A Small Embedding For Large Directed Even Cycle Systems

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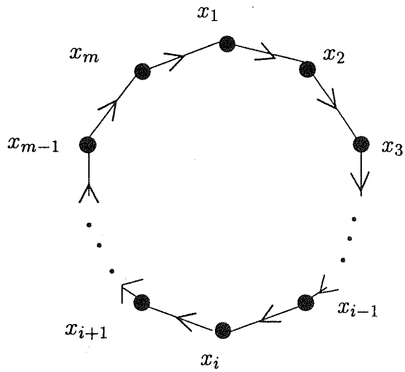
Abstract

A directed m-cycle system is a collection of directed edges of the form $\{(x_1, x_2), (x_2, x_3)(x_3, x_4), \ldots, (x_{m-1}, x_m), (x_m, x_1)\}$, where $x_1, x_2, x_3, \ldots, x_m$ are distinct. A *partial* directed m-cycle system of order n is a pair (S, C), where C is a collection of edge disjoint m-cycles of the complete directed graph D_n with vertex set S. If the cycles in C *partition* the edge set of D_n we have the definition of a directed m-cycle system. The object of this paper is the proof that for fixed $m = 2k$ and large n, a partial m-cycle of order n can be embedded in an m-cycle system of order *approximately* $(mn)/2$.

1 Introduction

We will denote the *directed* edge from x to y by (x, y). A *directed* m-cycle is a collection of m directed edges of the form $\{(x_1, x_2), (x_2, x_3), (x_3, x_4), \ldots, (x_{m-1}, x_m), (x_m, x_1)\}$, where $x_1, x_2, x_3, \ldots, x_m$ are distinct. We will denote this cycle by any cycle shift of $(x_1, x_2, x_3, \ldots, x_m)$.



A *directed m-cycle system* ($mDCS$) of *order* n is a pair (S, C), where C is a collection of directed m-cycles which partition the edge set of the complete directed graph D_n with vertex set S.

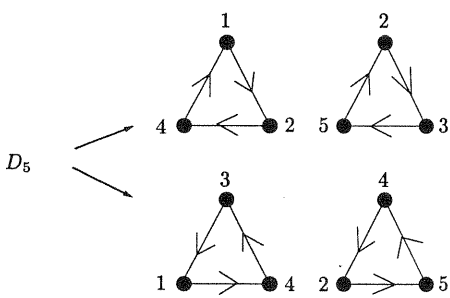
The necessary conditions for the existence of an $mDCS$ of order n are:

$$\begin{cases} (1) & n \geq m, \text{ and} \\ (2) & n(n-1)/m \text{ is an integer.} \end{cases}$$

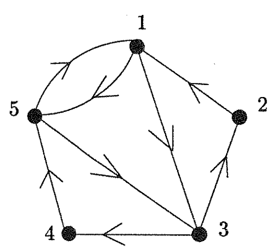
Whether or not these necessary conditions are sufficient is an open problem. However, for fixed m, R. M. Wilson [7] has shown these necessary conditions are sufficient for *sufficiently* large n.

A *partial* $mDCS$ of order n is a pair (S, P), where P is a collection of edge disjoint directed m-cycles of the edge set of D_n. The difference between a partial $mDCS$ of order n and a (complete) $mDCS$ of order n is that the cycles belonging to a partial system do not necessarily partition the edge set of D_n.

Example 1.1 (partial 3DCS (X, P) of order 5.)



