On the index of simple trades

Colin Ramsay

Centre for Discrete Mathematics and Computing,
The University of Queensland, Brisbane, Qld. 4072, Australia.
(email: cram@csee.uq.edu.au)

Abstract

A (v, k, t) trade of volume m consists of two disjoint collections T_1 and T_2, each of m k-subsets (blocks) of a v-set V, such that each t-subset of V is contained in the same number of blocks of T_1 and of T_2. A (v, k, t) trade is simple if it has no repeated blocks, and has index i if some t-subset occurs in i blocks of T_1 but no t-subset occurs in more than i blocks. In this paper we investigate the spectrum (that is, the set of possible volumes) of simple $(v, k, 2)$ trades of index i.

1 Introduction

Let V be a v-set and T_1, T_2 be collections of m k-subsets (**blocks**) of V. We say that T_1 and T_2 are t-**balanced** if each t-subset of V is contained in the same number of blocks of T_1 and of T_2. If T_1 and T_2 are disjoint and t-balanced, then $T = \{T_1, T_2\}$ is said to be a (v, k, t) **trade** of **volume** m. If $T_1 = T_2 = \emptyset$, the trade is said to be **void**.

Note that not all elements of V need appear in the blocks of T. The subset of V contained in T_1 is called the **foundation**, denoted by $F(T_1)$. If T is a trade then $F(T_1) = F(T_2)$, so we define $F(T) = F(T_1)$. We also write $f(T) = |F(T)|$ and $m(T) = m$. Where we do not know, or have no interest in, v, we speak of a (k, t) trade.

To avoid trivialities, we assume throughout that $k > t > 0$, and we ignore the void trade. When writing blocks and sets of blocks, we omit separating commas and braces where possible. It is convenient to think of the blocks of T_1 being labelled '+' and those of T_2 labelled '−', and to write a trade in the form $T = T_1 - T_2$.

EXAMPLE 1: *Let* $T = T_1 - T_2 = +135 + 146 + 236 + 245 - 136 - 145 - 235 - 246$. *Then* T *is a* $(3, 2)$ *trade, with* $m(T) = 4$, $F(T) = \{1, 2, 3, 4, 5, 6\}$ *and* $f(T) = 6$.

As well as being interesting in their own right, trades (also known as null t-designs) have many uses in design theory. They can be used to construct t-designs with different support sizes [5], and are related to the design intersection problem [2] and

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the problem of finding defining sets of designs [8]. Trades are also frequently used implicitly, in a variety of guises: e.g., in [1] (n,t)-partitionable sets are used in halving the complete design. When $n=2$, (n,t)-partitionable sets are trades.

Let $P=\{4,6,7,8,9,\ldots\}$. It is well-known that there is a $(k,2)$ trade of volume m if and only if $m \in P$. A trade T is said to be **simple** if both T_1 and T_2 are sets, as opposed to multisets. It is easy to establish that there is a simple $(k,2)$ trade of volume m if and only if $m \in P$. If T_1-T_2 is a (k,t) trade and no t-subset occurs in more than one block of T_1, then the trade is said to be **Steiner**. Steiner trades are obviously simple. For Steiner $(k,2)$ trades, $k \neq 3$, the set of possible volumes is a proper subset of P (see Theorem 4 below). In this paper, we consider the possible volumes of simple $(k,2)$ trades as a function of how 'non-Steiner' they are.

DEFINITION 2: *Suppose that B is a set of blocks, and that $S \subseteq F(B)$. We say that S has* ***multiplicity*** *r_S in B if S is in r_S blocks of B. We use r_x and r_{xy} for $r_{\{x\}}$ and $r_{\{x,y\}}$ respectively. If $T=T_1-T_2$ is a (k,t) trade, then we define*

$$i=\max\{r_S : S \subseteq F(T_1),\ |S|=t\}$$

to be the ***index*** *of T.*

DEFINITION 3: *For $k \geq t+1$ and $i \geq 1$, the* ***spectrum*** *of simple (k,t) trades of index i is*

$$\mathcal{S}_i(k,t)=\{m(T) : T \text{ is a simple } (k,t) \text{ trade of index } i\}.$$

It is the spectra $\mathcal{S}_i(k,2)$, $k \geq 3$, $i \geq 1$, which we study in this paper. For convenience, we also define $\overline{\mathcal{S}}_i(k,2)=P \setminus \mathcal{S}_i(k,2)$. Steiner trades obviously have index one, and the spectra $\mathcal{S}_1(k,2)$ are known for all $k \geq 3$.

THEOREM 4: ([3, 4, 7]) *For $k \geq 3$, let $N_k=\{m : 2k-2 \leq m < 3k-3,\ m \text{ even}\} \cup \{m : m \geq 3k-3\}$. Then:*
(1) If $k \neq 7$, then $\mathcal{S}_1(k,2)=N_k$;
(2) $\mathcal{S}_1(7,2)=N_k \cup \{15\}$. □

In the next section we review the necessary background material on trades. In Section 3 we present some basic results, and show how $(k,1)$ trades of particular forms can be used to construct $(k,2)$ trades with specific indices. As part of this, we determine $\mathcal{S}_i(k,1)$, for all $k \geq 2$, $i \geq 1$. We also show how two $(k,2)$ trades can be combined in various ways to generate $(k,2)$ trades of different volumes and indices. The values of k partition naturally into three classes, $k=3$, $k=4$ and $k \geq 5$, and Sections 4, 5 and 6 consider these in turn. We completely solve the $k=4$ case, and we discuss the work remaining in the other cases in Section 7. We also make some suggestions regarding other interesting spectra problems.

We use $\lceil x \rceil$ to denote the least integer greater than or equal to x. Set union is sometimes denoted by juxtaposition, and is assumed to 'distribute': so, e.g., $S_0S_1 = S_0 \cup S_1$ and $xT_1=\{\{x\} \cup A : A \in T_1\}$.

2 Review

We start our review of the basic properties of trades with a fundamental result.

LEMMA 5: ([5, 6]) *Let $T = T_1 - T_2$ be a non-void (k, t) trade. Then:*
(1) T is a (k, s) trade for all $0 < s < t$;
(2) $m(T) \geq 2^t$;
(3) $f(T) \geq k + t + 1$. □

The following result is an immediate consequence of Lemma 5(1).

LEMMA 6: *Suppose that $T = T_1 - T_2$ is a (k, t) trade, and x and y are distinct elements not in $F(T)$. Then:*
(1) $+xT_1 + yT_2 - yT_1 - xT_2$ is a $(k + 1, t + 1)$ trade of volume $2m(T)$;
(2) $xT_1 - xT_2$ is a $(k + 1, t)$ trade of volume $m(T)$. □

If A is a collection of blocks, then define $A^x = \{B \setminus \{x\} : x \in B,\ B \in A\}$ and $\overline{A^x} = \{B : x \notin B,\ B \in A\}$. Then we have the following result.

