The spectrum of rotational directed triple systems and rotational Mendelsohn triple systems

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Abstract

Necessary and sufficient conditions for the existence of k-rotational directed triple systems and k-rotational Mendelsohn triple systems are derived.

1. Introduction

Let V be a set of v *points* and $\mathcal{B}$ be a collection of 3-subsets (called *blocks* or *triples*) of V. A pair $(V, \mathcal{B})$ is called a *triple system*, denoted by $\mathrm{TS}(v, \lambda)$, if every pair of distinct points of V is contained in precisely λ blocks of $\mathcal{B}$. Furthermore, when $\lambda = 1$, it is called a *Steiner triple system* (STS) and when $\lambda = 2$, it is called a *twofold triple system* (TTS). There is a vast amount of literature on such generalized triple systems. As is well-known, directed triple systems [9] and Mendelsohn triple systems [12] are also included in such generalizations.

A *directed triple system* $\mathrm{DTS}(v, \lambda)$ is a pair $(V, \mathcal{B})$ such that $\mathcal{B}$ is a collection of edge-disjoint transitive tournaments of order 3 with vertices from V, having the property that every ordered pair of elements of V appears in precisely λ transitive tournaments. To distinguish a block (triple) of a $\mathrm{DTS}(v, \lambda)$ from a block $\{a, b, c\}$ of an ordinal triple system, we denote it by $\langle a, b, c\rangle$. In this case, the set of its ordered pairs is $\{(a, b), (a, c), (b, c)\}$, which is represented also as a *difference triple* $(b - a, c - b, c - a)$.

A *Mendelsohn triple system* $\mathrm{MTS}(v, \lambda)$ differs only in that the above $\mathcal{B}$ contains directed cycles of length 3. A triple of $\mathrm{MTS}(v, \lambda)$ is represented by $[a, b, c]$ and the set of its ordered pairs is given as $\{(a, b), (b, c), (c, a)\}$, which is represented also as a difference triple $(b - a, c - b, a - c)$. It is easy to see that $[a, b, c] = [b, c, a] = [c, a, b]$.

If one omits the directions in a $\mathrm{DTS}(v, \lambda)$ or a $\mathrm{MTS}(v, \lambda)$, then a $\mathrm{TS}(v, 2\lambda)$ can be obtained. Many researchers have investigated the existence problem of these triple systems. Hanani [8] determined the necessary and sufficient condition for the existence of $\mathrm{TS}(v, \lambda)$ for every λ. Similarly the necessary and sufficient condition was shown for $\mathrm{DTS}(v, 1)$ by Hung and Mendelsohn [9] and for $\mathrm{MTS}(v, 1)$ by Mendelsohn [12]. See, for the relevant results, [6] and [7].

Let G be an *automorphism group* of a generalized triple system $(V, \mathcal{B})$, that is, a group of permutations on the set V of v points leaving the collection of blocks $\mathcal{B}$ invariant. If there is an automorphism of order v, then the design is said to be *cyclic*. For a cyclic triple system $(V, \mathcal{B})$, the set V of v points can be identified with Z_v, i.e. the residue group of integers modulo v. In this case, the design has an automorphism $\sigma : i \mapsto i+1 \bmod v$ which is also represented by $\sigma = (0, 1, \ldots, v-1)$. Let B be a block of a cyclic triple system $(V, \mathcal{B})$. A *block orbit* of B is defined by $\{B + y : y \in Z_v\}$. The *length* of a block orbit is its cardinality. A block orbit of length v is said to be *full*, otherwise *short*. A *base block* of a block orbit $\mathcal{O}$ is a block $B \in \mathcal{O}$ which is chosen arbitrarily. For any cyclic triple system, the length of a short block orbit is $v/3$ if it exists.

If there is an automorphism consisting of a single fixed point and precisely k cycles of length $(v-1)/k$, then the design is said to be *k-rotational*. The automorphism can be represented by

$$\pi = (\infty)(0_1, 1_1, \ldots, (n-1)_1) \cdots (0_k, 1_k, \ldots, (n-1)_k)$$

on the point-set $V = \{\infty\} \cup (Z_n \times \{1, 2, \ldots, k\})$, where $n = (v-1)/k$ and x_i denotes the element $(x, i) \in Z_n \times \{i\}$. A block orbit of a k-rotational triple system is defined similarly to that of a cyclic triple system, but under the automorphism π. In this case, the length of a full block orbit is $(v-1)/k$ and the length of a short block orbit is $(v-1)/(3k)$ or $(v-1)/(2k)$ if it exists. Any cyclic or k-rotational triple system should be generated from base blocks. Note that a directed triple system has no short block orbit due to the order structure on its blocks.