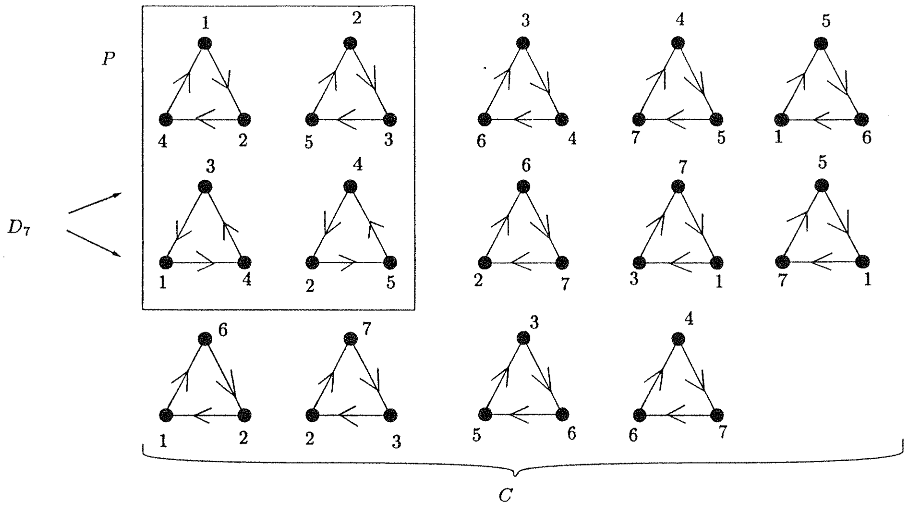
P



unused edges

It is immediately obvious that the unused edges in the above example cannot be partitioned into directed 3-cycles. For one thing, there are 8 unused edges! Since the partial $3DCS$ (X, P) in Example 1.1 cannot be "completed" to a $3DCS$, we can ask whether or not it can be "embedded" in a $3DCS$. That is, does there exists a $3DCS$ (S, C) such that $X \subseteq S$ and $P \subseteq C$? The following example shows that this is possible.

Example 1.2 (3DCS (S, C) of order 7 with (X, P) embedded in it.)



In general it is easy to construct partial $mDCS$s with the property that the unused edges cannot be partitioned into directed m-cycles. Put another way, it is easy to construct a partial $mDCS$ which cannot be completed to an $mDCS$. Hence we have the problem of embedding partial $mDCS$s into (complete) $mDCS$s. So that there is no confusion, the partial $mDCS$ (X, P) of order n is said to be *embedded* in the $mDCS$ (S, C) of order t provided that $X \subseteq S$ and $P \subseteq C$. Naturally, if an embedding is possible we would like t to be as small as possible.

Since it is always possible to embed a partial $mDCS$ in an $mDCS$ [3], the problem of embedding is reduced to the size of the containing system. The best results to date are given in the following (easy to read) table.

m odd	$\approx 4n+1$, $m=3$ [4] $(2n+1)m, m \geq 5$ [3]
m even	$\approx 2n+\sqrt{2n}$, $m=4$ [3] $\approx (mn)/2$, $m \equiv 0 \pmod 6$ [2] $= mn$, $m \geq 8$ [3]

The object of this paper is to substantially reduce the bound for partial $mDCS$s for ALL even m. In particular we will show that if $m = 2k$, $k \geq 4$, a partial

$2kDCS$ of order n can be embedded in a $2kDCS$ system of order $kn + (2\epsilon k + 1)\sqrt{k(k-1)n + 1/4} + \epsilon k(k-1)(\epsilon k + 1) + 1/2$ for some $0 \leq \epsilon < 1$, for *sufficiently large* n. For fixed $m = 2k$, this is asymptotic in n to $kn = (mn)/2$, and so for large n is roughly one-half of the best known bound of mn.

2 Preliminaries

We collect together here the ingredients necessary for the construction in Section 3. A $2k$-*bicycle system* of order (x, y) is a triple (X, Y, C), where C is a collection of $2k$-cycles which partitions the complete bipartite graph $K_{x,y}$ with parts X and Y; $|X| = x$, $|Y| = y$. A $2k$-*dicycle system* of order (x, y) is a triple (X, Y, D), where D is a collection of *directed* $2k$-cycles which partitions the complete *directed* bipartite graph $D_{x,y}$ with parts X and Y; $|X| = x$, $|Y| = y$.