LEMMA 7: ([6]) *Suppose that $T = T_1 - T_2$ is a (k, t) trade, and $x \in F(T)$. Then:*
(1) $T^x = T_1^x - T_2^x$ is a $(k - 1, t - 1)$ trade of volume r_x;
(2) $\overline{T^x} = \overline{T_1^x} - \overline{T_2^x}$ is a $(k, t - 1)$ trade of volume $m(T) - r_x$;
(3) $xT^x = xT_1^x - xT_2^x$ is a $(k, t - 1)$ trade of volume r_x. □

Note that if T is simple in Lemmas 6 and 7 then so are the trades constructed from T. We can also add trades, provided that we 'cancel' any blocks common to the two halves.

LEMMA 8: ([5]) *Suppose that $T_a = T_1 - T_2$ and $T_b = T_3 - T_4$ are (k, t) trades. Then $T = T_1 + T_3 - T_2 - T_4$ is a (k, t) trade of volume*

$$m(T_a) + m(T_b) - |T_1 \cap T_4| - |T_2 \cap T_3|.$$ □

Lemmas 5(2) and 7(1,2) easily yield the following result regarding the multiplicity of elements in the foundation.

LEMMA 9: *Suppose that T is a (k, t) trade and $x \in F(T)$. Then:*
(1) $r_x \geq 2^{t-1}$;
(2) If $t > 1$, then $r_x \neq 1,\ m(T) - 1$. □

By Lemma 5(2), non-void $(k, 2)$ trades must have volume at least four. Such trades have been completely characterised.

THEOREM 10: ([6]) *Volume four $(k, 2)$ trades exist for all $k \geq 3$, and necessarily have the following structure.*

$$\begin{aligned} T = &+S_0S_1S_3S_5 + S_0S_1S_4S_6 + S_0S_2S_3S_6 + S_0S_2S_4S_5 \\ &-S_0S_1S_3S_6 - S_0S_1S_4S_5 - S_0S_2S_3S_5 - S_0S_2S_4S_6, \end{aligned}$$

where: $S_i \subseteq F(T)$, $0 \leq i \leq 6$; $S_i \cap S_j = \emptyset$, $0 \leq i < j \leq 6$; $|S_0| \geq 0$; $|S_i| = |S_{i+1}| > 0$, $i = 1, 3, 5$; and $|S_0| + |S_1| + |S_3| + |S_5| = k$. □

3 General results

We begin with a basic result regarding 'small' volumes.

LEMMA 11: *Suppose that* $m \in \mathcal{S}_i(k,2)$. *Then:*
(1) $m \geq i$; *(2)* $m \neq i+1$.

PROOF: Part (1) is obvious, so suppose that $T = T_1 - T_2$ is a simple $(k,2)$ trade with index i and volume $i+1$. Suppose that $12 \subseteq F(T)$ has multiplicity i in T_1. Thus r_1 and r_2 are at least $i = m(T) - 1$. By Lemma 9(2), neither r_1 nor r_2 can equal i. Thus $r_1 = r_2 = i+1$, and so 12 has multiplicity $i+1$ in T_1, a contradiction. □

Our next result establishes when $4 \in \mathcal{S}_i(k,2)$. Note how this result partitions the possible values of k into three classes. As we will see, this partitioning is reflected in the differing structure of $\mathcal{S}_i(k,2)$ for $k=3$, $k=4$ and $k \geq 5$.

LEMMA 12: *Apart from the cases listed in (1)–(3) below,* $4 \in \overline{\mathcal{S}}_i(k,2)$.
(1) $4 \in \mathcal{S}_1(3,2)$;
(2) $4 \in \mathcal{S}_2(4,2)$;
(3) If $k \geq 5$, *then* $4 \in \mathcal{S}_2(k,2)$ *and* $4 \in \mathcal{S}_4(k,2)$.

PROOF: Consider Theorem 10, and note that any $(k,2)$ trade of volume 4 must have index 1, 2 or 4. (1) If $k=3$, then $S_0 = \emptyset$ and $|S_i| = 1$, $1 \leq i \leq 6$. So the trade has index 1. (2) If $k=4$ and $S_0 = \emptyset$, then $|S_i| = |S_{i+1}| = 2$ for some $i = 1$, 3 or 5, and so the trade has index 2. If $|S_0| = 1$, then $|S_i| = 1$, $1 \leq i \leq 6$, and the trade has index 2. (3) If $k \geq 5$ and $|S_0| \geq 2$, then the trade has index 4. If $|S_0| \leq 1$, then $|S_i| = |S_{i+1}| \geq 2$ for some $i = 1$, 3 or 5, and so the trade has index 2. □

The following pair of constructions enable us to construct simple $(k,2)$ trades with specified index from simple $(k,1)$ trades of a particular form.

LEMMA 13: *Let* $T = T_1 - T_2$ *be a simple* $(k,1)$ *trade of volume* m *and index* i.
(1) If $k \geq 2$, $i \geq 2$, *and* $f(T) = mk - i + 1$, *then* $2m \in \mathcal{S}_i(k+1,2)$;
(2) If $k \geq 3$, $i \geq 1$, $12 \subseteq F(T)$ *has multiplicity* i *in both* T_1 *and* T_2, *and* $f(T) = mk - 2i + 2$, *then* $2m \in \mathcal{S}_{2i}(k+1,2)$.

PROOF: Let x, y be distinct elements not in $F(T)$. By Lemma 7(2), $T^* = +xT_1 + yT_2 - yT_1 - xT_2$ is a simple $(k+1,2)$ trade of volume $2m$.

(1) By supposition, some element of $F(T)$, say 1, occurs in precisely i sets of T_1 and of T_2, and all other elements of $F(T)$ occur precisely once in T_1 and in T_2. Thus the pairs $x1$ and $y1$ have index i in T^*. Pairs of the form $x\alpha$ and $y\alpha$, where $1 \neq \alpha \in F(T)$, have multiplicity 1 in T^*. Pairs of the form 1α and $\alpha\beta$, where $1 \not\subseteq \{\alpha,\beta\} \subseteq F(T)$, have multiplicity at most 2 in T^*. Since $i \geq 2$, the result follows.

