d and the father of c, so that $v=v_2$. Then there exist two different blocks of $\mathcal{C}_X$ such that one contains α,b and the other c,d while $\alpha<c<b<v$. This contradicts the non crossing property of $\mathcal{C}_X$.

(I_2) If $d<y$ we use the same method. Here, the four elements that give the required contradiction are α,b,z,w where z is either c or v_2 and w is either d or v_2.

The proof of case II is similar and it is omitted.

We now show that $U\in GN_m$. Indeed, if α and b are connected in U, there exists a sequence $t_1,t_2,\ldots,t_n\in D(U)$ such that $t_1=\alpha$, $t_k=b$ and $\{t_i,t_{i+1}\}\in U$, $i\in[k-1]$. Then, since $D(S_{v_1})\cap D(S_{v_2})=\varnothing$ for $v_1\neq v_2$, we deduce that there exists $v\in V$ such that $\{t_\iota,t_{\iota+1}\}\in S_v$ for each $i\in[k-1]$. Thus, $\alpha S_v b$ and hence $\alpha U b$.

It remains to show that $\varphi(U)=(V,A)$. Indeed, since $A=\{i\in D(U) : d_U(i)=2\}$ and $D(V)=\{i\in D(U) : d_U(i)=1\}$, it is enough to show that for $x,y\in D(V)$ we have that $\{x,y\}\in V$ if and only if xUy.

If $\{x,y\}=v\in V$ we have xS_vy and hence xUy. Conversely, if xUy, then as we have shown in an earlier step of the proof, there exists $v\in V$ such that xS_vy. Then since $d_{S_v}(x)=d_{S_v}(y)=1$ and the only such elements of $D(S_v)$ are the two elements of v, we deduce that $\{x,y\}=v\in V$. ■

Proposition 2.3 : *Every $U\in GNm$ such that $\varphi(U)=(V,A)$, is deduced by the main construction.*

Proof. Given a $U\in GNm$ and the corresponding pair (V,A), it is enough to choose the appropriate family $\mathcal{C}$ and then the sets S_v for which $U=\bigcup_{v\in V} S_v$.

So, for each $X\in\mathcal{L}$ and for each $v=\{x,y\}\in\overline{X}\cap V$, let $B_v=\{v\}\cup\{\alpha\in X\cap A:\alpha Ux\}$ and $\mathcal{C}_X=\{B_v : v\in\overline{X}\cap V\}$.

It is easy to check that $\mathcal{C}_X\in R(\overline{X}\cap V,\overline{X})$.

We then define $A_v(\mathcal{C})=\{\alpha\in A: \alpha Ux\}$ for every $v=\{x,y\}\in V$ and $S_v=\{\{p,q\}\in U: p,q\in A_v(\mathcal{C})\cup\{x,y\}\}$. Using the fact that the degrees of the elements of V and A (and hence of $A_v(\mathcal{C})$) in U are 1 and 2 respectively, it is easy to see that for every $v\in V$, $S_v\in T(v,A_v(\mathcal{C}))$, finally giving that $U=\bigcup_{v\in V} S_v$. ■

3. Generalized planar permutations

Let σ be a permutation on $[m]$, with $\sigma(1)=1$, $\varepsilon=\varepsilon_1\varepsilon_2\ldots\varepsilon_m$ a finite sequence of $\{u,d\}^m$ and $U_\sigma(\varepsilon)=\{\{\sigma(i),\sigma(i+1)\}:\varepsilon_i=u\}$, $L_\sigma(\varepsilon)=\{\{\sigma(i),\sigma(i+1)\}:\varepsilon_i=d\}$. Here we assume that $\sigma(m+1)=1$. We say that σ,ε are *compatible* if and only if $U_\sigma(\varepsilon)$ and $L_\sigma(\varepsilon)$ are g.n.s.

A permutation σ is called a *generalized planar permutation* (g.p.p.) if there exists an $\varepsilon\in\{u,d\}^m$ such that σ,ε are compatible.

The notion of g.p.p. has been introduced in [4], using an equivalent geometrical definition.

Note that if $m=2n$ and $\varepsilon=udud\ldots ud$ we get that $U_\sigma(\varepsilon)=\{\{\sigma(2i-1),\sigma(2i)\}:\ i\in[n]\}$, $L_\sigma(\varepsilon)=\{\{\sigma(2i),\sigma(2i+1)\}:\ i\in[n]\}$ are nested, giving a p.p. (see [7]) On the other hand, it is possible for σ to be a g.p.p. although it is not compatible with udud...ud; e.g. $\sigma=13425687$ is a g.p.p. since it is compatible with the sequence uudduduu, whereas σ is not compatible with udududud and hence not planar.

The following two propositions extend some basic results on p.p. (see 3.4, 3.5 of [6]) to g.p.p. The first gives a necessary condition for a pair of sequences σ,ε to be compatible, while the second shows that two matching g.n.s. arise from a compatible pair σ,ε.