Theorem 2.1 (D. Sotteau [5].) *The necessary and sufficient conditions for the existence of a* $2k$*-bicycle system of order* (x, y) *are: (i)* x *and* y *are even, (ii)* $x \geq k$, $y \geq k$, *and (iii)* $2k \mid xy$. *The necessary and sufficient conditions for the existence of a* $2k$*-dicycle system of order* (x, y) *are: (i)* $x \geq k$, $y \geq k$, *and (ii)* $k \mid xy$. □

In everything that follows, all block designs have index 1.

Theorem 2.2 (R. M. Wilson [7].) *The necessary conditions for the existence of a block design with block size* k *are sufficient for sufficiently large* n. □

Corollary 2.3 *There exists a block design with block size* k *for every sufficiently large* $n \equiv 1$ *(mod* $k(k-1)$*).*

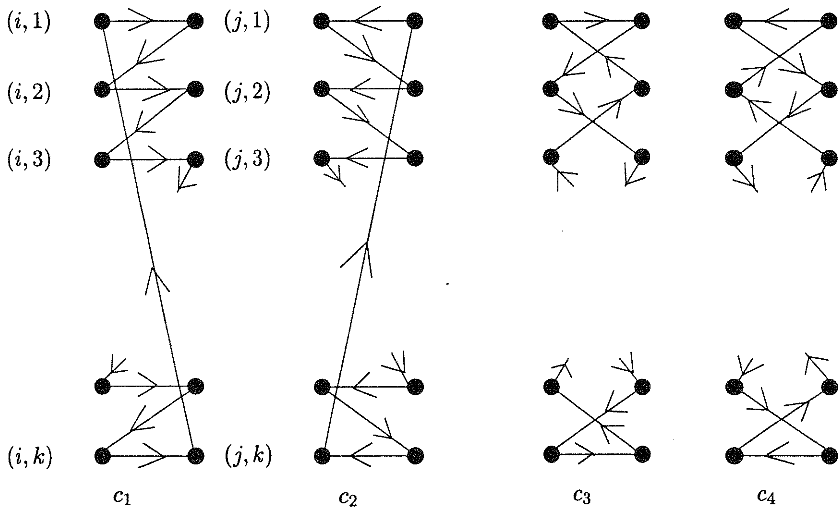
Proof: The necessary conditions for a block design with block size k and order $n \equiv 1$ (mod $k(k-1)$) are (i) $\binom{n}{2}/\binom{k}{2}$ is an integer and (ii) $(k-1) \mid (n-1)$. □

□

Theorem 2.4 (T. W. Tillson [6].) *There exists an* $mDCS$ *of order* m *for all even* $m \geq 8$. □

The following result is probably a Folk Theorem and can be found in [1].

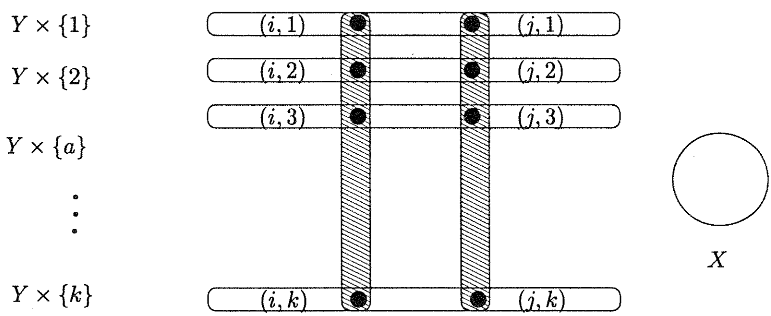
Folk Theorem 2.5 *Let the complete directed bipartite graph* $D_{k,k}$ *have parts* $\{(i,1), (i,2), (i,3), \ldots, (i,k)\}$ *and* $\{(j,1), (j,2), (j,3), \ldots, (j,k)\}$. *There exists a* $2k$*-dicycle system of order* (k, k) *containing the cycles* (i) c_1 *and* c_2 *if* k *is odd, and* (ii) *the cycles* c_3 *and* c_4 *if* k *is even.* □



