Matching Designs

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Abstract

A collection of k-matchings of K_n with the property that every pair of independent edges lies in exactly λ of the k-matchings is called a $MATCH(n,k,\lambda)$-design and the analogous design for the bipartite graph $K_{n,n}$ is called a $BIMATCH(n,k,\lambda)$-design. Constructions for various $MATCH(n,k,\lambda)$-designs and $BIMATCH(n,k,\lambda)$-designs are given. There is special emphasis on the case $k = 3$.

1. Introduction

Jungnickel and Vanstone [9] studied what have been called hyperfactorizations of index λ of the complete graph K_{2n}. A hyperfactorization consists of a family of perfect matchings of K_{2n} so that every pair of independent edges of K_{2n} lies in exactly λ of the perfect matchings. Their motivation for studying such designs was a desire to construct new t-designs. They showed that a hyperfactorization of K_{2n} of index λ yields a 5-$(2n,6,15\lambda)$ design. Thus, hyperfactorizations of index 1 have come to be of particular interest.

In the present paper, we study a natural generalization of hyperfactorizations, namely, given positive integers n and k, $n \geq 2k$, is it possible to find a family of k-matchings of K_n (a k-matching being a set of k independent edges) so that every pair of independent edges of K_n lies in precisely λ of the k-matchings? A different generalization has been considered by Stinson [12].

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1.1 Definition. By a $MATCH(n,k,\lambda)$-design we shall mean a collection of k-matchings of K_n (repetitions are allowed) so that every pair of independent edges of K_n lies in exactly λ members of the collection. Note that such a design is equivalent to a partition of the edges of λG, where G is the complement of the line graph of K_n and the prefix λ means that each edge has multiplicity λ, into subgraphs isomorphic to K_k.

Let $\mathbb{Z}$ denote the set of integers. There are two obvious necessary conditions for the existence of a $MATCH(n,k,\lambda)$-design:

$$\frac{\frac{\lambda}{2}\binom{n}{2}\binom{n-2}{2}}{\binom{k}{2}} \in \mathbb{Z} \qquad (1)$$

and

$$\frac{\lambda\binom{n-2}{2}}{k-1} \in \mathbb{Z}. \qquad (2)$$

Condition (1) is determined by the number of k-matchings required and condition (2) reflects the number of k-matchings in which a particular edge lies.

We now summarize the known results concerning matching designs. A $MATCH(n,2,1)$-design trivially exists by simply choosing each pair of independent edges exactly once. The trivial $MATCH(n,k,\lambda)$-design is obtained by taking all k-matchings of K_n. In such a design $\lambda = \frac{1}{(k-2)!}\binom{n-4}{2}\binom{n-6}{2}\cdots\binom{n-2(k-1)}{2}$. Any $MATCH(n,k,\lambda)$-design which does not have every k-matching of K_n occurring with the same multiplicity is called *non-trivial.*

The most extensively studied matching designs have been the $MATCH(2m,m,\lambda)$-designs. Note that the trivial design is the only $MATCH(6,3,1)$-design. It has been verified that neither a $MATCH(12,6,1)$-design nor a $MATCH(8,4,1)$-design exists [10,11] and Mathon [11] has shown that there are precisely two non-isomorphic $MATCH(10,5,1)$-designs. Using the Mathieu groups M_{12} and M_{24}, Jungnickel and Vanstone [9] showed that both a $MATCH(12,6,15)$-design and a $MATCH(24,12,495)$-design exist. In particular, for $\lambda = 1$, there is a well-known infinite family of designs. Since the nature of these designs will be required later we will describe their construction.

1.2 Theorem. (see Cameron [3, p. 133]) *A $MATCH(2^a+2, 2^{a-1}+1, 1)$-design exists for all $a \geq 2$.*

Proof. Take a hyperoval H in a projective geometry $PG(2,2^a)$. Let the 2^a+2 points of H be the vertices of the complete graph K_{2^a+2}. Each point of the projective geometry which does not belong to H determines a perfect matching of the complete graph. It is not difficult to show that these perfect matchings produce a $MATCH(2^a+2, 2^{a-1}+1, 1)$-design. ••

The following argument by Godsil, mentioned in [2], establishes that non-trivial $MATCH(2m,m,\lambda)$-designs exist for all m.

Let $A = (a_{ij})$ be the (0,1)-incidence matrix whose rows correspond to pairs of independent edges and whose columns correspond to perfect matchings, where $a_{ij} = 1$ if and only if the pair of independent edges corresponding to the i^{th} row lies in the perfect matching corresponding to the j^{th} column. Let $\mathbf{1}$ denote the column vector of length $(2m-1)(2m-3)\cdots5\cdot3$ all of whose entries are 1. Since each row of A has precisely $(2m-5)(2m-7)\cdots5\cdot3$ ones in it, $A\mathbf{1} = (2m-5)(2m-7)\cdots5\cdot3\mathbf{1}$ so that $Ax = \mathbf{1}$ has a solution over the rational numbers $\mathbb{Q}$. Hence, there is a basic solution over $\mathbb{Q}$. (A solution is basic when the columns of A corresponding to the non-zero entries of the solution are linearly independent.) When $m \geq 5$, $\frac{1}{2}\binom{2m}{2}\binom{2m-2}{2} < \frac{1}{m!}\binom{2m}{2}\binom{2m-2}{2}\cdots\binom{2}{2} = (2m-1)(2m-3)\cdots5\cdot3$. Thus, a basic solution has at most $\frac{1}{2}\binom{2m}{2}\binom{2m-2}{2}$ non-zero entries. Multiplying by an appropriate integer c yields an integer solution of $Ay = c\mathbf{1}$ for some $y \neq \mathbf{1}$ and the resulting design is non-trivial. However, it may have many repeated perfect matchings.

A $MATCH(n,k,\lambda)$-design is said to be *simple* if no k-matching appears more than once. Boros, Jungnickel and Vanstone [2] use Theorem 1.2 as a basis for constructing simple, non-trivial $MATCH(2m,m,\lambda)$-designs. As mentioned, Godsil's proof given above produces non-trivial matching designs but gives no information about simplicity.