(2) By supposition, elements 1 and 2 occur in precisely i sets of T_1 and of T_2 and are always paired, and all other elements of $F(T)$ occur precisely once in T_1 and in T_2. Obviously, 12 has index $2i$ in T^*, and it is easy to see that any other pair from $F(T^*)$ has multiplicity 1, 2 or i. □

It is trivial that $\mathcal{S}_1(k,1)$, the spectrum of Steiner $(k,1)$ trades, is equal to $\{2,3,4,\ldots\}$ for all $k \geq 2$. We will call a $(k,1)$ trade T of index $i \geq 2$ and $f(T) = mk - i + 1$ a **near-Steiner** $(k,1)$ trade of index i. We now determine $\mathcal{S}_i(k,1)$ for all $i \geq 2$ and $k \geq 2$, and show that in all cases we can construct a near-Steiner trade.

THEOREM 14: *Suppose that $i \geq 2$, and let $s = \lceil 3i/2 \rceil$. Then:*
(1) $\mathcal{S}_i(2,1) = \{s, s+1, s+2, \ldots\}$;
(2) If $k \geq 3$, then $\mathcal{S}_i(k,1) = \{i, i+1, i+2, \ldots\}$.
In all cases, a near-Steiner trade exists.

PROOF: Let $T = T_1 - T_2$ be a simple $(k,1)$ trade of index i and volume m. That $m \geq i$ is obvious. Suppose that $k = 2$, and let $1 \in F(T)$ be an element with multiplicity i. Then the i elements that occur with 1 in T_1 and in T_2 must all be distinct, since $T_1 \cap T_2 = \emptyset$ and $k = 2$. Thus, those elements which occur with 1 in T_2 must occur in sets not containing 1 in T_1. So $m \geq i + i/2$. It remains to construct a near-Steiner trade in all cases.

(1) We need only prove the cases $m = s$ and $m = s+1$. As $\mathcal{S}_1(2,1) = \{2,3,4,\ldots\}$, the other cases follow from Lemma 8 by adding a Steiner $(2,1)$ trade of appropriate volume and disjoint foundation. First note that, given $F(T)$, T_1 is fixed, up to a permutation of $F(T)$. By considering the cases i even or odd, and $m = s$ or $m = s+1$, it is easy to see that the blocks of T_1 that do not contain 1 contain a total of i, $i+1$, $i+2$ or $i+3$ distinct elements from $F(T)$. Further, it is always possible to pick i of these points such that at least one point from each of the blocks not containing 1 is chosen. Now use any bijection between these points and the i points which occur with 1 to form T_2 from T_1.

(2) As in (1), T_1 is fixed, up to a permutation of $F(T)$. Recall that a derangement is a permutation with no fixed points. To form T_2 from T_1, chose one element, not equal to 1, from each set of T_1 and apply any derangement. □

The $(k,1)$ trades required by Lemma 13(2) are also straightforward to construct.

LEMMA 15: *If $i \geq 1$ then there exists a simple $(k,1)$ trade $T = T_1 - T_2$ of volume m and $f(T) = mk - 2i + 2$, with some pair $xy \subseteq F(T)$ having multiplicity i in T_1 and in T_2, if and only if:*
(1) $k = 3$ and $m \geq \lceil 4i/3 \rceil = s$;
(2) $k \geq 4$ and $m \geq i$.

PROOF: Let $T = T_1 - T_2$ be a simple $(k,1)$ trade of volume m with $f(T) = mk - 2i + 2$, and suppose that $12 \subseteq F(T)$ has multiplicity i in T_1 and in T_2. Obviously, $k > 2$.

(1) The i elements which occur with 12 in T_1 must be distinct, and cannot occur with 12 in T_2; so $m \geq i + i/3$. As in the proof of Theorem 14(1), we need only prove existence for $m = s$ and $m = s+1$. By considering the cases $i \equiv 0,1,2 \pmod 3$, and $m = s$ or $m = s+1$, it is easy to see that the blocks of T_1 that do not contain 1 contain a total of $i, \ldots, i+5$ distinct elements from $F(T)$. Except when $i = 1$ and $m = s+1 = 3$, it is always possible to pick i of these points such that at least one point from each of the blocks not containing 1 is chosen. Now use any bijection

between these points and the i points which occur with 1 to form T_2 from T_1. For the $i = 1$ and $m = 3$ case, use the trade $+123 + 456 + 789 - 126 - 459 - 783$.

(2) Obviously, $m \geq i$ is necessary. To see sufficiency, note first that, given $F(T)$, T_1 is fixed, up to a permutation of $F(T)$. Now form T_2 from T_1 by chosing one element from $F(T)$, not equal to 1 or 2, from each set of T_1 and permuting these elements using any derangement. □

We now show how $(k, 2)$ trades can be combined to yield $(k, 2)$ trades of other volumes and indices. In particular, part (3) of the following result can be used to generate trades of odd volume from the even volume trades constructed using Lemma 13. Note also that if we set $i = j$ in part (1), then we see that $\mathcal{S}_i(k, 2)$ is closed under addition.

THEOREM 16: *Suppose that* $m \in \mathcal{S}_i(k,2)$ *and* $n \in \mathcal{S}_j(k,2)$. *Then:*
(1) $m + n \in \mathcal{S}_{\max(i,j)}(k, 2)$;
(2) $m + n \in \mathcal{S}_{i+j}(k, 2)$;
(3) $m + n - 1 \in \mathcal{S}_{i+j-1}(k, 2)$.

PROOF: Let $T_a = T_1 - T_2$ (resp. $T_b = T_3 - T_4$) be a simple $(k, 2)$ trade of volume m (resp. n) and index i (resp. j). We can assume that $F(T_a) \cap F(T_b) = \emptyset$; for if not, simply relabel the elements of, say, $F(T_b)$.

(1) $T_a + T_b = +T_1 + T_3 - T_2 - T_4$ is obviously a simple $(k, 2)$ trade with volume $m + n$ and index $\max(i, j)$.

(2) Let $xy \subseteq F(T_a)$ and $zw \subseteq F(T_b)$ have indices i and j in T_a and T_b respectively. Now relabel $\{z, w\}$ so that $\{x, y\} = \{z, w\}$. Since $k > 2$, then T_a and T_b have no blocks in common, so $T_a + T_b$ is a simple $(k, 2)$ trade with volume $m + n$; by our choice of foundations, it has index $i + j$.