3 The $k\binom{x}{2}/\binom{k}{2}+x$ Construction

Let $m=2k$, $k\geq 4$, and (X,B) a block design of order $x\equiv 1 \pmod{k(k-1)}$ with block size k (see Wilson's Theorem 2.2). Let Y be a set of size $\binom{x}{2}/\binom{k}{2}=|B|$, K a set of size k, and set $S=(Y\times K)\cup X$. For the purposes of the construction we can assume that $Y=\{1,2,3,\ldots,|B|\}$ and $K=\{1,2,3,\ldots,k\}$. Define a collection C of directed m-cycles of $D_{|S|}$ with vertex set S as follows

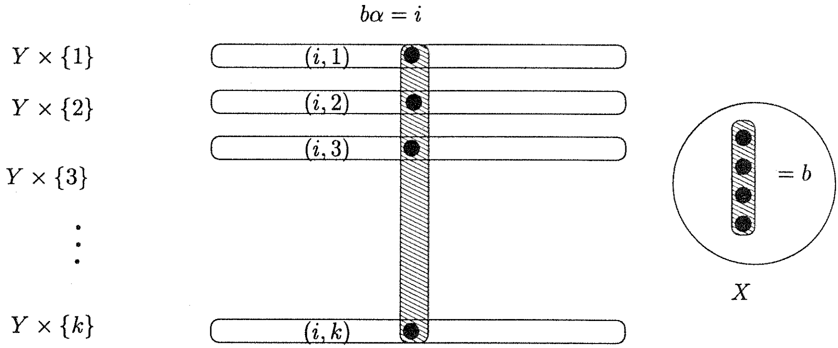
(1) For each $i\neq j\in Y$ let $(\{i\}\times K,\{j\}\times K, C(i,j))$ be a $2k$-dicycle system of order (k,k) and place the cycles belonging to $C(i,j)$ in C (see Sotteau's Theorem 2.1).



Let α be any $1-1$ mapping from B onto Y. For each block $b\in B$ let:

(2) $(((b\alpha)\times K)\cup b, D(b))$ be any $mDCS$ of order $2k$ (see [6]) and place the cycles belonging to $D(b)$ in C; and

(3) $((b\alpha) \times K, X\backslash b, D^*(b))$ a $2k$-dicycle system of order (k,k) and place the cycles belonging to $D^*(b)$ in C.

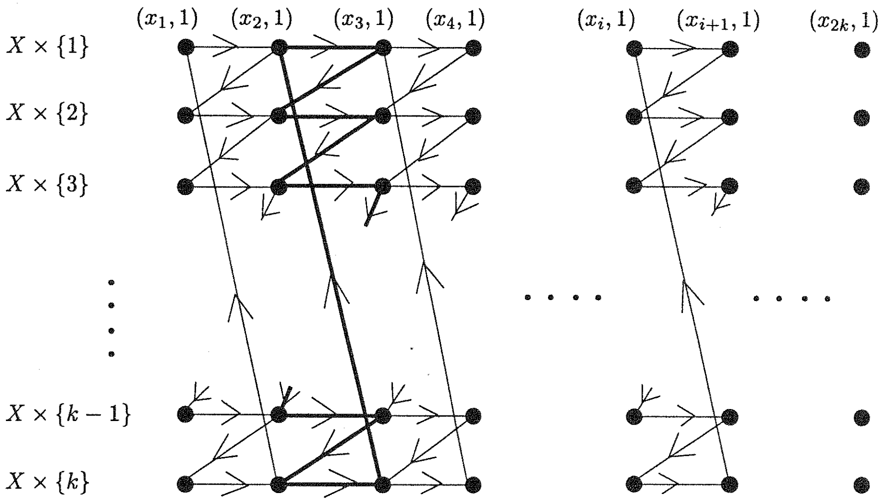


It is easy to see that (S, C) is an $mDCS$ of order $k\binom{x}{2}/\binom{k}{2} + x$.