The condition for the existence of a cyclic $\mathrm{TS}(v, \lambda)$ was determined by Colbourn and Colbourn [5], and that of a cyclic $\mathrm{DTS}(v, \lambda)$ was given by Cho, Han and Kang [4]. Quite recently, the spectrum of a cyclic $\mathrm{MTS}(v, \lambda)$ has been settled by Shen [14]. For the existence of a 1-rotational $\mathrm{TS}(v, \lambda)$, Kuriki and Jimbo [11], and Cho [2] gave the same result independently.

Theorem 1.1 ([2], [11]) *A 1-rotational* $\mathrm{TS}(v, \lambda)$ *exists if and only if*

(i) $\lambda = 1$ *and* $v \equiv 3, 9 \bmod 24$,
(ii) $\lambda \equiv 1, 5 \bmod 6$, $\lambda \neq 1$ *and* $v \equiv 1, 3 \bmod 6$,
(iii) $\lambda \equiv 2, 4 \bmod 6$ *and* $v \equiv 0, 1 \bmod 3$,
(iv) $\lambda \equiv 3 \bmod 6$ *and* $v \equiv 1 \bmod 2$, *or*
(v) $\lambda \equiv 0 \bmod 6$ *and* $v \geq 3$.

Remark. The terminology of 'k-rotational' was defined by Phelps and Rosa [13] who showed (i) of Theorem 1.1.

Our aim is to determine completely necessary and sufficient conditions for the existence of a k-rotational $\mathrm{DTS}(v, \lambda)$ and a k-rotational $\mathrm{MTS}(v, \lambda)$ for all λ.

In fact, only when $\lambda = 1$, we can find the necessary and sufficient conditions for the existence of a k-rotational directed triple system and a k-rotational Mendelsohn triple system in [3] and [10], respectively.

Theorem 1.2 (Cho, Chae and Hwang [3]) *A k-rotational* DTS$(v,1)$ *exists if and only if*

(i) $k \equiv 1,2 \bmod 3$, $v \equiv 0 \bmod 3$ *and* $v \equiv 1 \bmod k$, *or*
(ii) $k \equiv 0 \bmod 3$ *and* $v \equiv 1 \bmod k$.

Theorem 1.3 (Jiang and Colbourn [10]) *A k-rotational* MTS$(v,1)$ *exists if and only if* $v \equiv 0,1 \bmod 3$ *and* $v \equiv 1 \bmod k$, *except when* $k = 1$ *and* $v \equiv 0 \bmod 6$ *or* $v = 10$.

If there exists a k-rotational DTS(v,λ) or a k-rotational MTS(v,λ), then there exists a k-rotational TS$(v,2\lambda)$ without directions in the design, but it should be remarked that the converse is not necessarily true. This means that the condition for the existence of a k-rotational TS$(v,2\lambda)$ can be regarded as the necessary condition both for the existence of a k-rotational DTS(v,λ) and for the existence of a k-rotational MTS(v,λ). On the other hand, if α is a 1-rotational automorphism of a DTS(v,λ) or a MTS(v,λ), then α^k is also an automorphism of the design for any integer k. Since α^k is a k-rotational permutation provided $v \equiv 1 \bmod k$, we should note that any 1-rotational DTS(v,λ) or any 1-rotational MTS(v,λ) is also k-rotational if $v \equiv 1 \bmod k$.

2. A k-rotational DTS(v,λ)

First of all, we will show the following recursive construction, which will be useful for our further discussion.