(3) Let $xy \subseteq F(T_a)$ and $zw \subseteq F(T_b)$ have indices i and j in T_a and T_b respectively, and suppose that $xy \subseteq M \in T_1$ and $zw \subseteq N \in T_4$. Now relabel the elements of N so that $M = N$ and $\{x, y\} = \{z, w\}$. Since T_a and T_b are simple, and $|F(T_a) \cap F(T_b)| = k$, there is precisely one block common to $T_1 + T_3$ and $T_2 + T_4$. So $T_a + T_b$ is a simple $(k, 2)$ trade with volume $m + n - 1$. The pair xy obviously has index $i + j - 1$ in $T_a + T_b$. If a pair ab has index greater than $i + j - 1$ in $T_a + T_b$, it must have index i in T_b and index j in T_a, and $ab \subseteq F(T_a) \cap F(T_b)$. But any pair in $F(T_a) \cap F(T_b)$ can have index at most $i + j - 1$ in $T_a + T_b$, a contradiction. □

EXAMPLE 17: *Using Lemma 13(1) and Theorem 14(2) (resp. Lemmas 13(2) and 15(1)) we can construct the trades*

$$\begin{aligned} T_a &= +x123 + x145 + x678 + y125 + y148 + y673 \\ &\quad -y123 - y145 - y678 - x125 - x148 - x673, \\ T_b &= +z129 + z12a + zbcd + w12b + w12c + wa9d \\ &\quad -w129 - w12a - wbcd + z12b - z12c - za9d, \end{aligned}$$

which demonstrate that $6 \in \mathcal{S}_2(4, 2)$ *(resp.* $6 \in \mathcal{S}_4(4, 2)$*). These can be combined using Theorem 16(3), relabelling* w *and* 9 *in* T_b *to* x *and* 3 *respectively, to yield the* *trade*

$$+x145+x678+y125+y148+y673+z123+z12a+zbcd+x12b+x12c+xa3d$$
$$-y123-y145-y678-x125-x148-x673-x12a-xbcd+z12b-z12c-za3d.$$

So $11 \in \mathcal{S}_5(4,2)$. □

Lemma 13, with Theorem 14 and Lemma 15, provides all even volumes in $\mathcal{S}_i(k,2)$ which are 'large' in relation to i, for all $i \geq 2$ and $k \geq 3$. Theorem 16 can now be used with these, and the trades of Theorem 4 and Lemma 12, to fill in the 'large' odd volumes and many 'smaller' volumes. In the following three sections, we prove results concerning the volumes not covered by these theorems. Note that, as new volumes are proved to exist, Theorem 16 can be reapplied to fill in further missing volumes.

4 Results for $k = 3$

In this section, we prove the following result regarding $\mathcal{S}_i(3,2)$.

THEOREM 18:
(1) $\mathcal{S}_2(3,2) = P \setminus \{4\}$;
(2) $\mathcal{S}_3(3,2) = P \setminus \{4,6,7\}$;
(3) $\mathcal{S}_4(3,2) = P \setminus \{4,6,\ldots,10\}$;
(4a) For $i \geq 5$, *define*

$$r = 2i + \left\lceil \frac{i}{3} \right\rceil, \quad 2i+1+\left\lceil \frac{i-2}{3} \right\rceil$$

depending as i is even or odd, respectively. Then $\overline{\mathcal{S}}_i(3,2) \supseteq \{4,6,\ldots,r\}$.
(4b) For $i \geq 5$, *define*

$$s = \frac{8i-3}{3}, \quad \frac{8i-2}{3}, \quad \frac{8i-1}{3}$$

depending as $i \equiv 0,1,2 \pmod 3$, *respectively. Then* $\mathcal{S}_i(3,2) \supseteq P \setminus \{4,6,\ldots,s\}$.

LEMMA 19: *Suppose that* $m \in \mathcal{S}_i(3,2)$. *Then* $m \geq 2i + \lceil i/3 \rceil$ *if i is even, and* $m \geq 2i+1+\lceil (i-2)/3 \rceil$ *if i is odd.*

PROOF: Let $T = T_1 - T_2$ be a simple $(3,2)$ trade of volume m and index i. We can suppose, without loss of generality, that $\{12x_1,\ldots,12x_i\} \subseteq T_1$, $\{12y_1,\ldots,12y_i\} \subseteq T_2$, and that these $2i$ sets are distinct. Now the pairs $1y_j$ and $2y_j$, $1 \leq j \leq i$, must occur in T_1. Since the pair 12 cannot occur again, this requires two sets of blocks, each of at least $\lceil i/2 \rceil$ blocks. The x_j must occur at least once more in T_1. If i is even this requires at least $\lceil i/3 \rceil$ further blocks, and if i is odd it requires at least $\lceil (i-2)/3 \rceil$ further blocks. □

LEMMA 20: *Let r be as in Theorem 18(4a). If* $i \geq 4$, *then* $r \in \overline{\mathcal{S}}_i(3,2)$.

PROOF: Assume that $T = T_1 - T_2$ is a simple $(3,2)$ trade of index i and volume r, and note that T must conform to the structure discussed in Lemma 19. Put $F = F(T)$, $X = \{x_1, \ldots, x_i\}$ and $Y = \{x_1, \ldots, y_i\}$. After placing each of 1 and 2 in $i + \lceil i/2 \rceil$ blocks in T_1 and in T_2, and each element of $X \cup Y$ in two blocks, there are zero, one or two positions in each of T_1 and T_2 free, in the sense that these elements could be drawn from $\{1,2\} \cup X \cup Y$ or from some disjoint set Z. We consider the residue classes for i modulo 6, and obtain a contradiction in each case.

(1) $i = 6n$, $n \geq 1$: There are no positions of T_1 or T_2 free. So $F = \{1,2\} \cup X \cup Y$, each element in $X \cup Y$ has multiplicity two, and is paired with each of 1 and 2 precisely once. Now, T_1 contains $2n$ blocks all of whose elements are drawn from X. Consider $x_\alpha x_\beta x_\gamma \in T_1$. The pairs $x_\alpha x_\beta$, $x_\alpha x_\gamma$ and $x_\beta x_\gamma$ must occur in T_2, as the blocks with 1 or 2. But this is impossible, since, e.g., using $1x_\alpha x_\beta$ forces the block $2x_\alpha x_\gamma$ and now $x_\beta x_\gamma$ cannot be placed without repeating either $1x_\beta$ or $2x_\gamma$.

(2) $i = 6n+1$, $n \geq 1$: There is one position free in each of T_1 and T_2; let u be the element used to fill this position. Since $r_u \neq 1$, then $u \in \{1,2\} \cup X \cup Y$. If $u \in X$, then T_1 contains $2(3n)$ pairs of the form $y_\alpha y_\beta$. To balance these, T_2 must contain $2n$ blocks all of whose elements are from Y. But now, T_1 contains two $x_\alpha y_\beta$ pairs, while T_2 contains only one. Similarly if $u \in Y$. So $u \in \{1,2\}$; suppose, without loss of generality, that $u = 1$. Now count pairs of the form $1x_\alpha$. T_1 has $i+3$ such pairs, while T_2 has i.