4 Mutually balanced partial $mDCS$s

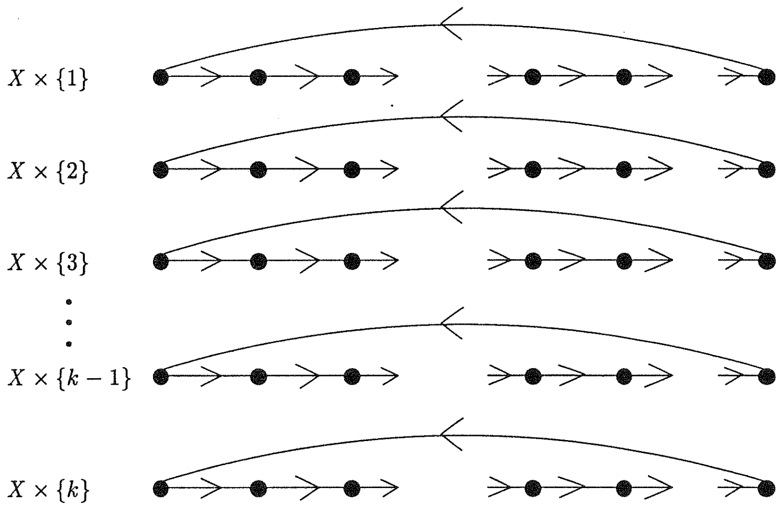
Two *partial* $mDCS$s (S, P_1) and (S, P_2) are said to be mutually balanced provided the cycles belonging to P_1 cover *exactly* the same edges as the cycles belonging to P_2. In order to obtain the embedding result in Section 5 we will need the following two collections of mutually balanced partial $mDCS$s.

m = 2k, k odd. Let $X = \{1, 2, 3, \ldots, 2k\}$, $K = \{1, 2, 3, \ldots, k\}$ and set $S = X \times K$. Let $c = (x_1, x_2, x_3, \ldots, x_{2k})$ and define collections $P_1(c)$ and $P_2(c)$ of $2k$ directed $2k$-cycles with vertex set S as follows:

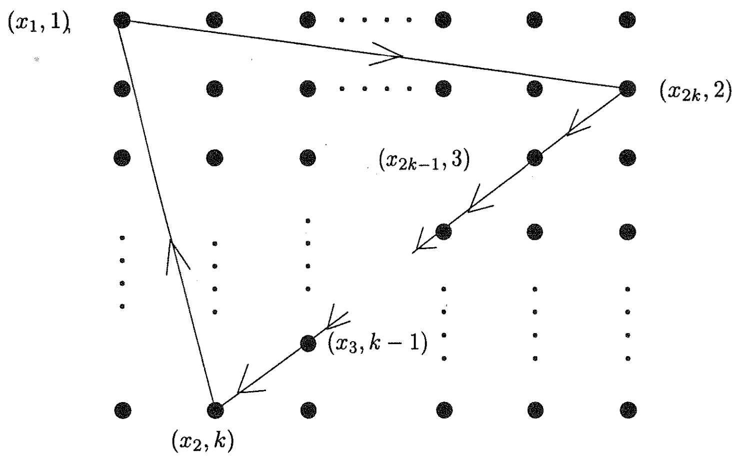


(1) $P_1(c)$ For each edge (x_i, x_{i+1}) belonging to c place a copy of c_1 or c_2, as the case may be, in $P_1(c)$ (Folk Theorem 2.5).

(2) $P_2(c)$ (a) Place the k cycles $((x_1, i), (x_2, i), (x_3, i), \ldots, (x_{2k}, i)), i = 1, 2, \ldots, k$, in $P_2(c)$.