We make the remark that if a point is deleted from a $MATCH(2m,m,\lambda)$-design, then a $MATCH(2m-1,m-1,\lambda)$-design results.

2. Some constructions

Much of the remainder of the paper will deal with the case when $k = 3$. Of course, by choosing all 3-matchings of K_n, we obtain a $MATCH(n,3,\binom{n-4}{2})$-design.

We would like to construct matching designs with smaller values of λ. A first step in this direction is the following construction based on matching designs for the complete bipartite graph.

2.1 Definition. By a *BIMATCH*(n,k,λ)-design we shall mean a collection of k-matchings of $K_{n,n}$ (repetitions are allowed) so that every pair of independent edges of $K_{n,n}$ lies in exactly λ members of the collection.

Let $\mathbb{Z}$ denote the set of integers. There are two obvious necessary conditions for the existence of a *BIMATCH*(n,k,λ)-design:

$$\frac{\lambda n^2(n-1)^2}{k(k-1)} \in \mathbb{Z} \tag{3}$$

and

$$\frac{\lambda(n-1)^2}{k-1} \in \mathbb{Z}. \tag{4}$$

2.2 Theorem. *If for $n \geq 2k$ there exists a BIMATCH(n,k,μ)-design and a BIBD$(v-n,n,\gamma)$, then there exists a MATCH(v,k,λ)-design, where* $\lambda = 4\mu\gamma\binom{v-4}{n-2}$.

Proof. Let the v-set be $V = \{1,2,\ldots,v\}$. For each n-set $A \subset V$, take a *BIBD*$(v-n,n,\gamma)$ on the set $V-A$. For each block B in the design, take a *BIMATCH*(n,k,μ)-design on the complete bipartite graph $K_{A,B}$. Consider a typical pair of independent edges 12 and 34. There are four types of n-subsets of V that contain exactly one endvertex from each edge. There are those that contain only 1 and 3, only 1 and 4, only 2 and 3, and only 2 and 4. The number of such subsets is thus $4\binom{v-4}{n-2}$. The total number of blocks containing the other two elements in the design is γ and, for each such block, the pair of independent edges appears in μ k-matchings of $K_{A,B}$. Therefore, we have a *MATCH*(v,k,λ)-design with the value of λ as claimed. • •

2.3 Corollary. *There exist MATCH$(n,3,\lambda)$-designs for all values of $n \geq 6$ where λ is as follows:*

1) $\lambda = 4(n-4)$ *when* $n \equiv 0,4 (mod\ 6)$,
2) $\lambda = 8(n-4)$ *when* $n \equiv 1,3 (mod\ 6)$,
3) $\lambda = 12(n-4)$ *when* $n \equiv 2 (mod\ 6)$, *or*
4) $\lambda = 24(n-4)$ *when* $n \equiv 5 (mod\ 6)$.

Proof. Trivially there is a $BIMATCH(3,3,1)$-design and it is well known (for example, see [6]) that if $n \equiv 0,4 \pmod 6$ there is a $BIBD(n-3,3,1)$, if $n \equiv 1,3 \pmod 6$ there is a $BIBD(n-3,3,2)$, if $n \equiv 2 \pmod 6$ there is a $BIBD(n-3,3,3)$, and if $n \equiv 5 \pmod 6$ there is a $BIBD(n-3,3,6)$. Applying Theorem 2.2 yields the result. ••

Notice that the trivial $MATCH(n,3,\lambda)$-design has $\lambda = \frac{1}{2}(n-4)(n-5)$, whereas, λ is linear in n in the preceding result.

2.4 **Corollary.** *There exist $MATCH(n,4,\lambda)$-designs for all values of $n \geq 8$ where λ is as follows:*

1) $\lambda = 2(n-4)(n-5)$ *when* $n \equiv 5,8 \pmod{12}$,
2) $\lambda = 4(n-4)(n-5)$ *when* $n \equiv 2,11 \pmod{12}$,
3) $\lambda = 6(n-4)(n-5)$ *when* $n \equiv 0,1,4,9 \pmod{12}$, *or*
4) $\lambda = 12(n-4)(n-5)$ *when* $n \equiv 3,6,7,10 \pmod{12}$.

Proof. A $BIMATCH(4,4,1)$-design is shown to exist in Theorem 4.2. A $BIBD(n-4,4,\lambda)$ exists if and only if $\lambda(n-4)(n-5) \equiv 0 \pmod{12}$ and $\lambda(n-5) \equiv 0 \pmod 3$ [6]. Using the smallest possible value of λ for each residue class of n modulo 12 and Theorem 2.2, the result follows. ••

Notice that the trivial $MATCH(n,4,\lambda)$-design has $\lambda = \frac{1}{8}(n-4)(n-5)(n-6)(n-7)$.

Since it is known [6,7,8] that there is a BIBD$(n-5,5,\gamma)$ if and only if $\gamma(n-6) \equiv 0 \pmod 4$, $\gamma(n-5)(n-6) \equiv 0 \pmod{20}$ and $(n-5,5,\gamma) \neq (15,5,2)$, and there is a $BIMATCH(5,5,1)$-design (see Theorem 4.2) a similar corollary can be obtained for $MATCH(n,5,\lambda)$-designs.

What we are particularly interested in are matching designs with $\lambda = 1$. In particular, consider a $MATCH(n,3,1)$-design. With $\lambda = 1$ and $k = 3$, conditions (1) and (2) imply that $n \equiv 2,3 \pmod 4$. We first give two results that yield infinitely many values of n for which such matching designs exist.