(3) $i = 6n+2$, $n \geq 1$: There is one position free in each of T_1 and T_2; let u be the element used to fill this position. Since $r_u \neq 1$, then $u \in \{1,2\} \cup X \cup Y$. If $u \in X$, then count pairs of the form $x_\alpha x_\beta$; T_1 contains $3(2n+1)$ such pairs, while T_2 contains $2(3n+1)$. Similarly if $u \in Y$. So $u \in \{1,2\}$; suppose, without loss of generality, that $u = 1$. Now count pairs of the form $1x_\alpha$. T_1 has $i+2$ such pairs, while T_2 has i.

(4) $i = 6n+3$, $n \geq 1$: There are two positions free in each of T_1 and T_2; let u and v be the elements used to fill these positions. Suppose that $u = v \in Z$. If T_1 and T_2 contain the pairs $1u$ and $2u$, then T_1 contains no ux_i pairs, while T_2 contains two such pairs. If T_1 and T_2 contain the pair $1u$ but not the pair $2u$, then T_1 contains two ux_i pairs, while T_2 contains only one; similarly if they contain $2u$ but not $1u$. If T_1 and T_2 do not contain either of the pairs $1u$ or $2u$, then T_1 contains four ux_i pairs, while T_2 contains none. Thus $u, v \in \{1,2\} \cup X \cup Y$, since $r_u, r_v \neq 1$. By symmetry, the only cases for (u,v) we need consider are: $(1,1)$, $(1,2)$, $(1,x_\alpha)$; (x_α, x_α), (x_α, x_β); (x_α, y_β).

(i) If $(u,v) = (1,1)$ (resp. $(1,2)$, $(1,x_\alpha)$), then T_1 has $i+5$ (resp. $i+3$, $i+3$) pairs of the form $1x_a$, while T_2 has only i (resp. i, i or $i+1$) such pairs. (ii) If $(u,v) = (x_\alpha, x_\alpha)$ or (x_α, x_β), then T_1 has $i+1$ pairs of each of the forms $1x_a$ and $2x_b$. To balance these in T_2, all the x_i must be in blocks with 1 or 2. But now T_1 has two pairs of the form $x_a y_b$, while T_2 has none. (iii) If $(u,v) = (x_\alpha, y_\beta)$ and y_β is not paired with 1 or 2, then T_1 contains $3(2n)+1$ pairs of the form $x_a x_b$, while T_2 contains at least $2(3n+1)$ such pairs. So suppose, without loss of generality, that T_1 and T_2 contain the pair $1y_\beta$. Note that $r_{y_\beta} = 3$, and count pairs of the form $y_\beta y_i$. T_1 contains either two or three such pairs, while T_2 contains four.

(5) $i = 6n + 4$, $n \geq 0$: There are two positions free in each of T_1 and T_2; let u and v be the elements used to fill these positions. Suppose that $u = v \in Z$; then T_1 contains $2(3n + 2)$ pairs of the form $y_\alpha y_\beta$, while T_2 contains $3(2n) + 2$ such pairs. Thus $u, v \in \{1, 2\} \cup X \cup Y$, since $r_u, r_v \neq 1$. By symmetry, the only cases for (u, v) we need consider are: $(1, 1)$, $(1, 2)$, $(1, x_\alpha)$; (x_α, x_α), (x_α, x_β); (x_α, y_β).

(i) If $(u, v) = (1, 1)$ (resp. $(1, 2)$, $(1, x_\alpha)$), then T_1 has $i+4$ (resp. $i+2$, $i+2$) pairs of the form $1x_a$, while T_2 has only i (resp. i, i or $i+1$) such pairs. (ii) If $(u, v) = (x_\alpha, x_\alpha)$ or (x_α, x_β), then T_1 contains $3(2n+2)$ pairs of the form $x_a x_b$, while T_2 contains either $2(3n + 2)$ or $2(3n + 2) + 1$ such pairs. (iii) If $(u, v) = (x_\alpha, y_\beta)$, then both T_1 and T_2 contain precisely one block containing elements from both X and Y. This block is of the form $x_a x_b y_\beta$ in T_1 and $x_\alpha y_a y_b$ in T_2. Balancing pairs forces $x_a = x_b = x_\alpha$ and $y_a = y_b = y_\beta$, which is not possible.

(6) $i = 6n + 5$, $n \geq 0$: There are no positions free in T_1 or T_2. Now count pairs of the form $x_\alpha x_\beta$. T_1 contains $3(2n + 1)$ such pairs, while T_2 contains $2(3n + 2)$. □

We are now in a position to prove Theorem 18. We work through the proof in some detail, to illustrate our methods. Similar techniques apply in Sections 5 and 6, but there we suppress much of the detail.

PROOF OF THEOREM 18(1): That $6 \in \mathcal{S}_2(3, 2)$ and $m \in \mathcal{S}_2(3, 2)$ for $m \geq 8$ follows from Lemma 13 and Theorem 14, $\mathcal{S}_1(3, 2)$ and Theorem 16. That $7 \in \mathcal{S}_2(3, 2)$ follows from considering the trade

$$T = +123 + 145 + 247 + 257 + 268 + 356 + 378$$
$$-124 - 135 - 237 - 256 - 278 - 368 - 457.$$

That $4 \in \overline{\mathcal{S}}_2(3, 2)$ follows from Lemma 12. □

PROOF OF THEOREM 18(2): That $8, 9 \in \mathcal{S}_3(3, 2)$ follows from considering the trades

$$T_a = +248 + 259 + 349 + 367 + 389 + 458 + 469 + 479$$
$$-249 - 258 - 348 - 369 - 379 - 459 - 467 - 489,$$
$$T_b = +128 + 139 + 147 + 158 + 168 + 249 + 256 + 278 + 348$$
$$-129 - 138 - 148 - 156 - 178 - 247 - 258 - 268 - 349.$$

That $m \in \mathcal{S}_3(3, 2)$ for $m \geq 10$ follows from $\mathcal{S}_2(3, 2)$ and $4 \in \mathcal{S}_1(3, 2)$ on applying Theorem 16(2). That $\{4, 6, 7\} \subseteq \overline{\mathcal{S}}_3(3, 2)$ follows from Lemmas 12 and 19. □

PROOF OF THEOREM 18(3): That $11 \in \mathcal{S}_4(3, 2)$ follows from considering the trade

$$T = +146 + 157 + 235 + 267 + 367 + 457 + 478 + 49a + 568 + 579 + 57a$$
$$-145 - 167 - 236 - 257 - 357 - 468 - 479 - 47a - 567 - 578 - 59a.$$

That $m \in \mathcal{S}_4(3, 2)$ for $m \geq 12$ follows from $\mathcal{S}_3(3, 2)$ and $4 \in \mathcal{S}_1(3, 2)$ on applying Theorem 16(2). That $\{4, 6, \ldots, 10\} \subseteq \overline{\mathcal{S}}_4(3, 2)$ follows from Lemmas 12 and 19. □

PROOF OF THEOREM 18(4): Part (4a) follows immediately from Lemmas 19 and 20. For part (4b), note that $8 \in \mathcal{S}_3(3, 2)$. Repeated addition of a simple $(3, 2)$ trade of index three and volume eight to the values in $\mathcal{S}_i(3, 2)$, $2 \leq i \leq 4$, using Theorem 16(2), now yields the result. □

5 Results for $k = 4$

We completely solve the spectrum problem for $k = 4$, proving the following result.