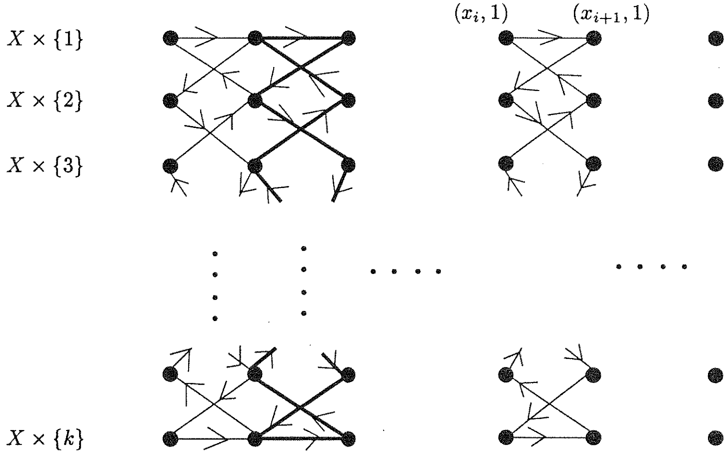
(b) Place the k cycles $((x_1, i), (x_{2k}, 1+i), (x_{2k-1}, 2+i), (x_{2k-2}, 3+i), \ldots, (x_2, k-1+i))$ in P_2.



It is straightforward, and not difficult, to see that $P_1(c)$ and $P_2(c)$ are mutually balanced; i.e., cover exactly the same edges. It is IMPORTANT to note that $P_2(c)$ contains k disjoint copies of the cycle $c = (x_1, x_2, x_3, x_4, \ldots, x_{2k})$.

m = 2k, k even. Define collections $E_1(c)$ and $E_2(c)$ of $2k$ directed $2k$-cycles as follows:

(1) $E_1(c)$: For each edge (x_i, x_{i+1}) belonging to c place a copy of c_3 or c_4, as the case may be, in $E_1(c)$ (Folk Theorem 2.5).

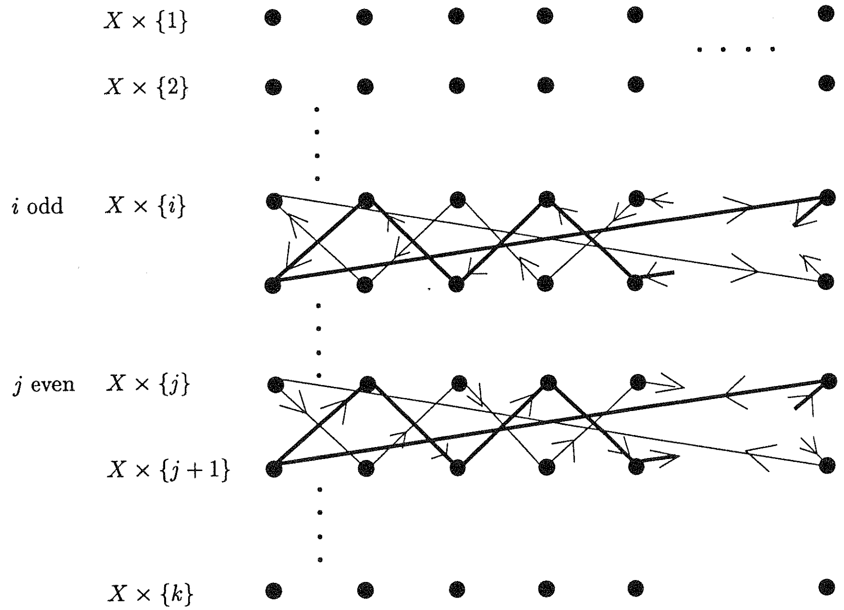


(2) $E_2(c)$ (a) Place the two $2k$-cycles $((x_1, 1), (x_2, 1), (x_3, 1), \ldots, (x_{2k}, 1))$ and $((x_1, k), (x_2, k),$
$(x_3, k), \ldots, (x_{2k}, k))$ in $E_2(c)$.

(b) For each 2-element subset $\{i, i+1\}$, $i = 1, 2, \ldots, k-1$, place the two $2k$-cycles $((x_1, i),$
$(x_{2k}, i+1), (x_{2k-1}, i), (x_{2k-2}, i+1), \ldots, (x_3, i), (x_2, i+1))$ and $((x_1, i+1), (x_{2k}, i),$ $(x_{2k-1}, i+1), (x_{2k-2}, i), \ldots (x_3, i+1), (x_2, i))$ in $E_2(c)$ if i is odd; and the two $2k$-cycles
$((x_1, j), (x_2, j+1), (x_3, j), (x_4, j+1), \ldots, (x_{2k-1}, j), (x_{2k}, j+1))$ and $((x_1, j+1),$ $(x_2, j), (x_3, j+1), (x_4, j), \ldots, (x_{2k-1}, j+1), (x_{2k}, j))$ in $E_2(c)$ if j is even.