2.5 **Theorem.** *If a is even and $a \geq 2$, then there is a $MATCH(2^a+2,3,1)$-design.*

Proof. By Theorem 1.2 we know that a $MATCH(2^a+2,2^{a-1}+1,1)$-design exists. When a is even, $2^{a-1}+1 \equiv 3 \pmod 6$ and there is a $BIBD(2^{a-1}+1,3,1)$. Replace each perfect matching in a $MATCH(2^a+2,2^{a-1}+1,1)$-design by a collection of 3-matchings as determined by the $BIBD(2^{a-1}+1,3,1)$. ••

2.6 Corollary. *If a is odd and $a \geq 2$, then there is a MATCH*$(2^a+2,3,6)$*-design.*

Proof. Using a $BIBD(2^{a-1}+1,3,6)$ instead of a $BIBD(2^{a-1}+1,3,1)$ in the preceding proof yields the result. ••

2.7 Definition. Let M be a subset of 3-matchings in a $MATCH(n,3,1)$*-design*. Let $M_x = \{\{ab,cd\}: \{ab,cd,xy\} \in M$ for some vertex $y\}$. If for each vertex x, M_x is the set of edges of $K_n - x$, then M is called a *core* of the $MATCH(n,3,1)$-design.

Remark. The $MATCH(2^a+2,3,1)$-design of Theorem 2.5 has a core M. To see this recall the proof of Theorem 1.2. Take the perfect matchings determined by the 2^a+1 points on a fixed line outside the hyperoval and observe that they constitute a 1-factorization of K_{2^a+2}. We obtain the core M by taking the 3-matchings that arise from each of these perfect matchings using a $BIBD(2^{a-1}+1,3,1)$ as in the proof of Theorem 2.5.

2.8 Theorem. *If there exists a* $MATCH(n,3,1)$*-design with a core, then there exists a* $MATCH(3n,3,1)$*-design.*

Proof. Take a $MATCH(n,3,1)$-design with vertex-set $\{1,2,...,n\}$ and core M. Consider K_{3n} and partition its vertex-set into three sets $A = \{a_1,a_2,...,a_n\}$, $B = \{b_1,b_2,...,b_n\}$ and $C = \{c_1,c_2,...,c_n\}$ (Figure 1). We will say that the vertices $\{a_i,b_i,c_i\}$ are the vertices of *level i*.

- For each edge $ij \in E(K_n)$, take a $MATCH(6,3,1)$-design on the vertices $\{a_i,a_j,b_i,b_j,c_i,c_j\}$ (Figure 2). In these 3-matchings we have every pair of independent edges whose vertices lie in precisely two levels of K_{3n}.

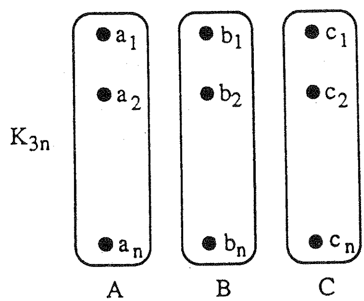

Figure 1: The vertices of K_{3n}

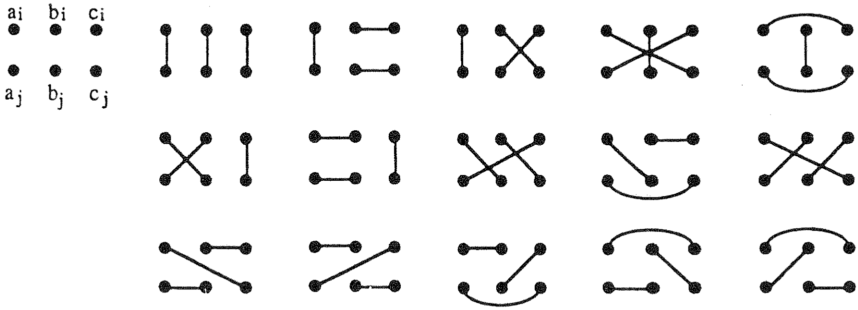

Figure 2: $MATCH(6,3,1)$-design

• For each 3-cycle $\{ij,jr,ir\} \subset E(K_n)$, take the following eighteen 3-matchings on the set of vertices $\{a_i,a_j,a_r,b_i,b_j,b_r,c_i,c_j,c_r\}$ (Figure 3):
$\{a_ia_r,b_jc_r,c_ic_j\}$, $\{a_ia_r,a_jb_r,b_ib_j\}$, $\{a_ia_r,b_ia_j,b_jb_r\}$, $\{a_ia_r,c_ib_j,c_jc_r\}$, $\{a_ia_r,a_jc_r,b_ic_j\}$, $\{a_ia_r,c_ia_j,c_jb_r\}$, $\{b_ib_r,a_ia_j,c_jc_r\}$, $\{b_ib_r,a_ja_r,c_ic_j\}$, $\{b_ib_r,a_ic_j,b_ja_r\}$, $\{b_ib_r,c_ia_j,b_jc_r\}$, $\{b_ib_r,a_jc_r,c_ib_j\}$, $\{b_ib_r,a_ib_j,c_ja_r\}$, $\{a_ia_j,b_ja_r,c_ic_r\}$, $\{b_ib_j,c_ja_r,c_ic_r\}$, $\{b_ic_j,b_jb_r,c_ic_r\}$, $\{a_ib_j,a_ja_r,c_ic_r\}$, $\{b_ia_j,c_jb_r,c_ic_r\}$, and $\{a_ic_j,a_jb_r,c_ic_r\}$.

Let σ be the permutation given by $\sigma = (a_r\ b_r\ c_r)$. Add to the above 3-matchings a further thirty-two obtained by applying σ and σ^2 to each of the eighteen 3-matchings given in Figure 3. For example, from the 3-matching $\{a_ia_r,\ b_jc_r,\ c_jc_i\}$ we obtain the 3-matchings $\{a_ib_r,\ b_ja_r,\ c_jc_i\}$ and $\{a_ic_r,\ b_jb_r,\ c_jc_i\}$.