Proposition 3.1 : *If σ,ε are compatible, then $U_\sigma(\varepsilon)$, $L_\sigma(\varepsilon)$ are matching g.n.s.*

Proof. Since it is clear that the sum of the degrees of each $i\in[m]$ in $U_\sigma(\varepsilon)$ and $L_\sigma(\varepsilon)$ is equal to 2, it is enough to prove that for $\emptyset\neq I\subseteq[m]$ with $I=N(I,U_\sigma(\varepsilon))\cup N(I,L_\sigma(\varepsilon))$ we have that $I=[m]$. Indeed, if $k=\sigma(j)\in I$, then we obviously get $\{\sigma(j),\ \sigma(j+1)\}\in U_\sigma(\varepsilon)\cup L_\sigma(\varepsilon)$, so that $\sigma(j+1)\in N(I,U_\sigma(\varepsilon))\cup N(I,L_\sigma(\varepsilon))=I$. This, recursively shows that $\sigma(i)\in I$ for each $i\in[m]$ and hence $I=[m]$. ∎

Proposition 3.2 : *If U,L are matching g.n.s. of $[m]$, with $D(U)\cup D(L)=[m]$, then there exist compatible σ,ε such that $U_\sigma(\varepsilon)=U$ and $L_\sigma(\varepsilon)=L$.*

Proof. We construct a mapping σ on $[m+1]$ as follows : $\sigma(1)=1$. Let $\sigma(2)$ be an element of $[m]$ such that $\{\sigma(1),\sigma(2)\}\in U\cup L$. Suppose that we have determined $\sigma(1),\sigma(2),\ldots,\sigma(i)$ for $1<i\leq m$. We define $\sigma(i+1)$ to be the unique element of $[m]$ with $\sigma(i+1)\neq\sigma(i-1)$ and $\{\sigma(i),\sigma(i+1)\}\in U\cup L$.

The restriction of σ on $[m]$ is a required permutation. Indeed, suppose that σ is not 1-1 and let $k\neq\lambda$ with $\sigma(k)=\sigma(\lambda)$, $1\leq k<\lambda\leq m$ and such that λ-k is minimum and k is minimum. If $k>1$, then $\{\sigma(k-1),\sigma(k)\},\{\sigma(k),\sigma(k+1)\}$ and $\{\sigma(\lambda-1),\sigma(k)\}$ belong to $U\cup L$, since $\sigma(k)=\sigma(\lambda)$. Then, since the sum of the degrees of $\sigma(i)$, $i=1,2,\ldots,m$ equals to 2 and $\sigma(k-1)\neq\sigma(k+1)$, we get that either $\sigma(k+1)=\sigma(\lambda-1)$ contradicting the minimality of λ-k, or $\sigma(k-1)=\sigma(\lambda-1)$ contradicting the minimality of k. If now $k=1$, let $I=\{\sigma(1),\sigma(2),\ldots,\sigma(\lambda-1)\}$. Since $|I|=\lambda-1<m=|D(U)\cup D(L)|$, we have that $I\neq D(U)\cup D(L)$; it is enough then to prove that $N(I,U)\cup N(I,L)=I$, in order to get a contradiction to the matching property of U,L. It is clear that $I\subseteq N(I,U)\cup N(I,L)$. Let now $x\in N(I,U)\cup N(I,L)$. Then, there exists $\varrho\in[\lambda-1]$ such that $\{x,\sigma(\varrho)\}\in U\cup L$. If $\varrho=1$, then given that $\{(\sigma(1),\sigma(2)\}$ and $\{\sigma(1),\sigma(\lambda-1)\}$ belong to $U\cup L$, we get that $x=\sigma(2)\in I$ or $x=\sigma(\lambda-1)\in I$. Similarly, we prove that $x\in I$ in the other cases giving $N(I,U)\cup N(I,L)=I$.

Now, we define $\varepsilon=\varepsilon_1\varepsilon_2\ldots\varepsilon_m\in\{u,d\}^m$ with

$$\varepsilon_r=\begin{cases} u\ , & \text{if } \{\sigma(r),\sigma(r+1)\}\in U \\ d\ , & \text{if } \{\sigma(r),\sigma(r+1)\}\in L \end{cases}\qquad,\ r=1,2,\ldots,m$$

Obviously, $U_\sigma(\varepsilon)=U$, $L_\sigma(\varepsilon)=L$ and σ,ε are compatible since U,L are matching.∎

From the above proof we realize that, although the selection of $\sigma(2),\sigma(3),\ldots,\sigma(m)$ is unique, we have two choices for $\sigma(1)$, thus getting exactly two compatible pairs (σ,ε) and (σ^*,ε^*) satisfying the result of Proposition 3.2, such that σ^* is the obverse of σ (see [6]) and $\varepsilon_i^*=\varepsilon_{n-i+1}$.

We can construct the set of all g.p.p. on [m] as follows :

Step 1 : We construct the set GN_m of all g.n.s. on [m], (see [9]).

Step 2 : We construct every pair U,L of matching g.n.s. of GN_m. For this, let $U\in GN_m$. Consider the corresponding pair (V,A), with $V\in N_m$ and $A\subseteq[m]$, as obtained by the transformation φ of section 2. Construct the nested sets of [m] that are matching with V (see [5]). For each such nested set W, find the set of all the members $L \in GN_m$ such that $\varphi(L)=(W,B)$, where $B=[m]\setminus D(U)$. Every such $L \in GN_m$ is matching with W, according to Proposition 2.1. Obviously, we thus find all the elements of GN_m that are matching with U.

Step 3 : We create all compatible pairs σ,ε (and hence all g.p.p.). This can be done as follows : For each pair U,L of matching g.n.s. with $D(U)\cup D(L)=[m]$, constructed by step 2, there exist, according to proposition 3.2 and Note 1, two compatible pairs σ,ε with $U_\sigma(\varepsilon)=U$ and $L_\sigma(\varepsilon)=L$.

Thus, we create all compatible pairs σ,ε. Indeed, if σ,ε are compatible, then according to Proposition 3.1, $U_\sigma(\varepsilon)$ and $L_\sigma(\varepsilon)$ are matching g.n.s. Then, by Proposition 3.2, there exist compatible σ',ε' with $U_{\sigma'}(\varepsilon')=U_\sigma(\varepsilon)$ and $L_{\sigma'}(\varepsilon')=L_\sigma(\varepsilon)$. But then, it is easy to prove that $\sigma=\sigma'$ or $\sigma=(\sigma')^*$.

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