As with the case when k is odd, it is easy to see that $E_1(c)$ and $E_2(c)$ are mutually balanced.

It is *IMPORTANT* to note that $E_2(c)$ contains *TWO* disjoint copies of the cycle
$c = (x_1, x_2, c_3, \ldots, x_{2k})$.



5 Embedding partial $2kDCS$s

Let (Z, P) be a partial $2kDCS$ of order n, $2k \geq 8$. Let (X, B) be a block design of order $x \equiv 1 \pmod{k(k-1)}$ and block size k such that $\binom{x}{2}/\binom{k}{2} \geq n$, Y a set of size $\binom{x}{2}/\binom{k}{2}$ such that $Z \subseteq Y$, and $S = (Y \times \{1, 2, 3, \ldots, k\}) \cup X$. Let (S, C) be the $2kDCS$ constructed using the $k\binom{x}{2}/\binom{k}{2} + x$ Construction using the $2k$-dicycle systems in Folk Theorem 2.5. So each dicycle systems contains the cycles c_1 and c_2 if k is odd and the cycles c_3 and c_4 if k is even.

We now resort to some Auburn/Catania trickery!

For each cycle $c = (x_1, x_2, x_3, \ldots, x_{2k}) \in P$, the $2kDCS$ (S, C) constructed using the $k\binom{x}{2}/\binom{k}{2} + x$ Construction contains a copy of the partial dicycle system $P_1(c)$ if k is odd and a copy of the partial dicycle system $E_1(c)$ if k is even. (See Section 4.) A bit of reflection reveals that if c and c^* are different cycles belonging to P, the edge sets of $P_1(c)$ and $P_1(c^*)$ ($E_1(c)$ and $E_2(c^*)$) are *disjoint*. Now set $C^* = (C \backslash \{P_1(c) \mid c \in P\}) \cup \{P_2(c) \mid c \in P\}$ if k is odd and $= (C \backslash \{E_1(c) \mid c \in P\}) \cup \{E_2(c) \mid c \in P\}$ if k is even. Then (S, C^*) is a $2kDCS$ of order $k\binom{x}{2}/\binom{k}{2} + x$ which contains at least two disjoint copies of the partial $2kDCS$ (Z, P). We have the following theorem.

Theorem 5.1 *A partial $2kDCS$ of order n can be embedded in a $2kDCS$ of order $k\binom{x}{2}/\binom{k}{2} + x$, for any positive integer $x \equiv 1 \pmod{k(k-1)}$ such that $\binom{x}{2}/\binom{k}{2} \geq n$ for which a block design of order x and block size k exists.* □

6 Bounds

In Theorem 5.1 if we take x to be as "small as possible", a straightforward calculation shows that for some ϵ, $0 \leq \epsilon < 1$, the containing system has size $kn + (2\epsilon k + 1)\sqrt{k(k-1)n + 1/4} + \epsilon k(k-1)(\epsilon k + 1) + 1/2$. For fixed $m = 2k$, this is asymptotic in n to $(mn)/2$. This bound along with the results in [2] and [3] for $m = 4$ and 6 gives the following result.

Theorem 6.1 *For n large enough with respect to k, a partial $2kDCS$ of order n can be embedded in a $2kDCS$ of order $kn+(2\epsilon k+1)\sqrt{k(k-1)n + 1/4}+\epsilon k(k-1)(\epsilon k+1)+\frac{1}{2}$ for some ϵ, $0 \leq \epsilon < 1$. For fixed $m = 2k$, this is asymptotic in n to $(mn)/2$.* □

Remark The bound obtained in this paper is not the best possible. Never-the-less, for large n, Theorem 6.1 is a substantial improvement over the previous known bound of mn.

Acknowledgement

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