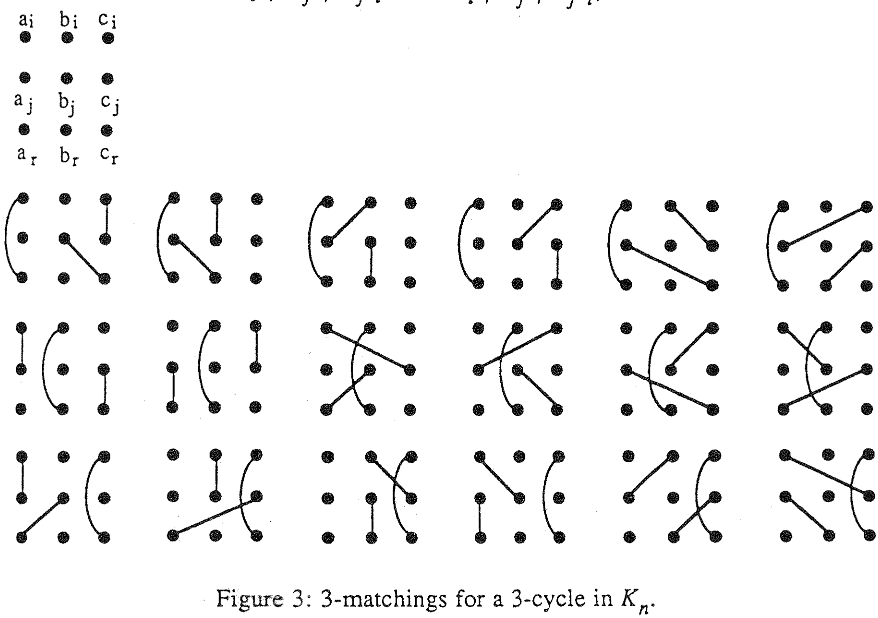

Figure 3: 3-matchings for a 3-cycle in K_n.

In the 3-matchings defined from a given 3-cycle in K_n no edge lies in a level, and any pair of independent edges they contain has vertices from exactly three levels. Moreover, a close inspection of these fifty-four 3-matchings shows that every possible pair of independent edges of this type occurs exactly once in one of these 3-matchings. Since such 3-matchings are defined for all possible 3-cycles, every such pair of independent edges in K_{3n} lies in one of these $54 \cdot \binom{n}{3}$ 3-matchings.

- For every 3-matching $\{ij,tr,ms\}$ in the matching design $MATCH(n,3,1)$, take the following twenty-seven 3-matchings given in nine groups of three each (Figure 4):

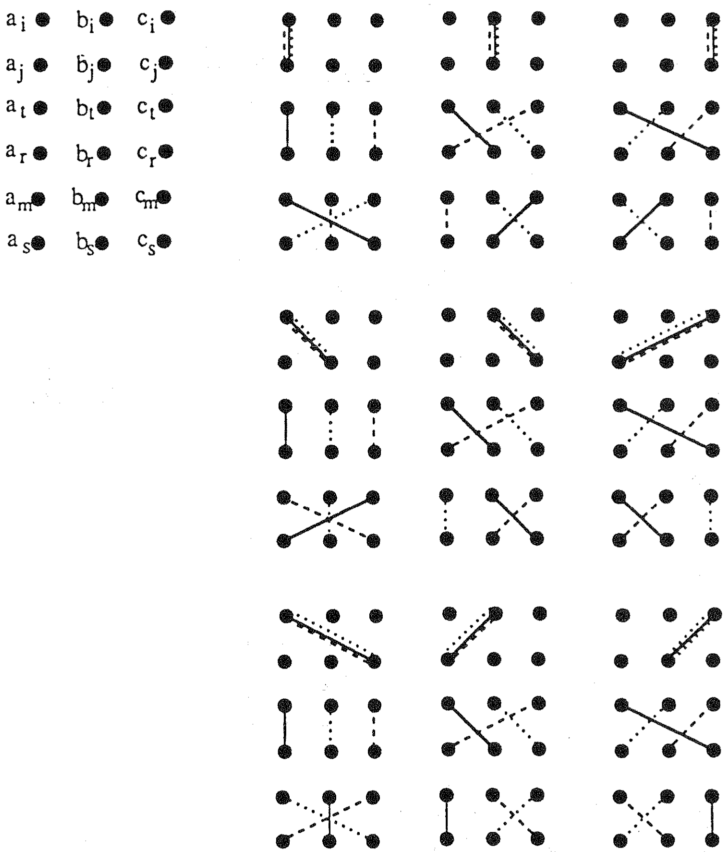

Figure 4: 3-matchings from the $MATCH$(n,3,1)-design

$M_1 = \{\{a_ia_j,a_ta_r,a_mc_s\},\ \{a_ia_j,b_tb_r,c_ma_s\},\ \{a_ia_j,c_tc_r,b_mb_s\}\}$;
$M_2 = \{\{b_ib_j,a_tb_r,c_mb_s\},\ \{b_ib_j,b_tc_r,b_mc_s\},\ \{b_ib_j,c_ta_r,a_ma_s\}\}$;
$M_3 = \{\{c_ic_j,a_tc_r,b_ma_s\},\ \{c_ic_j,b_ta_r,a_mb_s\},\ \{c_ic_j,c_tb_r,c_mc_s\}\}$;
$M_4 = \{\{a_ib_j,a_ta_r,c_ma_s\},\ \{a_ib_j,b_tb_r,b_mb_s\},\ \{a_ib_j,c_tc_r,a_mc_s\}\}$;
$M_5 = \{\{b_ic_j,a_tb_r,b_mc_s\},\ \{b_ic_j,b_tc_r,a_ma_s\},\ \{b_ic_j,c_ta_r,c_mb_s\}\}$;
$M_6 = \{\{c_ia_j,a_tc_r,a_mb_s\},\ \{c_ia_j,b_ta_r,c_mc_s\},\ \{c_ia_j,c_tb_r,b_ma_s\}\}$;
$M_7 = \{\{a_ic_j,a_ta_r,b_mb_s\},\ \{a_ic_j,b_tb_r,a_mc_s\},\ \{a_ic_j,c_tc_r,c_ma_s\}\}$;
$M_8 = \{\{b_ia_j,a_tb_r,a_ma_s\},\ \{b_ia_j,b_tc_r,c_mb_s\},\ \{b_ia_j,c_ta_r,b_mc_s\}\}$; and
$M_9 = \{\{c_ib_j,a_tc_r,c_mc_s\},\ \{c_ib_j,b_ta_r,b_ma_s\},\ \{c_ib_j,c_tb_r,a_mb_s\}\}$.