Lemma 2.1 *If there exist a k-rotational* DTS(v,λ_1) *and a k-rotational* DTS(v,λ_2), *then there exists a k-rotational* DTS$(v,n\lambda_1+m\lambda_2)$ *for any positive integers n and m.*

It is easy to see that $|\mathcal{B}| = \lambda v(v-1)/3$ for a DTS(v,λ) $(V,\mathcal{B})$. Since any DTS(v,λ) has no short block orbit, if a DTS(v,λ) is k-rotational, then $\lambda v(v-1)/3$ is divisible by $(v-1)/k$. Thus the basic necessary condition for the existence of a k-rotational DTS(v,λ) is that

$$kv\lambda \equiv 0 \bmod 3 \qquad \text{and} \qquad v \equiv 1 \bmod k. \tag{2.1}$$

Now, let us consider the existence of a 1-rotational DTS(v,λ). Remember that the underlying triple system of a 1-rotational DTS(v,λ) is a 1-rotational TS$(v,2\lambda)$. Noting this fact and Lemma 2.1, it suffices to take the cases when $\lambda = 1,2$ and 3. However, we already know from (i) of Theorem 1.2 that there exists a 1-rotational DTS$(v,1)$ if and only if $v \equiv 0 \bmod 3$. Thus we have only to consider two cases when $\lambda = 2$ and 3.

Lemma 2.2 *There exists a* 1*-rotational* DTS$(v,2)$ *if and only if* $v \equiv 0 \bmod 3$.

Proof. Since any DTS(v,λ) has no short block orbit, a 1-rotational DTS(v,λ) is generated by $\lambda v/3$ base blocks for full block orbits. So it is evident that v is divisible by 3 when $\lambda = 2$. Thus the necessity of the assertion follows from (iii) of Theorem 1.1. The sufficiency follows from the existence of a 1-rotational DTS$(v,1)$ for any $v \equiv 0 \bmod 3$, shown by (i) of Theorem 1.2. □

Lemma 2.3 *A* 1*-rotational* DTS$(v,3)$ *exists for any* $v \geq 3$.

To prove Lemma 2.3, we need the following result by Cho, Han and Kang [4].

Theorem 2.4 ([4]) *A cyclic* DTS(v,λ) *exists if and only if*

(i) $\lambda \equiv 1,5 \bmod 6$ *and* $v \equiv 1,4,7 \bmod 12$,
(ii) $\lambda \equiv 2,4 \bmod 6$ *and* $v \equiv 1 \bmod 3$,
(iii) $\lambda \equiv 3 \bmod 6$ *and* $v \equiv 0,1,3 \bmod 4$, *or*
(iv) $\lambda \equiv 0 \bmod 6$ *and* $v \geq 3$.

Proof of Lemma 2.3. We can find in [11] a 1-rotational TS$(v,3)$ for any $v \equiv 1 \bmod 2$ constructed by $(v-1)/2$ full block orbits one of which is generated from a base block including ∞, say, $\{\infty,0,x\}$ $(x \neq (v-1)/2)$ and a short block orbit generated from $\{\infty,0,(v-1)/2\}$. Replace each base block $\{a,b,c\}$ of a 1-rotational TS$(v,3)$ with two base blocks $\langle a,b,c\rangle$ and $\langle c,b,a\rangle$ for $a,b,c \neq \infty$, the base blcok $\{\infty,0,x\}$ with two base blocks $\langle 0,\infty,x\rangle$ and $\langle x,\infty,0\rangle$, and the base block $\{\infty,0,(v-1)/2\}$ for a short block orbit with a base block $\langle 0,\infty,(v-1)/2\rangle$, respectively. Then the v base blocks obtained above generate a 1-rotational DTS$(v,3)$.

Now it remains for us to consider the case when $v \equiv 0 \bmod 2$. A cyclic DTS$(v-1,3)$ can be modified to obtain a 1-rotational DTS$(v,3)$. Note that a cyclic DTS$(v-1,3)$ consists of $v-2$ full block orbits. Without loss of generality, let $\langle 0,a,b\rangle$ be a base block of a full block orbit chosen arbitrarily from a cyclic DTS$(v-1,3)$. Next, replace the base block $\langle 0,a,b\rangle$ with three base blocks $\langle 0,\infty,a\rangle$, $\langle 0,\infty,b\rangle$ and $\langle 0,\infty,b-a\rangle$. Then these three base blocks and the rest $v-3$ base blocks of a cyclic DTS$(v-1,3)$ generate a 1-rotational DTS$(v,3)$. Thus (iii) of Theorem 2.4 implies the existence of a 1-rotational DTS$(v,3)$ for $v \equiv 0,2,3 \bmod 4$, which covers $v \equiv 0 \bmod 2$. The lemma is proved. □

With Lemma 2.1, the case (i) of Theorem 1.2, and Lemmas 2.2 and 2.3 can show the following theorem.