THEOREM 21:
(1) $\mathcal{S}_2(4,2) = P$;
(2) $\mathcal{S}_3(4,2) = P \setminus \{4\}$;
(3) $\mathcal{S}_4(4,2) = P \setminus \{4\}$;
(4) $\mathcal{S}_5(4,2) = P \setminus \{4,6,7\}$;
(5) $\mathcal{S}_6(4,2) = P \setminus \{4,6,7\}$;
(6) For $i \geq 7$, *define* $s = \lceil 7i/6 \rceil$. *Then* $\mathcal{S}_i(4,2) = P \setminus \{4, 6, \ldots, s-1\}$, *except that* $m \in \overline{\mathcal{S}}_i(4,2)$ *for the following* (m,i) *pairs:* $(9,7)$; $(11,9)$; $(12,10)$; $(13,11)$; $(14,12)$; $(19,16)$; $(20,17)$; $(21,18)$; $(27,23)$.

LEMMA 22: *Suppose that* $m \in \mathcal{S}_i(4,2)$. *Then* $m \geq \lceil 7i/6 \rceil$.

PROOF: Let $T = T_1 - T_2$ be a simple $(4,2)$ trade of volume m and index i, and suppose that 12 is a pair which occurs in i blocks of T_1 and T_2. Now consider the i pairs which occur with 12 in blocks of T_1. Since $T_1 \cap T_2 = \emptyset$, none of these pairs can occur as a block with 12 in T_2. To balance pairs, these i pairs must occur in blocks of T_2 which do not contain 12. Each such block can contain at most 6 pairs, so $6(m - i) \geq i$. □

Theorem 21(2) follow from our results so far by repeated application of Theorem 16. For Theorem 21(1), it remains only demonstrate that $7 \in \mathcal{S}_2(4,2)$. Consider

$$\begin{aligned} T = &+3459 + 3468 + 3567 + 3789 + 4578 + 4679 + 5689 \\ &-3469 - 3478 - 3568 - 3579 - 4567 - 4589 - 6789. \end{aligned}$$

To complete our proof of Theorem 21 we will use some structural properties of simple $(4,2)$ trades which enable the problem to be reduced to the question of the existence of certain $(4,1)$ trades. To motivate what follows, consider the following example, which completes the proof of Theorem 21(3).

EXAMPLE 23: *That* $7 \in \mathcal{S}_4(4,2)$ *follows from considering the trade*

$$\begin{aligned} T = &+xy27 + xy37 + xy46 + xy56 + 1236 + 1345 + 1467 \\ &-xy26 - xy35 - xy47 - xy67 - 1237 - 1346 - 1456. \end{aligned}$$

Note how the sets not containing xy *form a* $(4,1)$ *trade of volume three. Of the eighteen pairs in each half of this* $(4,1)$ *trade, fourteen appear in the other half, and the remaining four pairs are those occurring with* xy *in the other half of* T. *Further, the four pairs occurring with* xy *in each half form a simple* $(2,1)$ *trade of volume four.*

DEFINITION 24: *Let* $T = T_1 - T_2$ *be a simple* $(4,1)$ *trade of volume* m. *Suppose that, of the* $6m$ *pairs in the blocks of* T_1, *precisely* e *of them appear in the blocks of* T_2. *Then* e *is called the* **excess** *of* T. *The* $6m - e$ *pairs in* T_1 *(resp.* T_2*) that do not appear in* T_2 *(resp.* T_1*) are called* ***non-balanced*** *pairs.*

We can think of the excess as measuring how close to being 2-balanced T is, since $e = 6m$ if T is also a $(4,2)$ trade. The $(4,1)$ trade of volume three in Example 23 has an excess of fourteen, and the sets of non-balanced pairs are $26, 35, 47, 67$ and $27, 37, 46, 56$. Note that, in an arbitrary simple $(4,1)$ trade, the set of excess pairs and the set of non-balanced pairs can contain repeated pairs, and need not be disjoint.

LEMMA 25: *Suppose that $S = S_1 - S_2$ is a simple $(4,1)$ trade of volume n and excess e. Put $i = 6n - e$ and $m = i + n$. If $i \geq n$ and the i non-balanced pairs in S_1 and in S_2 are distinct, then there is a simple $(4,2)$ trade of volume m and index i.*

PROOF: Let R_1^* (resp. R_2^*) be the set of i pairs which are in S_2 but not in S_1 (resp. in S_1 but not in S_2). Then $R_1^* - R_2^*$ is a simple $(2,1)$ trade of volume i. To see this, simply note that: each element of $F(S)$ is in the same number of pairs in S_1 and in S_2; pairs common to S_1 and S_2 are not in R_1^* or R_2^*; the non-balanced pairs are distinct. By construction, $S_1 \cup R_1^*$ and $S_2 \cup R_2^*$ are 2-balanced. Now choose distinct x and y not in $F(S)$, and let $R_1 = xyR_1^*$ and $R_2 = xyR_2^*$. Since $R_1^* - R_2^*$ is 1-balanced, then $T = R_1 + S_1 - R_2 - S_2$ is a simple $(4,2)$ trade of volume m, with $r_x = r_y = r_{xy} = i$. Since $i \geq n$, any pair in S_1 (which must be in either S_2 or R_2^*) has multiplicity at most i. Similarly for pairs in S_2. So T has index i. □

LEMMA 26: *Let $T = T_1 - T_2$ be a simple $(4,2)$ trade of volume m and index i, and let $n = m - i$ and $e = 6n - i$. Suppose that $xy \subseteq F(T)$ has $r_{xy} = i$, and let S_1 and S_2 be the sets of n blocks, from T_1 and T_2 respectively, which do not contain the pair xy. Then $S = S_1 - S_2$ is a simple $(4,1)$ trade of volume n and excess e, with distinct non-balanced pairs, if either of the following holds:*
(1) $m = \lceil 7i/6 \rceil$; (2) $n \leq 2$.