From each of these 3-matchings we obtain a further two by letting the permutations β and β^2 act on it, where $\beta = (a_i\ b_i\ c_i)(a_j\ b_j\ c_j)(a_t\ b_t\ c_t)(a_r\ b_r\ c_r)(a_m\ b_m\ c_m)(a_s\ b_s\ c_s)$; for a total of eighty-one 3-matchings. Note that every pair of independent edges of K_{3n} which has vertices from four different levels lies in one of the 3-matchings. To see this note that if the edges are $\{x_iy_j, z_tw_r\}$, then there is a 3-matching in the $MATCH(n,3,1)$-design containing the edges $\{ij, tr\}$ and careful scrutiny shows that among the eighty-one 3-matchings arising from it, is one that contains the pair of edges $\{x_iy_j, z_tw_r\}$. Reflection shows that in the 3-matchings so far described we have every pair of independent edges occuring except when the edges cover three levels and one of them lies entirely in a level. To take care of this we are now going to delete some of these 3-matchings and replace them by others.

For each 3-matching in the core M, replace the fifty-four 3-matchings $\{X, \beta(X): X \in \bigcup_{i=1}^{9} M_i\}$ obtained from it, by the following one hundred and sixty-two 3-matchings. Begin with three copies of $I = \{X: X \in \bigcup_{i=1}^{9} M_i\}$ and three copies of $\beta(I) = \{\beta(X): X \in \bigcup_{i=1}^{9} M_i\}$. Let τ be the permutation $\tau = (a\ b\ c)$, and let $x \in \{a,b,c\}$.

In the first copy of I, if x_ix_j is an edge of a 3-matching replace it by the edge a_ib_i; if $x_i\tau(x)_j$ is an edge of a 3-matching replace it by the edge b_ic_i; and if $x_i\tau^2(x)_j$ is an edge of a 3-matching replace it by the edge a_ic_i. In the second copy of I, if x_ty_r is an edge of the 3-matching replace it by the edge $x_t\tau(x)_t$. In the third copy, if x_mx_s is an edge of the 3-matching replace it by the edge a_mb_m; if $x_m\tau(x)_s$ is an edge of a 3-matching replace it by the edge b_mc_m; and if $x_m\tau^2(x)_s$ is an edge of a 3-matching replace it by the edge a_mc_m. In the first copy of $\beta(I)$, if x_ix_j is an edge of a 3-matching replace it by the edge a_jb_j; if $x_i\tau(x)_j$ is an edge of a 3-matching replace it by the edge b_jc_j; and if $x_i\tau^2(x)_j$ is an edge of a 3-matching replace it by the edge a_jc_j. In the second copy of $\beta(I)$, if $x_t y_r$ is an edge of the 3-matching replace it by the edge $x_r \tau(x)_r$. In the third copy, if $x_m x_s$ is an edge of the 3-matching replace it by the edge $a_s b_s$; if $x_m \tau(x)_s$ is an edge of a 3-matching replace it by the edge $b_s c_s$; and if $x_m \tau^2(x)_s$ is an edge of a 3-matching replace it by the edge $a_s c_s$. For example if $\{ij, tr, ms\}$ is a 3-matching in M, then the three 3-matchings derived from those of $M_1 \cup \{\beta(X): X \in M_3\}$ are shown in Figure 5.

There are two points to be noted. First, any two independent edges in K_{3n}, where one edge lies in a level and the other covers an additional two levels, occurs in one of the 3-matchings arising from the 3-matchings of M. Suppose one edge lies in level i and the other covers levels t and r. In M there is a 3-matching containing the pair $\{ij, tr\}$ for exactly one value of j. So we need to study all 3-matchings arising from this 3-matching in M. A close study reveals that every edge in a level occurs with every other edge that covers a further two levels. The second point is that any pair of independent edges that occurred in one of the fifty-four 3-matchings of $I \cup \beta(I)$ also occurs in one of the one hundred and sixty-two 3-matchings defined from $I \cup \beta(I)$. Again this is revealed by a close study of the 3-matchings. These observations combined with our earlier comments imply that every pair of independent edges in K_{3n} lies in a 3-matching.

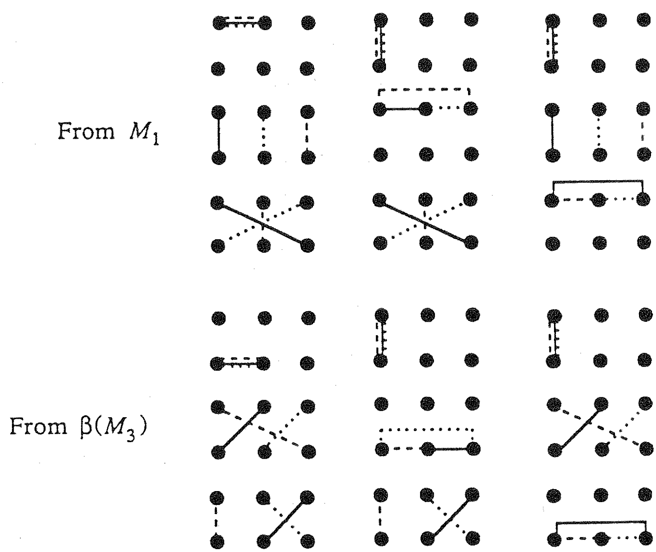

Figure 5: The 3-matchings derived from $M_1 \cup \{\beta(X): X \in M_3\}$

In fact, we have shown that every pair of independent edges in K_{3n} lies in exactly one 3-matching. However, this also follows by counting the total number of 3-matchings: From the first step we have $\binom{n}{2}\cdot 15$; from the second $\binom{n}{3}\cdot 18\cdot 3$; and from the third $((\binom{n}{2}\binom{n-2}{2}/6) - |M|)\cdot 81 + |M|\cdot(162+27)$. Noting that $|M| = n\cdot\binom{n-1}{2}/12$, we find that the total is indeed $\binom{3n}{2}\binom{3n-2}{2}/6$ as required. ••