Theorem 2.5 *A* 1*-rotational* DTS(v,λ) *exists if and only if*

(i) $\lambda \equiv 1,2 \bmod 3$ *and* $v \equiv 0 \bmod 3$, *or*
(ii) $\lambda \equiv 0 \bmod 3$ *and* $v \geq 3$.

By remembering the fact that any 1-rotational DTS(v,λ) is k-rotational if $v \equiv 1 \bmod k$, we can establish one of the main theorems of the present paper.

Theorem 2.6 *A* k*-rotational* DTS(v,λ) *exists if and only if*

(i) $\lambda \equiv 1,2 \bmod 3$, $k \equiv 1,2 \bmod 3$, $v \equiv 0 \bmod 3$ *and* $v \equiv 1 \bmod k$,
(ii) $\lambda \equiv 1,2 \bmod 3$, $k \equiv 0 \bmod 3$ *and* $v \equiv 1 \bmod k$, *or*
(iii) $\lambda \equiv 0 \bmod 3$ *and* $v \equiv 1 \bmod k$.

Proof. When $\lambda \equiv 1, 2 \bmod 3$ and $k \equiv 1, 2 \bmod 3$, the basic necessary condition (2.1) for the existence of a k-rotational DTS(v, λ) is that $v \equiv 0 \bmod 3$ and $v \equiv 1 \bmod k$. From (i) of Theorem 1.2 and the fact that a 1-rotational DTS$(v, 1)$ has a k-rotational automorphism if $v \equiv 1 \bmod k$, the sufficiency of (i) of Theorem 2.6 follows.

If $\lambda \equiv 1, 2 \bmod 3$ and $k \equiv 0 \bmod 3$, then (2.1) reduces to $v \equiv 1 \bmod k$. Since (ii) of Theorem 1.2 describes the existence of a k-rotational DTS$(v, 1)$ with the same condition, the sufficiency is also satisfied.

For the case when $\lambda \equiv 0 \bmod 3$, (2.1) is simplified as $v \equiv 1 \bmod k$ again. Since (ii) of Theorem 2.5 ensures the sufficiency of the last case, which completes the proof. □

3. A k-rotational MTS(v, λ)

In a manner similar to Section 2, we will provide a necessary and sufficient condition for the existence of a k-rotational MTS(v, λ).

Lemma 3.1 *If there exist a k-rotational* MTS(v, λ_1) *and a k-rotational* MTS(v, λ_2), *then there exists a k-rotational* MTS$(v, n\lambda_1 + m\lambda_2)$ *for any positive integers n and m.*

Firstly, the existence of a 1-rotational MTS(v, λ) will be considered. The following can be obtained easily from Theorem 1.3, but originally it was proved by Cho [1].

Lemma 3.2 ([1]) *A 1-rotational* MTS$(v, 1)$ *exists if and only if* $v \equiv 1, 3, 4 \bmod 6$ *and* $v \neq 10$.

A 1-rotational MTS(v, λ) can be obtained from a 1-rotational TS(v, λ) by replacing every block $\{a, b, c\}$ with two blocks $[a, b, c]$ and $[a, c, b]$. On the other hand, the underlying triple system of a 1-rotational MTS(v, λ) is a 1-rotational TS$(v, 2\lambda)$. Hence the condition (ii) of Theorem 1.1 implies the following.

Lemma 3.3 *There exists a 1-rotational* MTS$(v, 2)$ *if and only if* $v \equiv 0, 1 \bmod 3$.

Next, the existence of a 1-rotational MTS$(v, 3)$ will be examined. For that purpose, the following theorem is needed.

Theorem 3.4 (Shen [14]) *A cyclic* MTS(v, λ) *exists if and only if*

(i) $\lambda \equiv 1, 5 \bmod 6$ *and* $v \equiv 1, 3 \bmod 6$,
(ii) $\lambda \equiv 2, 4 \bmod 6$ *and* $v \equiv 0, 1 \bmod 3$,
(iii) $\lambda \equiv 3 \bmod 6$ *and* $v \equiv 1 \bmod 2$, *or*
(iv) $\lambda \equiv 0 \bmod 6$ *and* $v \geq 3$

with only three exceptions: $(v, \lambda) = (9, 1), (6, 2)$ *and* $(9, 2)$.