PROOF: Let R_1 (resp. R_2) be the set of i blocks in T_1 (resp. T_2) which contain the pair xy, and let R_1^* (resp. R_2^*) be the set of pairs formed by removing the pair xy from each of the blocks of R_1 (resp. R_2). Note that $S_1 - S_2$ is a $(4,1)$ trade if and only if $R_1 - R_2$ is a $(4,1)$ trade, and if and only if $R_1^* - R_2^*$ is a $(2,1)$ trade. Given this, the excess of $S_1 - S_2$ follows from the 2-balancing of T, since the only pairs not balanced in $R_1 - R_2$ are those in R_1^* and R_2^*, and these must come from S_2 and S_1 respectively. R_1^* and R_2^* are the non-balanced pairs in S, and are obviously distinct. So it remains to prove that one of $S_1 - S_2$, $R_1 - R_2$ or $R_1^* - R_2^*$ is 1-balanced.

(1) If at most one of x and y is in $F(S_1)$, then at least one of x and y (say x) occurs only in the blocks of R_1 and R_2. Balancing pairs of the form $x\alpha$, $\alpha \notin \{x, y\}$, now forces $R_1^* - R_2^*$ to be 1-balanced. So, if $R_1^* - R_2^*$ is not 1-balanced, then $\{x, y\} \subseteq F(S_1)$. Since x and y cannot occur together in a set from S_1, there must be at least six pairs in the blocks of S_1 which are not in R_2^*. This contradicts $m = \lceil 7i/6 \rceil$, which allows at most five such pairs (recall Lemma 22).

(2) As in (1), if $R_1^* - R_2^*$ is not 1-balanced, then $\{x, y\} \subseteq F(S_1)$ and x and y cannot occur together in a set from S_1. So $n \geq 2$. If $n = 2$, then $r_x = r_y = m - 1$, which contradicts Lemma 9(2). □

Now, let $m = \lceil 7i/6 \rceil$, and consider simple $(4,2)$ trades with volume m and index i. Suppose that $i = 6s + \delta$, $1 \leq \delta \leq 6$. For fixed s, as δ runs through $1, \ldots, 6$, then the

TABLE 1: Simple $(4,1)$ trades of volume three and excess 3–13, 15

e	$S = S_1 - S_2$
3	$+ab12 + cd34 + ef56 - ab35 - cd16 - ef24$
4	$+1234 + 5678 + 9abc - 1259 - 346a - 78bc$
5	$+123x + 145y + 67uv - 124u - 135v - 67xy$
6	$+1234 + 5678 + 9abc - 1256 - 349a - 78bc$
7	$+1234 + 5678 + 9abc - 1235 - 4679 - 8abc$
8	$+1234 + 1235 + 6789 - 1236 - 1247 - 3589$
9	$+1234 + 5678 + 9abc - 1238 - 567c - 49ab$
10	$+1234 + 1567 + 589a - 1235 - 149a - 5678$
11	$+1234 + 1235 + 6789 - 1236 - 1245 - 3789$
12	$+1234 + 1235 + 4678 - 1236 - 1245 - 3478$
13	$+1468 + 2568 + 3578 - 1568 - 2458 - 3678$
15	$+1234 + 1256 + 1278 - 1236 - 1247 - 1258$

excess of the trade, e, runs through $5, \ldots, 0$, while the value of $n = m - i$ is fixed. To complete our proof of Theorem 21, for 'large' values of i we will prove the existence of simple $(4,1)$ trades of excess e, $0 \leq e \leq 5$, and with distinct non-balanced pairs, for all 'large enough' n. The spectra $\mathcal{S}_i(4,2)$ now follow from Lemma 25 and repeated application of Theorem 16. For the smaller values of i, we need to prove the existence or non-existence of the appropriate $(4,1)$ trades for various 'small' values of n. For convenience, we use $\mathcal{S}_e$ to denote the spectrum of simple $(4,1)$ trades with excess e and with distinct non-balanced pairs.

LEMMA 27:
(1) $2 \in \mathcal{S}_e$ *if and only if* $e \in \{4,6,8,10\}$;
(2) For $i \geq 2$, $i+2 \in \mathcal{S}_i(4,2)$ *if and only if* $i \in \{2,4,6,8\}$.

PROOF: For (1), suppose that $S = S_1 - S_2$ is a simple $(4,1)$ trade of volume two. The two blocks of S_1 can intersect in 0, 1 or 2 points. These yield, respectively, 2, 1 and 1 non-isomorphic forms for S_2, with excesses of 4 and 6, 8 and 10. In all cases, the non-balanced pairs are distinct. Part (2) now follows from Lemmas 25 and 26. □

LEMMA 28:
(1) $3 \notin \mathcal{S}_e$ *if* $e \in \{0,1,2\}$, *and* $3 \in \mathcal{S}_e$ *if* $e \in \{3,\ldots,15\}$;
(2) For $i \geq 1$, $i+3 \in \mathcal{S}_i(4,2)$ *if and only if* $i \in \{3,\ldots,15\}$.

PROOF: Suppose that $S = S_1 - S_2$ is a simple $(4,1)$ trade of volume three and excess e. Since $k = 4$, at least one pair from each block of S_1 occurs in a block of S_2, so $e \geq 3$. For $e = 14$ use Example 23. To complete (1), consider the trades of Table 1. Part (2) now follows for $i+3 \geq 6$ from Lemmas 25 and 26, for $i+3=5$ from Theorem 21(1), and for $i+3=4$ from Lemma 12. □

LEMMA 29:
(1) $\mathcal{S}_0 = \{4,5,6,\ldots\}$; *(2)* $\mathcal{S}_1 = \{5,6,7,\ldots\}$; *(3)* $\mathcal{S}_2 = \{4,5,6,\ldots\}$;
(4) $\mathcal{S}_3 = \{3,4,5,\ldots\}$; *(5)* $\mathcal{S}_4 = \{2,3,4,\ldots\}$; *(6)* $\mathcal{S}_5 = \{3,4,5,\ldots\}$.

TABLE 2: Simple $(4,1)$ trades of excess e and volume n

e	n	$S = S_1 - S_2$
3	4	$+ab12 + cd34 + ef56 + 789x - ab67 - cd28 - ef49 - 135x$
	5	$+ab12 + cd34 + ef56 + 789x + yzuv - ab7y - cd8z - ef9u - 135x - 246v$
	6	$+12ab + 34cd + 56ef + ghij + klmn + opqr$
		$-12gm - 34hn - 56io - adjp - bekq - cflr$
4	4	$+ab12 + cd34 + ef56 + gh78 - ab35 - cd17 - ef28 - gh46$
	5	$+ab12 + cd34 + ef56 + gh78 + 9xyz - ab89 - cd2x - ef4y - gh6z - 1357$
5	4	$+abcd + ef12 + gh34 + ij56 - ab13 - ef45 - gh26 - ijcd$
	5	$+ab12 + cd34 + ef56 + gh78 + ij9x - ab35 - cd17 - ef29 - gh4x - ij68$
	6	$+ab12 + cd34 + ef56 + gh78 + ij9x + yzuv$
		$-ab3y - cd1z - ef7u - gh9v - ij58 - 246x$

PROOF: Let $S = S_1 - S_2$ be a simple $(4,1)$ trade of volume n and excess e, with distinct non-balanced pairs. Obviously, $n \neq 1$. The $n = 2$ and 3 cases are covered by Lemmas 27 and 28. In the constructions which follow, note that each element of $F(S)$ except x and y is used once only, so the non-balanced pairs will be distinct.