2.9 Corollary. *If a is even, there is a MATCH*$(3\cdot 2^a+6,3,1)$*-design.*

Proof. This follows from Theorem 2.8 and the remark after Definition 2.7. ••

3. *MATCH*(*n*,3,1)-designs for small values of *n*

Recall from Section 2 that a *MATCH*(n,3,1)-design can exist only if $n \equiv 2, 3 \pmod 4$. We will now consider the existence of these designs for small values of n. As we have already observed in Figure 2, there is a *MATCH*(6,3,1)-design.

3.1 Theorem. *There is no MATCH*(7,3,1)*-design.*

Proof. If there exists a *MATCH*(7,3,1)-design M, there are thirty-five 3-matchings in M according to condition (1). Condition (2) implies that each edge xy of K_7 lies in five 3-matchings of M and, thus, these five 3-matchings must cover all the edges of K_7-$\{x,y\}$.

Each vertex x lies in precisely thirty 3-matchings because each of the six edges incident with x lies in five 3-matchings. Consider the five 3-matchings of M that miss the vertex x. We prove they must be edge-disjoint. If not, two of them have an edge yz in common. The union of the remaining two edges of these two 3-matchings must be a 4-cycle on the vertices $V(K_7)$-$\{x,y,z\}$. The remaining three 3-matchings containing the edge yz must cover the edges of K_7-$\{y,z\}$ with a 4-cycle removed, but this is impossible.

Now add a new vertex w to K_7. To all five 3-matchings missing the vertex x, add the edge wx. Do this for each vertex of K_7. This gives a *MATCH*(8,4,1)-design which, as mentioned in Section 1, does not exist. Therefore, there is no *MATCH*(7,3,1)-design. ••

The next value is n = 10, and we do not know of the existence of a *MATCH*(10,3,1)-design.

3.2 Theorem. *There is a MATCH(11,3,1)-design.*

Proof. Let $V(K_{11}) = \{u_1, u_2, \ldots, u_{11}\}$ and let σ' be the permutation given by $\sigma' = (u_1\ u_2\ u_3 \cdots u_{11})$. Consider the permutation σ acting on $E(K_{11})$ induced by σ'. In the decomposition of σ into disjoint cycles, there are five cycles of length 11. One cycle contains the edge u_1u_2, another cycle contains the edge u_1u_3, another contains u_1u_4, another u_1u_5, and the last one u_1u_6.

We take the following 3-matchings and let σ act on each of them:

$\{u_1u_2, u_5u_9, u_7u_{11}\}$, $\{u_1u_2, u_6u_{10}, u_{11}u_4\}$, $\{u_1u_6, u_4u_7, u_{10}u_2\}$, $\{u_1u_6, u_7u_{10}, u_{11}u_3\}$,
$\{u_1u_3, u_4u_5, u_9u_{10}\}$, $\{u_1u_3, u_6u_7, u_8u_9\}$, $\{u_1u_3, u_5u_8, u_6u_9\}$, $\{u_1u_3, u_8u_{11}, u_{10}u_2\}$,
$\{u_1u_4, u_5u_6, u_8u_9\}$, $\{u_1u_4, u_6u_7, u_{10}u_{11}\}$, $\{u_1u_2, u_6u_{11}, u_9u_3\}$, $\{u_1u_2, u_4u_9, u_5u_{10}\}$,
$\{u_1u_5, u_3u_8, u_{10}u_4\}$, $\{u_1u_5, u_2u_7, u_4u_9\}$, $\{u_1u_5, u_6u_8, u_7u_9\}$, $\{u_1u_5, u_4u_6, u_{11}u_2\}$,
$\{u_1u_6, u_2u_4, u_8u_{10}\}$, $\{u_1u_6, u_3u_5, u_{11}u_2\}$, $\{u_1u_4, u_2u_6, u_{10}u_3\}$, $\{u_1u_4, u_6u_{10}, u_7u_{11}\}$,
$\{u_1u_2, u_{10}u_3, u_6u_8\}$, $\{u_1u_2, u_3u_8, u_4u_7\}$, $\{u_1u_2, u_{11}u_5, u_8u_{10}\}$, $\{u_1u_5, u_6u_{11}, u_{10}u_2\}$,
$\{u_1u_5, u_9u_{11}, u_4u_7\}$, $\{u_1u_2, u_3u_7, u_{10}u_4\}$, $\{u_1u_2, u_4u_8, u_{11}u_3\}$, $\{u_1u_2, u_3u_5, u_6u_9\}$,
$\{u_1u_5, u_9u_3, u_2u_4\}$, and $\{u_1u_6, u_7u_9, u_8u_{11}\}$.

This gives a suitable family of 3-matchings. ••

At this point recall that there is a *MATCH*$(n,3,1)$-design if and only if there is a partition of the edges of the complement of the line-graph of K_n into 3-cycles. Nash-Williams (see [1, p. 237]) has conjectured that if G is a graph with $|V(G)| = n \geq 15$, in which each vertex has even degree at least $\frac{3n}{4}$ and $|E(G)|$ is a multiple of 3, then the edges of G can be partitioned into triangles. Should this conjecture be true it would immediately imply that *MATCH*$(n,3,1)$-designs exist for all n, $n \equiv 2,3 \pmod 4$, except $n = 7$,and perhaps $n = 10$ and $n = 14$.

The next design we have been able to construct is a *MATCH*(15,3,1)-design. We do not know if a *MATCH*(14,3,1)-design exists.