Lemma 3.5 *A 1-rotational* MTS$(v, 3)$ *exists for any* $v \geq 3$.

Proof. It is easy to see that (iv) of Theorem 1.1 ensures the existence of a 1-rotational MTS$(v, 3)$ whenever $v \equiv 1 \bmod 2$. To complete the proof, we still need to take into account the case when $v \equiv 0 \bmod 2$. Choose a base block for a full block orbit of a cyclic MTS$(v-1, 3)$ arbitrarily, say, $[0, a, b]$. Replace the block $[0, a, b]$ with three base blocks $[\infty, 0, a]$, $[\infty, 0, b-a]$ and $[\infty, b, 0]$. Note that any of them cannot be for a short block orbit since $(v-1)/2$ is not an integer. Then it is readily checked that these three base blocks and the rest of base blocks of a cyclic MTS$(v-1, 3)$ generate a 1-rotational MTS$(v, 3)$. Thus (iii) of Theorem 3.4 shows the existence of a 1-rotational MTS$(v, 3)$ for $v \equiv 0 \bmod 2$, which completes the proof. □

Applying Lemma 3.1 to Lemmas 3.2, 3.3 and 3.5, a necessary and sufficient condition for the existence of a 1-rotational MTS(v, λ) is obtained.

Theorem 3.6 *A 1-rotational* MTS(v, λ) *exists if and only if*

(i) $\lambda = 1$, $v \equiv 1, 3, 4 \bmod 6$ *and* $v \neq 10$,
(ii) $\lambda \neq 1$, $\lambda \equiv 1, 2 \bmod 3$ *and* $v \equiv 0, 1 \bmod 3$, *or*
(iii) $\lambda \equiv 0 \bmod 3$ *and* $v \geq 3$.

Since any 1-rotational MTS(v, λ) is also k-rotational if $v \equiv 1 \bmod k$, we can state the sufficiency for the existence of a k-rotational MTS(v, λ).

Lemma 3.7 *A k-rotational* MTS(v, λ) *exists whenever*

(i) $\lambda = 1$, $v \equiv 1, 3, 4 \bmod 6$, $v \equiv 1 \bmod k$ *and* $v \neq 10$,
(ii) $\lambda \neq 1$, $\lambda \equiv 1, 2 \bmod 3$, $v \equiv 0, 1 \bmod 3$ *and* $v \equiv 1 \bmod k$, *or*
(iii) $\lambda \equiv 0 \bmod 3$ *and* $v \equiv 1 \bmod k$.

The necessity of (i) of Lemma 3.7 is shown in Theorem 1.3. Since $v \equiv 1 \bmod k$ should hold for a MTS(v, λ) to have a k-rotational automorphism, (iii) of Lemma 3.7 is also necessary. Hence the only case we need to concern is that $\lambda \neq 1$, $\lambda \equiv 1, 2 \bmod 3$, $v \equiv 2 \bmod 3$ and $v \equiv 1 \bmod k$. However, if $\lambda \equiv 1, 2 \bmod 3$ and $v \equiv 2 \bmod 3$, $\lambda v(v-1)/3$ is not an integer, which contradicts the existence of a MTS(v, λ). Thus there is no MTS(v, λ) when $v \equiv 2 \bmod 3$ and $\lambda \not\equiv 0 \bmod 3$. Therefore the necessity of (ii) of Lemma 3.7 follows. Finally the other main theorem can be established.

Theorem 3.8 *A k-rotational* MTS(v, λ) *exists if and only if*

(i) $\lambda = 1$, $v \equiv 1, 3, 4 \bmod 6$, $v \equiv 1 \bmod k$ *and* $v \neq 10$,
(ii) $\lambda \neq 1$, $\lambda \equiv 1, 2 \bmod 3$, $v \equiv 0, 1 \bmod 3$ *and* $v \equiv 1 \bmod k$, *or*
(iii) $\lambda \equiv 0 \bmod 3$ *and* $v \equiv 1 \bmod k$.

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