For $e = 0$ and $n \geq 4$, form an $n \times 4$ array of distinct points. Take the rows of this array as the blocks of S_1. For block j of S_2, $1 \leq j \leq n$, take the points at positions $(j,1)$, $(j+1,2)$, $(j+2,3)$ and $(j+3,4)$, reducing the first subscript modulo n to lie in $\{1,\ldots,n\}$. Obviously, no pair in a block of S_1 occurs in a block of S_2.

Suppose that $e = 1$ and $n = 4$, and let xy be the pair which occurs in both S_1 and S_2. Consider the three blocks in S_1 which do not contain the pair xy. The four points in each of these must be in separate blocks in S_2, else $e > 1$. However, the block in S_2 which contains xy has positions for only two points, a contradiction. For $e = 1$ and $n \geq 5$, form an $n \times 4$ array $A = [a_{i,j}]$ where $a_{1,1} = x$, $a_{1,2} = y$ and the remaining positions are filled in row-major order with $1, \ldots, 4n-2$. Take the rows of A as the blocks of S_1. For the blocks of S_2, take the rows of an $n \times 4$ array B where $b_{1,1} = x$, $b_{1,2} = y$ and the remaining positions are filled in column-major order with $1, \ldots, 4n-2$. It is easy to see that $S_1 - S_2$ is a simple $(4,1)$ trade of volume n, and the only pair common to S_1 and S_2 is xy.

For $e = 2$ and $n \geq 4$, proceed as for $e = 1$, except that $a_{1,1} = a_{2,1} = b_{1,1} = b_{2,1} = x$, $a_{1,2} = a_{2,2} = b_{1,2} = b_{2,2} = y$ and using the points $1, \ldots, 4n-4$ for the remaining positions.

To demonstrate existence for the $e \geq 3$ cases, we make use of the fact that, if $a \in \mathcal{S}_u$ and $b \in \mathcal{S}_v$, then $a + b \in \mathcal{S}_{u+v}$. So we need only exhibit an appropriate S for a small number of cases. The required trades as given in Table 2. □

Repeated application of Theorem 16 now proves existence for all required (m, i) pairs except $(21, 17)$, $(22, 18)$ and $(29, 24)$. These are easily dealt with using Lemma 25 and the following result.

LEMMA 30:
(1) $4 \in \mathcal{S}_7$; *(2)* $4 \in \mathcal{S}_6$; *(3)* $5 \in \mathcal{S}_6$.

PROOF: Consider the (4, 1) trades:
(1) $+1234 + 1567 + 39ab + cdef - 123c - 159d - 34ae - 67bf$;
(2) $+1234 + 5678 + 9abc + defg - 123b - 564f - 9a7g - de8c$;
(3) $+1234 + 5678 + 9abc + defg + hijk - 123h - 56gi - 9a8j - deck - 47bf$. □

6 Results for $k \geq 5$

Our result for $k \geq 5$ is the following.

THEOREM 31: *Suppose that $k \geq 5$. Then:*
(1) $\mathcal{S}_2(k,2) \supseteq P \setminus \{m : 7 \leq m \leq 2k-1,\ m \text{ odd}\}$;
(2) $\mathcal{S}_3(k,2) = P \setminus \{4\}$;
(3) $\mathcal{S}_4(k,2) = P$;
(4) $\mathcal{S}_5(k,2) = P \setminus \{4,6\}$;
(5) $\mathcal{S}_6(k,2) = P \setminus \{4,7\}$;
(6) If $i \geq 7$, then $\mathcal{S}_i(k,2) = P \setminus \{4,6,\ldots,i-1,i+1\}$.

Parts (1), (2), (5) and (6) of Theorem 31 follow immediately from Theorem 4 and the results of Section 3. To complete parts (3) and (4), we need only the following result.

LEMMA 32: *For all $k \geq 5$:*
(1) $7 \in \mathcal{S}_4(k,2)$; (2) $8 \in \mathcal{S}_5(k,2)$.

PROOF: Let A and B be disjoints sets of cardinality $k-3$, each disjoint from $\{1,\ldots,9\}$, and consider the trades

$$\begin{aligned} T_a &= +A146 + A236 + A139 + A789 + B245 + B158 + B357 \\ &\quad -A136 - A246 - A189 - A379 - B145 - B235 - B578, \\ T_b &= +A136 + A157 + A189 + A234 + A256 + B245 + B238 + B279 \\ &\quad -A156 - A138 - A179 - A236 - A245 - B234 - B257 - B289. \end{aligned}$$

These are simple $(k,2)$ trades of the required volumes. Since $|A| \geq 2$, pairs from A have multiplicity 4 or 5. To see that no pair has higher multiplicity, simply note that no element occurs more than four times in T_a, or more than five times in T_b. □

7 Concluding remarks

For $k = 3$, the arguments of Lemma 20 could perhaps be extended to prove non-existence in further cases. Whether or not this would be sufficient to close the gap of Theorem 18(4) is not clear. For $k \geq 5$ we are unable to determine whether $m \in \mathcal{S}_2(k,2)$ or $m \in \overline{\mathcal{S}}_2(k,2)$ for $m \in \{n : 7 \leq n \leq 2k-1,\ n \text{ odd}\}$. Our inability to construct trades of these volumes using Theorem 16 is due to the non-existence of Steiner trades with volumes less than $2k-2$.

Our results demonstrate that, in general, $\mathcal{S}_i(k,2) \subset P$. Any $(k,2)$ trades with index i and volumes in $\overline{\mathcal{S}}_i(k,2)$ must be non-simple. It would be interesting to know when such trades can be constructed, and how many repeated blocks are necessary. Finally, recall the near-Steiner $(k, 1)$ trades of Section 3. Given that the index $i > 1$, these are as 'Steiner' as possible, in the sense that only one element has multiplicity greater than 1. It would be interesting to extend the definition of near-Steiner to $(k, 2)$ trades and to investigate their spectra.

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