3.3 Theorem. *There is a MATCH(15,3,1)-design.*

Proof. Let G denote the complement of $L(K_{15})$. Let σ denote the permutation acting on $V(G)$ induced by the permutation $\sigma' = (v_0\ v_1 \cdots v_{14})$ on the vertices of K_{15}. The disjoint cycle decomposition of σ has seven cycles of length 15. Let G_i, $1 \leq i \leq 7$, denote the graph K_{15} with all edges of length i removed. The graphs $G_1, G_2, \ldots, G_7$ are the subgraphs induced on the cycles of σ by G. Colbourn and Rosa [3] have proved that such a graph has an edge-partition into triangles. Thus, it suffices to prove that the remainder of the edges of G can be partitioned into triangles.

We shall say that the edge from $u_k u_{i+k}$ of G_i to $u_r u_{r+j}$ of G_j has *jump* $r - k$ computed modulo 15 on the residues 0, 1, ..., 14. The edges joining G_i to G_j in G are all possible edges except those of jumps 0, i, i-j, and -j from G_i to G_j.

There exists a $BIBD(7,3,1)$ on the element-set $G_1, G_2, \ldots, G_7$. Let $\{G_i, G_j, G_k\}$ be one of the blocks, and consider the subgraph of $K_{15,15,15}$ defined by it. Under the action of σ, the missing edges in this subgraph can be partitioned into the triangles:

$$\{u_1u_{i+1}, u_1u_{j+1}, u_1u_{k+1}\}, \qquad \{u_1u_{i+1}, u_{i+1}u_{i+j+1}, u_{i+1-k}u_{i+1}\},$$
$$\{u_1u_{i+1}, u_{i-j+1}u_{i+1}, u_{i+1}u_{i+k+1}\}, \text{ and } \{u_1u_{i+1}, u_{-j+1}u_1, u_{-k+1}u_1\}.$$

These triangles define a partial latin square of order 15, say $A = (a_{rs})$, by $a_{rs} = t$ if and only if $\{u_r u_{i+r}, u_s u_{j+s}, u_t u_{k+t}\}$ is one of the triangles. Clearly, a completion of this square to a latin square yields a partition of the edges of the tripartite subgraph on $G_i \cup G_j \cup G_k$ into triangles.

Shown below are the seven triples of an $BIBD(7,3,1)$ and the first row of a latin square associated with each, where the boldface entries correspond to the missing triples. The remainder of each square is determined by $a_{i+t,j+t} = a_{ij} + t$, with calculations modulo 15.

G_1,G_2,G_4:	**0**	**12**	7	10	8	4	9	2	14	6	3	5	13	**11**	**1**
G_2,G_3,G_5:	**0**	3	**12**	14	8	6	11	13	5	1	9	4	**10**	7	**2**
G_3,G_4,G_6:	**0**	4	13	**12**	6	11	7	14	1	8	5	**9**	2	10	**3**
G_4,G_5,G_7:	**0**	13	11	7	**12**	1	9	6	3	10	**8**	2	14	5	**4**
G_5,G_6,G_1:	**0**	13	12	11	8	**4**	7	3	10	**14**	2	9	6	1	**5**
G_6,G_7,G_2:	**0**	7	3	11	8	2	**4**	1	**13**	12	5	10	14	9	**6**
G_7,G_1,G_2:	**0**	9	6	2	10	14	7	**4**	13	11	5	3	8	1	**12**

This completes the proof of the theorem. ••

The proof technique used for Theorem 3.3 can be generalized whenever $n \equiv 3 \pmod{12}$. Then n is odd, $\binom{n}{2} - n \equiv 0 \pmod 3$ and $\frac{n-1}{2} \equiv 1 \pmod 6$. Thus, each G_i has a partition into triangles [4] and there is a $BIBD(\frac{n-1}{2},3,1)$. All that needs to be shown is that the partial latin squares corresponding to the blocks of the $BIBD(\frac{n-1}{2},3,1)$ can be completed. This suggests the following question.

3.4 Question. Does there exist an N such that if four transversals of a partial latin square of order $n \geq N$ are prescribed, then the square can always be completed? More generally, does there exists an $N(k)$ such that if k transversals of a partial latin square of order $n \geq N(k)$ are prescribed, the square can always be completed?

The following shows that $N(4) \geq 10$. Let the first row of a 9×9 partial latin square $A = (a_{ij})$ be 1 7 * * * * * 6 2, where * denotes an empty cell, and let the remaining entries be defined by $a_{i+t,j+t} = a_{ij} + t$ if $a_{ij} \neq *$ and $a_{i+t,j+t} = *$ if $a_{ij} = *$. This cannot be completed to a 9×9 latin square.

4. Bipartite matching designs

It is apparent from Theorem 2.2 that bipartite matching designs are important in the construction of matching designs. This naturally leads to the consideration of the existence of $BIMATCH(n,k,\lambda)$-designs with λ small. In the important special case that $k = n$, conditions (3) and (4) are always satisfied for $\lambda = 1$. This suggests the possibility that a $BIMATCH(n,n,1)$-design exists for all $n \geq 2$. The following result gives one way to obtain bipartite matching designs.

4.1 Proposition. *If a $BIMATCH(n,n,1)$-design exists and there is a $BIBD(n,k,\lambda)$, then there exists a $BIMATCH(n,k,\lambda)$-design.*

Proof. For each perfect matching M of a $BIMATCH(n,n,1)$-design, take a $BIBD(n,k,\lambda)$ D with the n edges of M as the points of D. For each block of D, take the k-matching made up of the edges corresponding to the set of points in the block. It is easy to see that each pair of independent edges in $K_{n,n}$ lies in precisely λ k-matchings. ••

If we knew that a $BIMATCH(n,n,1)$-design existed for all $n \geq 2$, then we could apply Proposition 4.1 to prove a variety of results. However, we do not know whether or not a $BIMATCH(n,n,1)$-design always exists. Nevertheless, we can prove the following result.

4.2 Theorem. *There exists a $BIMATCH(n,n,1)$-design whenever n is a prime power.*

Proof. Since n is a prime power, there exists a complete set $A_1, A_2, \ldots, A_{n-1}$ of mutually orthogonal latin squares of side n. Let $M_{i,1}, M_{i,2}, \ldots, M_{i,n}$ be the n perfect matchings of $K_{n,n}$ corresponding to the latin square A_i, $i = 1, 2, \ldots, n-1$. This yields

$n(n-1)$ perfect matchings. If the same pair of independent edges xy and uv appears in two distinct perfect matchings in this set, there are two distinct latin squares each of which has the same entry in the (x,y) and (u,v) cells. This contradicts the orthogonality of the two squares. It then follows that we have a $BIMATCH(n,n,1)$-design whenever n is a prime power. ••

While the existence of a complete set of mutually orthogonal latin squares of side n implies the existence of a $BIMATCH(n,n,1)$-design the converse need not be true. There must be $n(n-1)$ perfect matchings in the $BIMATCH(n,n,1)$-design, but there is no requirement that they have a partition into $n-1$ sets of n perfect matchings so that every set corresponds to a latin square.

4.3 Corollary. *If n is a prime power and $n \geq 3$, there exists a $BIMATCH(n,3,\lambda)$-design with λ taking on the following values:*

(i) $\lambda = 1$, *when* $n \equiv 1$ *or* $3(mod\ 6)$,

(ii) $\lambda = 2$, *when* $n \equiv 4(mod\ 6)$,

(iii) $\lambda = 3$, *when* $n \equiv 5(mod\ 6)$, *and*

(iv) $\lambda = 6$, *when* $n \equiv 2(mod\ 6)$.

There is another method for constructing $BIMATCH(n,3,1)$-designs. Conditions (3) and (4) imply that $n \equiv 1$ or 3(mod 6) must hold for a $BIMATCH(n,3,1)$-design to exist. Let the bipartition sets of $K_{n,n}$ be $\{i_1: 1 \leq i \leq n\}$ and $\{i_2: 1 \leq i \leq n\}$. Take a $BIBD(n,3,1)$ with blocks $\mathcal{T}$. For each block $T = \{a,b,c\} \in \mathcal{T}$, take all six 3-matchings on $K_{3,3}$ with vertex bipartition $\{a_1,b_1,c_1\}$ and $\{a_2,b_2,c_2\}$. For each triple $\{a,b,c\}$ of $\{1,2,...,n\}$ which is not a block of $\mathcal{T}$, take the two 3-matchings $\{a_1b_2, b_1c_2, c_1a_2\}$ and $\{a_1c_2, b_1a_2, c_1b_2\}$. At this point all pairs of independent edges in $K_{n,n}$ of the form x_1y_2 and u_1v_2, where $|\{x,y,u,v\}| \leq 3$, appear in one 3-matching. Only pairs of edges of the form x_1y_2 and u_1v_2, where $|\{x,y,u,v\}| = 4$, do not appear in any 3-matching.

To finish the construction, we use a $MATCH(n,3,1)$-design which requires that $n \equiv 2$ or 3(mod 4) and, because of the above conditions on n, we see that we are restricted to $n \equiv 3$ or 7(mod 12). For each triple $\{ab,cd,ef\}$ in the $MATCH(n,3,1)$-design, we take the 3-matchings $\{a_1b_2, c_1d_2, e_1f_2\}$, $\{a_1b_2, d_1c_2, f_1e_2\}$, $\{b_1a_2, c_1d_2, f_1e_2\}$, and $\{b_1a_2, d_1c_2, e_1f_2\}$. Every pair of independent edges of $K_{n,n}$ now appears precisely once.

Unfortunately, none of the $MATCH(n,3,1)$-designs arising from Theorem 2.5 and Corollary 2.9 satisfy $n \equiv 3$ or $7 \pmod{12}$. The only one available to us is a $MATCH(15,3,1)$-design so that the following result is proved.

4.4 Proposition. *There exists a $BIMATCH(15,3,1)$-design.*

5. A comment on t-$MATCH(n,k,\lambda)$-designs

In the same spirit as $MATCH(n,k,\lambda)$-designs were defined, we can also define a design in which we ask that every subset of t independent edges lies in exactly λ k-matchings. The notation for this will be t-$MATCH(n,k,\lambda)$-design. Godsil's proof, given in Section 1, can be generalized to show that non-trivial t-$MATCH(n,k,\lambda)$-designs exist. The number of columns in the incidence matrix is

$$\frac{n!}{2^k(n-2k)!k!}$$

and the number of rows is

$$\frac{n!}{2^t(n-2t)!t!}.$$

Thus, the number of rows is smaller than the number of columns when $k > t$ and $n \geq 2(k+t)$.

We can generalize the idea used in Corollary 2.3 for constructing $MATCH(n,3,\lambda)$-designs to construct t-$MATCH(n,k,\lambda)$-designs. We illustrate this using Steiner quadruple systems 3-$(v,4,1)$, which exist exactly when $v \equiv 2$ or $4 \pmod 6$ [5]. Let $n \equiv 0$ or $2 \pmod 6$ and let $N = \{1,2,\ldots,n\}$. For each $\{a,b,c,d\} \in \binom{N}{4}$, choose a 3-$(n-4,4,1)$ design on $N-\{a,b,c,d\}$. For every block $\{i,j,k,r\}$ of the design, take the 4! 4-matchings in $K_{4,4}$ with vertex partitions $\{a,b,c,d\}$ and $\{i,j,k,r\}$.

To see that this gives us a 3-$MATCH(n,4,8(n-6))$-design, consider the three independent edges 12, 34 and 56. They come from designs on $N-\{1,3,5,x_1\}$, $N-\{1,3,6,x_2\},\ldots,$ $N-\{2,4,6,x_8\}$, where $x_i \notin \{1,2,3,4,5,6\}$, and so has $n-6$ possible values in each case. Notice that the trivial 3-$MATCH(n,4,\lambda)$-design has $\lambda = \binom{n-6}{2}$.

The preceding method will not be fruitful in general because little is known about t-designs for $t > 4$.

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